Event-Triggered Stabilization of Linear Systems Under Bounded Bit Rates

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Abstract—**This paper addresses the problem of exponential practical stabilization of linear time-invariant systems with disturbances using event-triggered control and bounded communication bit rate. We consider both the case of instantaneous communication with finite precision data at each transmission and the case of non-instantaneous communication with bounded communication rate. Given a prescribed rate of convergence, the proposed event-triggered control implementations opportunistically determine the transmission instants and the finite precision data to be transmitted on each transmission. We show that our design exponentially practically stabilizes the origin while guaranteeing a uniform positive lower bound on the inter-transmission and inter-reception times, ensuring that the number of bits transmitted on each transmission is upper bounded uniformly in time, and allowing for the possibility of transmitting fewer bits at any given time if more bits than prescribed were transmitted earlier. We also characterize the necessary and sufficient average data rate for exponential practical stabilization. Several simulations illustrate the results.**

I. Introduction

The digital nature of communication in networked control systems naturally induces sampling and quantization of signals. The increasing ubiquity of these systems, particularly in resource-constrained domains where communication channels have low, time-varying, and possibly unreliable channel capacity, has brought to the forefront the need for integrated and systematic design methodologies that go beyond adhoc approaches. This paper is a contribution to the modern body of research that seeks to fundamentally address the problem of control under constrained resources. Specifically, we seek to combine the strengths of event-triggered control and information theory to efficiently stabilize linear time-invariant systems under communication constraints.

Literature Review: The need for systems integration and the importance of bridging the gap between computing, communication, and control in the study of cyberphysical systems cannot be overemphasized [1], [2]. The present work builds on two areas of research that address the stabilization of control systems under limited information from different and complementary perspectives. In the information-theoretic approach to control under communication constraints, the focus is on determining sufficient and necessary conditions on the *bit data rates* (i.e., the number of bits transmitted over possibly multiple transmissions during an arbitrary time interval) that guarantee stabilization under varying assumptions on the communication channels. The works [3], [4] provide comprehensive accounts of this by now vast literature, and we highlight next a few references most relevant to the discussion here. Early data rate results appeared in [5]–[7], which employ the idea of countering the information generated (the growth in the uncertainty of the system state) with a sufficiently high data rate of the encoded feedback. This approach has been successful in providing tight necessary and sufficient conditions on the bit rate of the encoded feedback for asymptotic stabilization in the discrete time setting. Subsequently, similar ideas have been used to provide data rate theorems also for stochastic rate channels [8] and extended to vector systems and time-varying feedback channels [9] and Markov feedback channels [10]. In the continuous-time setting, the problem has been mainly studied under the assumption of periodic sampling or aperiodic sampling with known upper and lower bounds on the sampling period for single input systems [11], [12], nonlinear feedforward systems (single input systems [13], and switched linear systems [14]. In this context, it is not known if and how a best sampling period may be designed or if state-based aperiodic sampling can provide any advantages in the efficiency and performance of the resulting implementation. With a few exceptions, see e.g., [14], the works above do not characterize the convergence rates or explore the problem of guaranteeing a desired performance.

Event-triggered control, instead, seeks to trade computation and decision-making for less communication, sensing, or actuation effort, while guaranteeing a desired level of performance. This literature, see e.g. [15]–[17] and references therein, exploits the tolerance to measurement errors to design goal-driven state-based aperiodic sampling for the efficient use of the system resources. The main focus of this body of work is on minimizing the number of updates while guaranteeing the feasibility of the resulting real-time implementation. When interpreted in terms of communication, this results in a paradigm where one seeks to minimize the number of transmissions while largely ignoring the quantization aspect and allowing the data at each transmission to be of infinite precision. Among the few exceptions, we mention event-triggered schemes with static logarithmic quantization [18], [19] and dynamic quantization [20]–[23]. In [18], events are defined as the system state crossing static quantization cells, communication is assumed to be instantaneous and there are no disturbances. [19] considers the problem with modeling errors and communication delays. Both these papers do not explicitly study the notion of *communication bit rate* (i.e., the number of bits per transmission). In [20]–[23], the events are defined as the infinity norm of the encoding error crossing a fixed or piecewise-constant threshold. [20] considers instantaneous communication and

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external disturbances, although the use of a fixed threshold in the event-triggering condition results in practical stability even under no disturbance. In addition, if the channel imposes a bound on the communication bit rate, then it also affects the ultimate bound on the state. [21] addresses the problem for nonlinear systems and with communication delays, while [22], [23] extend these results to the case with external disturbance. All these works guarantee a positive lower bound on the inter-transmission times, while [20]–[23] also provide a uniform bound on the communication bit rate. Interestingly, these references do not address the inverse problem of triggering and quantization given a limit on the communication bit rate imposed by the channel. While guarantees on the uniform boundedness of the communication bit rate are useful, they do not characterize either necessary or sufficient conditions on the required data rates, i.e., the number of bits averaged over a finite or infinite time horizon. In fact, this has been a shortcoming of the event-triggered control literature on the whole, where the availability of such analytical results would be useful in the design of networked control systems as well as in quantifying their improvement over time-triggered implementations. Finally, the common underlying approach in the event-triggered works mentioned above is based on the notion of input-to-state-stability with respect to measurement errors for both event-triggering and quantization. This is in contrast with the information-theoretic data rate approach to quantization and encoding we adopt in the present work.

Statement of Contributions: This paper designs event-triggered controllers for linear-time invariant systems under bounded communication bit rate. We focus on the control goal of exponential practical stabilization, in the presence of disturbance and with a prescribed rate of convergence. The first contribution is the identification of a necessary condition on the average bit rate required for all solutions of a linear-time invariant system to exponentially converge with a prescribed convergence rate. Our second set of contributions pertain to the design of event-triggered controllers that guarantee exponential convergence with a desired performance by adjusting the communication rate in accordance with state information in an opportunistic fashion. We consider increasingly realistic scenarios, ranging from instantaneous transmissions with arbitrary, but finite communication rate, through instantaneous transmissions with uniformly bounded communication rate, to finally non-instantaneous transmissions with arbitrary bounded communication rate imposed by the channel. In all cases, our design guarantees the existence of a uniform positive lower bound on inter-transmission and inter-reception times, and ensures that the number of bits transmitted at each transmission is upper bounded. An overarching contribution of the paper is the introduction of the information-theoretic data rate approach to quantization and encoding to complement event-triggering for data rate limited feedback control. From an event-triggered control perspective, our key contribution is going beyond the paradigm of infinite precision at each transmission and adopting the information-theoretic approach to quantization, encoding, and triggering. This allows us to characterize necessary and sufficient data rates averaged over time, and quantify the capability to transmit fewer bits if more bits than prescribed were transmitted earlier. From an information-theoretic perspective, our key contribution is the efficient use of the communication resources by exploiting state-based opportunistic sampling. This allows us to tune the operation of the control system to the desired level of performance and guarantee a desired convergence rate.

Organization: Section II formally states the asymptotic stabilization problem under event-triggered control and finite communication bit rate. Section III identifies a necessary condition on the average bit rate required for all solutions to asymptotically converge with a prescribed convergence rate. Sections IV and V present our event-triggered control design with bounded communication rate under instantaneous and non-instantaneous communication, respectively. Section VI presents simulation results. Finally, Section VII gathers our conclusions and ideas for future work.

Notation: We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of real, nonnegative real, positive integer, and nonnegative integer numbers, respectively. We let I_n and $0_n \in \mathbb{R}^{n\times n}$ denote the identity and zero matrix, respectively, of dimension n. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we let $\lambda_m(A)$ and $\lambda_M(A)$ denote its smallest and largest eigenvalues, respectively. For a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and all $x \in \mathbb{R}^n$,

$$\sqrt{\lambda_m(P)}\|x\|_2 \leq \sqrt{x^T P x} \leq \sqrt{\lambda_M(P)}\|x\|_2. \quad (1)$$

Given $A_1, A_2 \in \mathbb{R}^{n\times n}$, $A_1 \prec A_2$ denotes that $A_1 - A_2$ is negative definite. Similarly, the symbols $\preceq$, $\succ$ and $\succeq$ stand for negative semi-definiteness, positive definiteness and positive semi-definiteness, respectively. We denote by $\|.\|_2$ and $\|.\|_\infty$ the Euclidean and infinity norm of a vector, respectively, or the corresponding induced norm of a matrix. For $A \in \mathbb{R}^{n\times m}$, we let A^+ denote the pseudoinverse. For $A \in \mathbb{R}^{n\times n}$, note that $\|e^{A\tau}\|_2 \leq e^{\|A\|_2\tau}$. Finally, for a function $f : \mathbb{R} \mapsto \mathbb{R}^n$ and any $t \in \mathbb{R}$, we let $f(t^-)$ denote the limit from the left, $\lim_{s\uparrow t} f(s)$.

II. Problem Statement

Consider a plant whose dynamics is given by a linear time-invariant control system,

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t), \quad (2)$$

where $x \in \mathbb{R}^n$ denotes the state of the plant, $u \in \mathbb{R}^m$ is the control input and $v \in \mathbb{R}^n$ is an unknown disturbance. Here, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrices. We assume that the pair (A, B) is stabilizable, i.e., there exists a control gain matrix $K \in \mathbb{R}^{m\times n}$ such that the matrix $\bar{A} = A + BK$ is Hurwitz, and that the disturbance is uniformly bounded by a known constant, i.e.,

$$\|v(t)\|_2 \leq \nu, \ \forall t \in [0, \infty). \quad (3)$$

Under these assumptions, $u(t) = Kx(t)$ renders the origin of (2) globally exponentially practically stable.

The plant is equipped with a sensor and an actuator, which we assume are not co-located. Further, we assume that the sensor can measure the state exactly, and that the actuator can exert the input to the plant with infinite precision. However, the sensor has the ability to transmit state information to the controller at the actuator only at discrete time instants (of its choice) and using only a finite number of bits. In this sense we refer to the sensor as the encoder and the actuator as the decoder. We let $\{t_k\}_{k\in\mathbb{N}}$ be the sequence of *transmission (or encoding) times* at which the sensor decides to sample the plant state, encode it, and transmit it. We denote by np_k the number of bits used to encode the plant state at the transmission time t_k. The process of encoding, transmission by the sensor, reception of a complete packet of encoded data at the controller, and decoding may take non-zero time. We let $\{r_k\}_{k\in\mathbb{N}}$ be the sequence of *reception (or update) times* at which the decoder receives a complete packet of data, decodes it, and updates the controller state. Therefore, $r_k \geq t_k$. The k^{th} communication time $\Delta_k \triangleq r_k - t_k$ is then a function of t_k and the packet size (of np_k bits) represented by p_k,

$$\Delta_k = r_k - t_k \triangleq \Delta(t_k, p_k).$$

In general, the time Δ_k could include communication time, computation time and other delays. We use the term instantaneous communication to refer to the case $\Delta \equiv 0$. To keep things simple, we assume that the encoder and the decoder have synchronized clocks and that they synchronously update their states at update times $\{r_k\}_{k\in\mathbb{N}}$. The latter assumption is justified in situations where the function $t \mapsto \Delta(t, p)$ is independent of t or where the encoder and decoder send short synchronization signals to indicate the start of encoding and the end of decoding, respectively.

We use dynamic quantization for finite-bit transmissions from the encoder to the decoder. In dynamic quantization, there are two distinct phases: the zoom out stage, during which no control is applied while the quantization domain is expanded until it captures the system state at time $r_0 = t_0 \in \mathbb{R}_{\geq 0}$; and the zoom in stage, during which the encoded feedback is used to asymptotically stabilize the system. A detailed description of the zoom out stage can be found in the literature, e.g., [24]. In this paper, we focus exclusively on the zoom-in stage, i.e., for $t \geq t_0$ for which we use a hybrid dynamic controller. We assume that both the encoder and the decoder have perfect knowledge of the plant system matrices. The state of the encoder/decoder is composed of the controller state $\hat{x} \in \mathbb{R}^n$ and an upper bound $d_e \in \mathbb{R}_{\geq 0}$ on the encoding error $x_e \triangleq x - \hat{x}$. Thus, the actual input to the plant is given by $u(t) = K\hat{x}(t)$. During inter-update times, the state of the dynamic controller evolves as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t) = \bar{A}\hat{x}(t), \quad t \in [r_k, r_{k+1}) \tag{4a}$$

Let the encoding and decoding functions at k^{th} iteration be represented by $q_{E,k} : \mathbb{R}^n \times \mathbb{R}^n \mapsto G_k$ and $q_{D,k} : G_k \times \mathbb{R}^n \mapsto \mathbb{R}^n$, respectively, where G_k is a finite set of 2^{p_k} symbols. At t_k, the encoder encodes the plant state as $z_{E,k} \triangleq q_{E,k}(x(t_k), \hat{x}(t_k^-))$, where $\hat{x}(t_k^-)$ is the controller state just prior to the encoding time t_k, and sends it to the controller. This signal is decoded as $z_{D,k} \triangleq q_{D,k}(z_{E,k}, \hat{x}(t_k^-))$ by the decoder at time r_k. Then at the update time r_k, the sensor and the controller update $\hat{x}$ using the jump map,

$$\begin{aligned}\hat{x}(r_k) &= e^{\bar{A}\Delta_k}\hat{x}(t_k^-) + e^{A\Delta_k}(z_{D,k} - \hat{x}(t_k^-)) \\ &\triangleq q_k(x(t_k), \hat{x}(t_k^-)).\end{aligned} \tag{4b}$$

We use the shorthand notation $q_k : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ to represent the quantization that occurs as a result of the finite-bit coding. We allow the quantization domain, the number of bits and the resulting quantizer, q_k, at each transmission instant $t_k \in \mathbb{R}_{\geq 0}$ to be variable. Note that the evaluation of the map q_k is inherently from the encoder's perspective because it depends on the plan state $x(t_k)$, which is unknown to the decoder. Also, while the encoder could store $\hat{x}(t_k^-)$, the decoder has to infer its value if $\Delta_k > 0$. We detail the specifics of the decoder's procedure to implement (4b) when communication is not instantaneous later.

The evolution of the plant state x and the encoding error x_e on the time interval $[r_k, r_{k+1})$ can be written as

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t) + v(t), \tag{5a}$$

$$\dot{x}_e(t) = Ax_e(t) + v(t). \tag{5b}$$

Note that while the controller state $\hat{x}$ is known to both the encoder and the decoder, the plant state (equivalently, the encoding error x_e) is known only to the encoder. However, at t_0, if a bound on $\|x_e(t_0)\|_\infty$ is available, then both the encoder and the decoder can compute a bound $d_e(t)$ on $\|x_e(t)\|_\infty$ for any $t \in \mathbb{R}_{\geq 0}$, as we explain later.

Finally, in order to formalize the control goal, we select an arbitrary symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$. Because $\bar{A}$ is Hurwitz, there exists a symmetric positive definite matrix P that satisfies the Lyapunov equation

$$P\bar{A} + \bar{A}^T P = -Q. \tag{6}$$

Consider then the associated candidate Lyapunov function $x \mapsto V(x) = x^T P x$. Given a desired "control performance"

$$V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0 \tag{7}$$

with $V_0 \geq 0$ (the steady state value of V_d) and $\beta > 0$ (rate of convergence) constants, the *control objective* is as follows: recursively determine the sequence of transmission times $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ and encoded messages $\hat{x}(t_k)$ so that $V(x(t)) \leq V_d(t)$ holds for all $t \geq t_0$, while also ensuring that the inter-transmission times $\{t_k - t_{k-1}\}_{k\in\mathbb{N}}$ are uniformly lower bounded by a positive quantity and that the number of bits transmitted at any instant is uniformly upper bounded. We structure our solution to this problem in several stages. Section III presents a necessary condition on the average bit rate required to meet the control objective under the assumption of zero disturbance. In Section IV we address the problem under instantaneous communication. Finally, we address the problem in all its generality in Section V.

III. Lower Bound on the Necessary Data Rate

Here we seek to determine the amount of information, in terms of the number of bits transmitted, necessary to meet the control goal stated in Section II for arbitrary initial conditions when no disturbances are present and communication is instantaneous. In the presence of unknown disturbances and/or non-instantaneous communication, the necessary bit rate is at least as much as in the case treated here, so our necessary condition holds in the more general case too. For convenience, let $\mathcal{B}(t, t_0)$ denote the number of bits transmitted in the time interval $[t_0, t]$. We are also interested in characterizing the data rate (i.e., the average number of bits transmitted) asymptotically,

$$R_{\text{as}} \triangleq \lim_{t\to\infty} \frac{\mathcal{B}(t,t_0)}{t-t_0}.$$

Since encoding is not exact, the decoder at the controller has knowledge of the plant state only up to some set $\mathcal{S}(t) \subset \mathbb{R}^n$, i.e., $x(t) \in \mathcal{S}(t)$. Equivalently, the decoder has knowledge of the encoding error $x_e(t)$ only up to some set $E(t) \subset \mathbb{R}^n$, i.e., $x_e(t) \in E(t)$. Because $\hat{x}$ is known to both the encoder and the decoder, $\mathcal{S}(t)$ is simply obtained as a coordinate shift of the set $E(t)$,

$$\mathcal{S}(t) = \{\xi \in \mathbb{R}^n : \xi = \hat{x}(t) + \xi_e,\ \xi_e \in E(t)\}.$$

Since $x_e(t_k) \in E(t_k)$ for each $k \in \mathbb{N}_0$, then equation (5b), with $v(t) \equiv 0$, implies that, for $t \in [t_k, t_{k+1})$,

$$E(t) = \{\xi \in \mathbb{R}^n : \xi = e^{A(t-t_k)}\xi_0,\ \xi_0 \in E(t_k)\}. \tag{8}$$

If A is not Hurwitz, then this set grows with time unless some new information is communicated to the controller. To meet the specified control goal, the idea is to keep the encoding error set $E(t)$ sufficiently small at all times by having the sensor transmit information to the controller at the time instants t_k.

Remark III.1. *(Reduction in the Bound on the Encoding Error with Communication).* Suppose the sensor encodes the state $x(t_k)$ at t_k using p_k bits by partitioning the set $E(t_k^-)$ (or equivalently $S(t_k^-)$) into 2^{p_k} subsets in a predetermined manner. The string of p_k bits informs the decoder the specific subset that $x(t_k)$ lies in. Further, suppose that $\hat{x}(t_k)$ is chosen as a nominal point of $\mathcal{S}(t_k)$ according to some predetermined rule. Then, note that there is some $x_e(t_k) \in E(t_k^-)$ such that, after performing the quantization,

$$\text{vol}(E(t_k)) \geq \frac{\text{vol}(E(t_k^-))}{2^{p_k}},$$

with the equality being achieved when the quantization (partitioning of the quantization domain) is uniform. •

The following result precisely characterizes the number of bits that *must* be transmitted to make it possible for the set $\mathcal{S}(t)$ (which has the same volume as $E(t)$) to be contained in $\mathcal{V}_d(t) = \{\xi \in \mathbb{R}^n : V(\xi) \leq V_d(t)\}$ as a means to ensure for *every* solution satisfying $V(x(t_0)) \leq V_d(t_0)$ at time t_0 to also satisfy $x(t) \in \mathcal{V}_d(t)$ for all $t \geq t_0$. Note that $\mathcal{V}_d(t)$ is a sub-level set of the quadratic function $V(x) = x^T P x$. Thus, $\mathcal{V}_d(t)$ is an n-dimensional ellipsoid whose volume is given by

$$\text{vol}(\mathcal{V}_d(t)) = c_P (V_d(t))^{\frac{n}{2}} \tag{9}$$

where c_P is a constant of proportionality that depends on the matrix P. Now, we are ready to state the result.

Proposition III.2. *(Necessary Number of Bits Transmitted and Asymptotic Data Rate). Consider the system (2), with $v(t) \equiv 0$ and $V_0 = 0$, and under the feedback law $u(t) = K\hat{x}(t)$, where $t \mapsto \hat{x}(t)$ evolves according to (4). A necessary condition for all solutions satisfying $V(x(t_0)) \leq V_d(t_0)$ at time t_0 to satisfy $V(x(t)) \leq V_d(t)$ for $t \geq t_0$ is*

$$\mathcal{B}(t,t_0) \geq \left(\text{tr}(A) + \frac{n\beta}{2}\right)\log_2(e)(t-t_0) + \log_2\left(\frac{\text{vol}(E(t_0))}{c_P(V_d(t_0))^{\frac{n}{2}}}\right). \tag{10}$$

Consequently, $R_{as} \geq \left(\text{tr}(A) + \frac{n\beta}{2}\right)\log_2(e)$.

Proof: Given a sequence of transmission times $\{t_k\}_{k\in\mathbb{N}}$, we deduce from (8) that for $t \in [t_k, t_{k+1})$,

$$\frac{\text{vol}(E(t))}{\text{vol}(E(t_k))} = \det\left(e^{A(t-t_k)}\right) = e^{\text{tr}(A)(t-t_k)},$$

where $\text{vol}(S)$ denotes the volume of the set S. Further, if $\mathcal{B}(t,t_0)$ number of bits are transmitted in the time interval $[t_0, t]$, then as a consequence of Remark III.1 it follows that there exists some $x(t_0)$ such that

$$\text{vol}(E(t)) \geq \frac{e^{\text{tr}(A)(t-t_0)}\,\text{vol}(E(t_0))}{2^{\mathcal{B}(t,t_0)}}, \tag{11}$$

Next, note that in order for all solutions satisfying $V(x(t_0)) \leq V_d(t_0)$ at time t_0 to satisfy $V(x(t)) \leq V_d(t)$ for $t \geq t_0$, it must hold true that $\mathcal{S}(t)$ is a subset of $\mathcal{V}_d(t)$ for $t \geq t_0$. In particular, this implies that the volume of $\mathcal{S}(t)$ (which is the same as that of $E(t)$) is no larger than the volume of the set $\mathcal{V}_d(t)$. Using $V_d(t) = V_d(t_0)e^{-\beta(t-t_0)}$ and (9), one can deduce that

$$\text{vol}(\mathcal{V}_d(t)) = c_P(V_d(t_0))^{\frac{n}{2}} e^{-\frac{n\beta}{2}(t-t_0)}.$$

Combining these observations with (11), we get

$$\begin{aligned} 2^{\mathcal{B}(t,t_0)} &\geq \frac{e^{\text{tr}(A)(t-t_0)}\,\text{vol}(E(t_0))}{\text{vol}(\mathcal{V}_d(t))} \\ &= \frac{e^{(\text{tr}(A)+\frac{n\beta}{2})(t-t_0)}\,\text{vol}(E(t_0))}{c_P(V_d(t_0))^{\frac{n}{2}}}, \end{aligned}$$

from which the result follows. ■

There are a few observations of note regarding Proposition III.2. First, the condition is dependent on the control goal but not on the control input itself. Since the result only relies on comparing the volumes of the sets $\mathcal{S}(t)$ and $\mathcal{V}_d(t)$, rather than on ensuring the stricter condition $\mathcal{S}(t) \subset \mathcal{V}_d(t)$ for $t \geq t_0$, it remains to be seen how a necessary or even a sufficient bit rate condition would depend on the control gain K and the sequence of communication times $\{t_k\}_{k\in\mathbb{N}_0}$. In general, a time-triggered implementation with the given control goal and communication constraints could be very conservative. This motivates our forthcoming investigation of event-triggered designs. Furthermore, note that Proposition III.2 is a necessary condition to meet the control goal *for every possible solution*. It is true that if the decoder at the controller were deciding the transmission time instants, then the condition $\mathcal{S}(t) \subset \mathcal{V}_d(t)$, $t \geq t_0$, would have to be enforced (given that it has no access to the actual plant state). However, when the encoder at the sensor is deciding the transmission time instants, as in our case, then it is sufficient to ensure $x(t) \in \mathcal{V}_d(t)$, $t \geq t_0$. This is yet another significant motivation to investigate event-triggered designs under bounded data rate constraints.

IV. Event-Triggered Control with Bounded Data Rate and Instantaneous Communication

In this section, we seek to design event-triggered laws for deciding the transmission times and the number of bits used per transmission based on feedback. We achieve this by letting the encoder at the sensor, which has access to the exact plant state, make these decisions in an opportunistic fashion. Here, we consider the simplified scenario of instantaneous communication and tackle the more general case of non-instantaneous communication in the next section.

A. Requirements on the Encoding Scheme

Here, we specify the basic requirements of the encoding scheme essential for our purposes. Consider the system defined by (5) where the controller state evolves according to (4). Assume that, at the beginning $t_0 \in \mathbb{R}_{\geq 0}$ of the zoom in stage, the encoder and decoder have a common knowledge of a constant $d_e(t_0)$ such that $\|x_e(t_0)\|_\infty \leq d_e(t_0)$. Given this common knowledge, the encoder and the decoder *inductively* construct a signal $d_e(.)$ such that $\|x_e(t)\|_\infty \leq d_e(t)$ is satisfied for all $t \geq t_0$ as follows. First, note that as a consequence of (5b), we have that

$$x_e(t) = e^{A(t-t_k)} x_e(t_k) + \int_{t_k}^{t} e^{A(t-s)} v(s) \mathrm{d}s,$$

which in turn implies

$$\begin{aligned}\|x_e(t)\|_\infty &\leq \|e^{A(t-t_k)} x_e(t_k)\|_\infty + \int_{t_k}^{t} \|e^{A(t-s)} v(s)\|_2 \mathrm{d}s \\ &\leq \|e^{A(t-t_k)}\|_\infty \|x_e(t_k)\|_\infty + \int_{t_k}^{t} e^{\|A\|_2 (t-s)} \nu \mathrm{d}s,\end{aligned}$$

where ν is the uniform bound on the disturbance v, cf. (3). Now, assuming that the encoder and the decoder know $d_e(t_k) \geq 0$ at time t_k such that $\|x_e(t_k)\|_\infty \leq d_e(t_k)$, then both can compute

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty d_e(t_k) + \frac{\nu}{\|A\|_2}[e^{\|A\|_2 (t-t_k)} - 1], \quad (12a)$$

for $t \in [t_k, t_{k+1})$. The above discussion guarantees that $\|x_e(t)\|_\infty \leq d_e(t)$ for $t \in [t_k, t_{k+1})$. Next, at time t_{k+1}, if np_{k+1} is the number of bits used to quantize and transmit information, then the encoder and the decoder update the value of $d_e(t_{k+1})$ by the jump,

$$d_e(t_{k+1}) = \frac{1}{2^{p_{k+1}}} d_e(t_{k+1}^-). \quad (12b)$$

Assuming the quantization at time t_k is such that $\|x_e(t_k)\|_\infty \leq d_e(t_k)$ given $\|x_e(t_k)\|_\infty \leq d_e(t_k^-)$, then it is straightforward to verify by induction that the so constructed signal d_e ensures $\|x_e(t)\|_\infty \leq d_e(t)$ for all $t \geq t_0$.

As an example, we next specify (up to the number of bits) an encoding scheme that satisfies the above requirements. Given $d_e(t_k)$ such that $\|x_e(t_k)\|_\infty \leq d_e(t_k)$, for $k \in \mathbb{N}_0$, the plant state satisfies

$$x(t) \in S(\hat{x}(t), d_e(t)) = \{\xi \in \mathbb{R}^n : \|\xi - \hat{x}(t)\|_\infty \leq d_e(t)\},$$

for all $t \in [t_k, t_{k+1})$. At time t_{k+1}, the sensor/encoder encodes the plant state and transmits using np_{k+1} bits. In this encoding scheme, the set $S(\hat{x}(t_{k+1}^-), d_e(t_{k+1}^-))$ is divided uniformly into $2^{np_{k+1}}$ hypercubes and $\hat{x}(t_{k+1})$ is chosen as the centroid of the hypercube containing the plant state $x(t_{k+1})$. This results in $d_e(t_{k+1})$ being updated as in (12b). Formally, we can express the quantization at time t_k as

$$q_k(x(t_k), \hat{x}(t_k^-)) \in \underset{\xi \in \mathcal{X}_k}{\operatorname{argmin}} \{\|x(t_k) - \xi\|_\infty\}, \quad (13)$$

where $\mathcal{X}_k$ is the set of centroids of the 2^{np_k} hypercubes that the set $S(\hat{x}(t_k^-), d_e(t_k^-))$ is divided into. We assume that if $x(t_k)$ lies on the boundary of two or more hypercubes, then the encoder and decoder choose the value of $q_k(x(t_k), \hat{x}(t_k^-))$ according to a common deterministic rule. As a result, given $\hat{x}(t_0)$ and $d_e(t_0)$ at time t_0, $\hat{x}(t)$ and $d_e(t)$ are known to both the encoder and the decoder at all times $t \geq t_0$.

In the remainder of the paper, we make no reference to this specific encoding scheme. Instead it is sufficient for us to use the properties of the encoding scheme specified by (12).

B. Event-Triggered Design with Arbitrary Finite Data Rate

Here, we solve the problem stated in Section II in a way that guarantees that the number of bits at each transmission is finite, although not necessarily uniformly upper bounded across all transmissions. We build on these developments in Section IV-C to address the problem when there exists an explicit uniform bound across all transmissions.

We start by defining the *performance-trigger* function, measuring the difference between the quadratic Lyapunov function V and the desired performance V_d,

$$h_V(t) = V(x(t)) - V_d(t).$$

We use this function to determine the transmission times in an opportunistic fashion in the following result.

Theorem IV.1. *(Control with Arbitrary Finite Data Rate). Consider the system (2) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k \in \mathbb{N}_0}$ determined recursively by*

$$t_{k+1} = \min \left\{ t \geq t_k \,:\, h_V(t) \geq 0, \; \dot{h}_V(t) \geq 0 \right\}. \quad (14)$$

Assume the encoding scheme is such that (12) holds for all $t \geq t_0$. Further assume that $V(x(t_0)) \leq V_d(t_0)$ and that

$$W \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - a\beta > 0, \quad (15a)$$

$$\sqrt{V_0} \geq \frac{2\|P\|_2 \nu}{\sigma(a-1)\beta\sqrt{\lambda_m(P)}}, \quad (15b)$$

where $a > 1$ and $\sigma \in (0, 1)$ are arbitrary constants. If the number of bits $p_k n$ transmitted at time t_k satisfies

$$p_k \geq \underline{p_k} \triangleq \left\lceil \log_2 \left(\frac{d_e(t_k^-)}{c\sqrt{V_d(t_k)}} \right) \right\rceil, \quad (16)$$

where $c \triangleq \frac{W\sqrt{\lambda_m(P)}}{2\sqrt{n}\|PBK\|_2}$, then the following holds:

(i) the inter-transmission times $\{T_k\}_{k\in\mathbb{N}} \triangleq \{t_{k+1} - t_k\}_{k\in\mathbb{N}}$ have a uniform positive lower bound,

(ii) the origin is exponentially practically stable for the closed-loop system, with $V(x(t)) \le V_d(t)$ for all $t \ge t_0$.

Proof: We first show that if the number of bits $p_k n$ transmitted at t_k satisfy (16), then the inter-event time $t_{k+1}-t_k$ is strictly positive. From (16), we deduce

$$2^{p_k} \ge 2^{(p_k-\underline{p_k})}\left(\frac{d_e(t_k^-)}{c\sqrt{V_d(t_k)}}\right).$$

Then, (12b) and the fact that $p_k - \underline{p_k} \ge 0$ in turn imply that for $k \in \mathbb{N}$

$$d_e(t_k) \le \frac{c\sqrt{V_d(t_k)}}{2^{(p_k-\underline{p_k})}} \le c\sqrt{V_d(t_k)}. \tag{17}$$

Next, from (5a), the derivative of V along the flow of the plant dynamics is

$$\begin{aligned}\dot V(t) &= -x^T(t)Qx(t) - 2x^T(t)PBKx_e(t) + 2x^T(t)Pv(t)\\ &\le -\frac{\lambda_m(Q)}{\lambda_M(P)}V(x(t)) + 2\frac{\sqrt{V(x(t))}}{\sqrt{\lambda_m(P)}}\|PBK\|_2\|x_e(t)\|_2 +\\ &\quad 2\frac{\sqrt{V(x(t))}}{\sqrt{\lambda_m(P)}}\|P\|_2\nu \end{aligned}\tag{18}$$

where we have used the fact that P satisfies (6) as well as (1) and (3). Then, using the definition of $d_e(t)$ and c, the derivative of h_V along the flow of the plant dynamics is

$$\begin{aligned}\dot h_V(t) \le& -\frac{\lambda_m(Q)}{\lambda_M(P)}V(x(t)) + \frac{W}{c}\sqrt{V(x(t))}d_e(t)+\\ & 2\frac{\sqrt{V(x(t))}}{\sqrt{\lambda_m(P)}}\|P\|_2\nu + \beta(V_d(t) - V_0).\end{aligned}$$

Evaluating at time t_k, for $k \in \mathbb{N}$, using (17), the definition of W in (15a), and the fact that $V(x(t_k)) = V_d(t_k)$ from (14), we obtain

$$\begin{aligned}\dot h_V(t_k) &= \dot h_V(t)\Big|_{t=t_k}\\ &\le \left(-\frac{\lambda_m(Q)}{\lambda_M(P)} + W + 2\frac{\|P\|_2}{\sqrt{\lambda_m(P)}}\frac{\nu}{\sqrt{V_d(t_k)}}\right)V_d(t_k)+\\ &\quad \beta(V_d(t_k) - V_0)\\ &\le \left(-(a-1)\beta + 2\frac{\|P\|_2}{\sqrt{\lambda_m(P)}}\frac{\nu}{\sqrt{V_0}}\right)V_d(t_k) - \beta V_0,\end{aligned}$$

where we have used in the second inequality that $V_d(t) \ge V_0$ for all time $t \ge t_0$. Finally, invoking (15b), we have

$$\dot h_V(t_k) \le -(1-\sigma)(a-1)\beta V_d(t_k) - \beta V_0. \tag{19}$$

Since $\dot h_V(t_k)$ is strictly negative and h_V is a continuous function, we deduce that the inter-event time $t_{k+1} - t_k$ is strictly positive.

Next, we establish the existence of a uniform positive lower bound on the inter-event times. To do so, we first provide an alternative description of the triggering function h_V. We begin by rewriting the system dynamics (5) in terms of $\zeta(t) \triangleq [x(t)^T, x_e(t)^T]^T$ as

$$\begin{aligned}\dot\zeta(t) &= \begin{bmatrix}\dot x(t)\\ \dot x_e(t)\end{bmatrix} = \begin{bmatrix}\bar A & -BK\\ 0_n & A\end{bmatrix}\begin{bmatrix}x(t)\\ x_e(t)\end{bmatrix} + \begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}v(t)\\ &\triangleq \mathcal{A}\zeta(t) + \begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}v(t).\end{aligned}$$

Next, in terms of the variable $\tau \triangleq t - t_k$, we see that

$$\begin{aligned}\zeta(\tau + t_k) &= e^{\mathcal{A}\tau}\zeta(t_k) + \int_0^\tau e^{\mathcal{A}(\tau-s)}\begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}v(s)\mathrm{d}s\\ &\triangleq e^{\mathcal{A}\tau}\zeta(t_k) + \mathcal{D}(\tau),\end{aligned}$$

from which we rewrite the triggering function h_V as

$$\begin{aligned}h_V(\tau + t_k) &= \zeta(t_k)^T R_1(\tau)\zeta(t_k) + 2\zeta(t_k)^T e^{\mathcal{A}^T\tau}C^T PC\mathcal{D}(\tau)\\ &\quad + \mathcal{D}(\tau)^T C^T PC\mathcal{D}(\tau) - V_0(1 - e^{-\beta\tau}),\\ R_1(\tau) &\triangleq e^{\mathcal{A}^T\tau}C^T PCe^{\mathcal{A}\tau} - C^T PCe^{-\beta\tau},\end{aligned}$$

where $C = \begin{bmatrix}\mathrm{I}_n & 0_n\end{bmatrix}$ and we have used $V_d(t) = V_d(t_k)e^{-\beta\tau} + V_0(1 - e^{-\beta\tau}) = V(x(t_k))e^{-\beta\tau} + V_0(1 - e^{-\beta\tau})$. Note that $R_1(\tau)$ is symmetric. From (17), we have

$$\|x_e(t_k)\|_2 \le \sqrt{n}\|x_e(t_k)\|_\infty \le \sqrt{n}d_e(t_k) \le c\sqrt{n}\sqrt{V(x(t_k))}.$$

This can be expressed as

$$\begin{aligned}\zeta(t_k)^T R_2\zeta(t_k) &\ge 0, \\ R_2 &\triangleq \begin{bmatrix}c^2 nP & 0_n\\ 0_n & -\mathrm{I}_n\end{bmatrix}.\end{aligned}\tag{20}$$

Note that $R_2(\tau)$ is also symmetric. Next, since we know that $V(x(t_k)) = V_d(t_k) \ge V_0$ for each $k \in \mathbb{N}$, we also have

$$\zeta(t_k)^T C^T PC\zeta(t_k) - V_0 \ge 0. \tag{21}$$

Therefore, our objective can be equivalently formulated as finding a lower bound on $\tau > 0$ such that $h_V(\tau+t_k) = 0$ under the constraints (20) and (21). Notice that $\tau \mapsto h_V(\tau + t_k)$ is a continuous function with $\dot h_V(t_k) < 0$ by (19). Therefore, $\dot h_V$ has to evolve from its initial negative value to zero before $h_V(\tau + t_k) = 0$ is satisfied. We build on this observation to identify the lower bound. Note,

$$\dot h_V(\tau + t_k) = \zeta(t_k)^T R_3(\tau)\zeta(t_k) + \eta(\tau)^T\zeta(t_k) + \gamma(\tau),$$

where

$$\begin{aligned}R_3(\tau) &\triangleq \mathcal{A}^T e^{\mathcal{A}^T\tau}C^T PCe^{\mathcal{A}\tau} + e^{\mathcal{A}^T\tau}C^T PCe^{\mathcal{A}\tau}\mathcal{A}\\ &\quad + \beta C^T PCe^{-\beta\tau}.\\ \eta(\tau) &\triangleq 2\big[\mathcal{A}^T e^{\mathcal{A}^T\tau}C^T PC + e^{\mathcal{A}^T\tau}C^T PC\mathcal{A}\big]\mathcal{D}(\tau)+\\ &\quad 2e^{\mathcal{A}^T\tau}C^T PC\begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}v(\tau),\\ \gamma(\tau) &\triangleq 2\mathcal{D}(\tau)^T C^T PC\Big(\mathcal{A}\mathcal{D}(\tau) + \begin{bmatrix}\mathrm{I}_n\\ \mathrm{I}_n\end{bmatrix}v(\tau)\Big) - \beta V_0 e^{-\beta\tau}.\end{aligned}$$

Note that $R_3(\tau)$ is symmetric. According to this expression, if we define for each $\tau \in \mathbb{R}_{\ge 0}$,

$$\begin{aligned}\mathcal{M}(\tau) \triangleq \max\{&\xi^T R_3(\tau)\xi + \eta(\tau)^T\xi + \gamma(\tau) :\\ &\xi^T R_2\xi \ge 0,\ \xi^T C^T PC\xi - V_0 \ge 0\},\end{aligned}$$

the time it takes $\dot{h}_V(\tau+t_k)$ to reach zero is lower bounded by

$$\tau_1^* \triangleq \min\{\tau \geq 0 : \mathcal{M}(\tau) \geq 0\}.$$

The Lagrangian of the maximization problem in the definition of $\mathcal{M}$ is

$$L_\tau(\xi,\mu_1,\mu_2) = \xi^T\Phi(\tau)\xi + \eta(\tau)^T\xi + \gamma(\tau) - \mu_2 V_0,$$
$$\Phi(\tau) \triangleq R_3(\tau) + \mu_1 R_2 + \mu_2 C^T P C,$$

and its dual function is

$$g_\tau(\mu_1,\mu_2) = \sup_{\xi\in\mathbb{R}^{2n}} L_\tau(\xi,\mu_1,\mu_2) = \begin{cases} -\frac{1}{4}\eta(\tau)^T\Phi(\tau)^+\eta(\tau) + \gamma(\tau) - \mu_2 V_0, & \text{if (22) holds,} \\ \infty, & \text{otherwise} \end{cases}$$

where (22) is defined by

$$\Phi(\tau) \preceq 0 \quad \text{and} \quad \Phi(\tau)\eta(\tau) \neq 0. \tag{22}$$

Consequently,

$$\mathcal{M}(\tau) \leq \min_{\substack{\mu_1\geq 0\\ \mu_2 \geq 0}} g_\tau(\mu_1,\mu_2).$$

Now, since we want to find a lower bound on inter-transmission times for all disturbance signals $v(.)$ satisfying (3), we restrict our attention to τ for which $\Phi(\tau) \prec 0$ (because if $\Phi(\tau)$ does have zero eigenvalues, then it is possible that a disturbance signal exists so that $\Phi(\tau)\eta(\tau) = 0$). We show later that there is no loss of generality in doing so. Note that when $\Phi(\tau) \prec 0$, then $\Phi^+ = \Phi^{-1}$.

Now, our next task is to find a bound on $g_\tau(\mu_1,\mu_2)$ independent of the disturbance. First, note that

$$\|\mathcal{D}(\tau)\|_2 \leq \sqrt{2}\int_0^\tau e^{\|\mathcal{A}\|_2(\tau-s)}\nu \mathrm{d}s = \frac{\sqrt{2}\nu}{\|\mathcal{A}\|_2}(e^{\|\mathcal{A}\|_2\tau} - 1).$$

Then, we note that $\|\eta(\tau)\|_2 \leq \tilde{\eta}(\tau)$, with

$$\tilde{\eta}(\tau) \triangleq \Big(2\sqrt{2}\|\mathcal{A}^T e^{\mathcal{A}^T\tau}C^TPC + e^{\mathcal{A}^T\tau}C^TPC\mathcal{A}\|_2\frac{(e^{\|\mathcal{A}\|_2\tau}-1)}{\|\mathcal{A}\|_2} + 2\sqrt{2}\|e^{\mathcal{A}^T\tau}C^TPC\|_2\Big)\nu.$$

Similarly, $|\gamma(\tau)| \leq \tilde{\gamma}(\tau)$, with

$$\tilde{\gamma}(\tau) \triangleq 4\|C^TPC\|_2\frac{\nu^2}{\|\mathcal{A}\|_2}e^{\|\mathcal{A}\|_2\tau}(e^{\|\mathcal{A}\|_2\tau}-1) - \beta V_0 e^{-\beta\tau}.$$

Thus, if $\Phi(\tau) \prec 0$ then

$$g_\tau(\mu_1,\mu_2) \leq \tilde{g}_\tau(\mu_1,\mu_2) \triangleq \frac{1}{4}\lambda_M(\Phi^{-1})\tilde{\eta}^2(\tau) + \tilde{\gamma}(\tau) - \mu_2 V_0 = \frac{1}{4\lambda_m(\Phi)}\tilde{\eta}^2(\tau) + \tilde{\gamma}(\tau) - \mu_2 V_0.$$

From this expression, we deduce that

$$\tau_1^* \geq \tau_2^* \triangleq \min\Big\{\tau \geq 0 \,:\, \min_{\substack{\mu_1\geq 0\\ \mu_2\geq 0}} \tilde{g}_\tau(\mu_1,\mu_2) \geq 0\Big\}.$$

Note that $\tilde{g}_\tau$, as a function of τ, is continuous. Thus, to show that τ_2^* is positive, it is sufficient to ensure that $\tilde{g}_0(\mu_1,\mu_2) < 0$ for some $\mu_1,\mu_2 \geq 0$. Clearly, $\tilde{\gamma}(0) \leq 0$ and thus it is sufficient to show that $\Phi(0) \prec 0$. Note that

$$\Phi(0) = \begin{bmatrix} -Q + (\beta + \mu_1 c^2 n + \mu_2)P & -PBK \\ -(PBK)^T & -\mu_1 \mathrm{I}_n \end{bmatrix},$$

whose Schur complement form [25] is

$$\mathcal{C} = -Q + \Big(\beta + \mu_1\frac{W^2\lambda_m(P)}{4\|PBK\|_2^2} + \mu_2\Big)P + \frac{1}{\mu_1}(PBK)(PBK)^T.$$

Letting $\mu_1 = \dfrac{\|PBK\|_2^2}{\lambda_m(P)}r$ and using (15a), we see that

$$\mathcal{C} \prec \Big(-(W + (a-1)\beta) + \frac{W^2}{4}r + \mu_2\Big)P + \frac{\lambda_m(P)}{r}\mathrm{I_n} \prec \frac{1}{r}\Big(-(W + (a-1)\beta + \mu_2)r + \frac{W^2}{4}r^2 + 1\Big)P.$$

Since $-\mu_1 I_n \prec 0$, the condition $\mathcal{C} \prec 0$ guarantees that $\Phi(0)$ is negative definite. Clearly, there exists a choice of $\mu_2 \geq 0$ such that the quadratic expression of r within the braces has two distinct positive zeros, between which $\mathcal{C}$ is negative definite. Thus, $\Phi(0) \prec 0$ and, by the discussion above, τ_2^* is positive. Since $T_k \geq \tau_1^* \geq \tau_2^*$, for $k \in \mathbb{N}$, this proves claim (i). As a consequence, we have that $V(x(t)) \leq V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0$ for all $t \geq t_0$, which proves claim (ii). ■

Remark IV.2. *(Tighter uniform bound on inter-transmission times).* Following the argument in the proof of Theorem IV.1, one can obtain a better uniform lower bound on the inter-transmission times by reasoning directly on the time it takes $h_V(\tau+t_k)$ to reach zero. Following steps analogous to those in the proof, one can define a maximization problem whose dual function $\tilde{\mathrm{g}}_\tau$ is similar in form to $\tilde{g}_\tau$. Then, the same reasoning leads to the lower bound

$$\tau^* = \min\Big\{\tau \geq \tau_2^* \,:\, \min_{\substack{\mu_1\geq 0\\ \mu_2\geq 0}} \tilde{\mathrm{g}}_\tau(\mu_1,\mu_2) \geq 0\Big\}. \quad \bullet$$

Remark IV.3. *(No disturbance case).* In the special case of no disturbance, $\nu = 0$, the statement of Theorem IV.1 still holds true. However, its proof needs to be modified in case $V_0 = 0$ because the constraint (21) reduces to the trivial constraint. If $V_0 = \nu = 0$, we have

$$h_V(\tau + t_k) = \zeta(t_k)^T R_1(\tau)\zeta(t_k),$$
$$\dot{h}_V(\tau + t_k) = \zeta(t_k)^T R_3(\tau)\zeta(t_k).$$

It is important to note that the zero crossing times of these functions, given any initial condition, are independent of $\|\zeta(t_k)\|_2$. Therefore, in this case, we replace the constraint (21) by $\zeta(t_k)^T\zeta(t_k) = 1$. With this in place, one can show that $T_k \geq \tau^* > 0$, for all $k \in \mathbb{N}$, with

$$\tau^* = \min\Big\{\tau > \tau_1^* : \Big(\min_{\mu_1\geq 0}\lambda_M(R_1(\tau) + \mu_1 R_2)\Big) \geq 0\Big\},$$
$$\tau_1^* = \min\Big\{\tau \geq 0 : \Big(\min_{\mu_1\geq 0}\lambda_M(R_3(\tau) + \mu_1 R_2)\Big) \geq 0\Big\}. \quad \bullet$$

The quantity $\underline{p_k}$ in Theorem IV.1 can be interpreted as the minimum number of bits to be transmitted sufficient to ensure that, after transmission, $\dot{h}_V(t_k) < 0$. The recursive nature of the inequalities (16) can be leveraged to better understand the relationship across different times among the bounds on the number of bits sufficient for stability. First, using (12a), we can upper bound $\underline{p_{k+1}}$, for each $k \in \mathbb{N}_0$, as follows,

$$\underline{p_{k+1}} \leq \left\lceil \log_2 \left(\frac{\|e^{AT_k}\|_\infty d_e(t_k) + \frac{\nu}{\|A\|_2}(e^{\|A\|_2 T_k} - 1)}{c\sqrt{(V_d(t_k) - V_0)e^{-\beta T_k} + V_0}} \right) \right\rceil. \tag{23}$$

In order to provide an intuitive interpretation, we assume in the following two results that there is no disturbance in the system ($\nu = 0$ and $V_0 = 0$). The following result states that the upper bound (23) can be made smaller if more bits have been transmitted in the past.

Corollary IV.4. *(Less bits are sufficient now if more bits were transmitted before). Under the assumptions of Theorem IV.1 and no disturbances, the following holds for any $k \in \mathbb{N}$,*

$$\underline{p_{k+1}} \leq \log_2 \left(e^{(\|A\|_2 + \frac{\beta}{2})(t_{k+1} - t_0)} \right) + (k+1) - \sum_{i=1}^{k} p_i.$$

Proof: Using (17) in (23) (with $\nu = 0$ and $V_0 = 0$), together with the fact that $V_d(t_{k+1}) = V_d(t_k)e^{-\beta T_k}$, for $k \in \mathbb{N}$, implies

$$\begin{aligned} \underline{p_{k+1}} &\leq \left\lceil \log_2 \left(\frac{\|e^{AT_k}\|_\infty}{2^{(p_k - \underline{p_k})}} \sqrt{\frac{V_d(t_k)}{V_d(t_{k+1})}} \right) \right\rceil \\ &\leq \log_2 \left(\frac{\|e^{AT_k}\|_\infty}{e^{\frac{-\beta}{2} T_k}} \right) - (p_k - \underline{p_k}) + 1 \\ &\leq \log_2 \left(\frac{e^{\|A\|_2 T_k}}{e^{\frac{-\beta}{2} T_k}} \right) - (p_k - \underline{p_k}) + 1, \end{aligned}$$

which, when recursively expanded, gives the stated result. ∎

The next result gives insight into the total number of bits sufficient for stability as a function of time.

Corollary IV.5. *(Upper bound on the bit rate sufficient for stability). Under the assumptions of Theorem IV.1 and no disturbances, the following holds for any $k \in \mathbb{N}$,*

$$\begin{aligned} &n(\underline{p_k} + \sum_{i=1}^{k-1} p_i) \\ &\leq n\Big(\|A\|_2 + \frac{\beta}{2}\Big) \log_2(e)(t_k - t_0) + n \log_2 \left(\frac{d_e(t_0)}{c\sqrt{V_d(t_0)}} \right) + n. \\ &\leq n \log_2 \left(e^{(\|A\|_2 + \frac{\beta}{2})(t_k - t_0)} \right) + n \log_2 \left(\frac{d_e(t_0)}{c\sqrt{V_d(t_0)}} \right) + n. \end{aligned}$$

Proof: Using (12) (with $\nu = 0$ and $V_0 = 0$) recursively gives

$$\begin{aligned} d_e(t_k^-) &= \|e^{AT_{k-1}}\|_\infty d_e(t_{k-1}) = \frac{\|e^{AT_{k-1}}\|_\infty d_e(t_{k-1}^-)}{2^{p_{k-1}}} \\ &= \prod_{i=1}^{k-1} \frac{\|e^{AT_i}\|_\infty}{2^{p_i}} \|e^{AT_0}\|_\infty d_e(t_0) \\ &\leq \frac{e^{\|A\|_2 (t_k - t_0)}}{\prod_{i=1}^{k-1} 2^{p_i}} d_e(t_0), \end{aligned}$$

for $k \in \mathbb{N}$. Substituting this bound in (16) (and multiplying by n to give us the number of bits), we arrive at

$$n\underline{p_k} \leq n \left\lceil \log_2 \left(\frac{e^{\|A\|_2 (t_k - t_0)} d_e(t_0)}{e^{\frac{-\beta}{2}(t_k - t_0)} c\sqrt{V_d(t_0)}} \right) \right\rceil - n \sum_{i=1}^{k-1} p_i,$$

where we have used $V_d(t_k) = V_d(t_0)e^{-\beta(t_k - t_0)}$. Upper bounding the $\lceil . \rceil$ function and rearranging the terms yields the result. ∎

This result is interesting for two reasons. First, observe that the upper bound on the number of bits to be transmitted up to time t_k, for any $k \in \mathbb{N}$, to meet the control goal depends only on the length of the time interval $t_k - t_0$, the initial conditions $d_e(t_0)$ and $V_d(t_0)$ and the plant system matrix. Second, the expression, albeit only being valid at the transmission times $\{t_k\}_{k \in \mathbb{N}_0}$, has a similar form to the lower bound (10) on the number of bits transmitted over the time interval $[t_0, t]$ stated in Proposition III.2. In fact, the occurrence of $\|A\|_2$ in Corollary IV.5 is a by-product of our use of the norm $\|.\|_\infty$ and hypercubes as our quantization domains. In comparison with (10), $n\|A\|_2$ plays the role of $\mathrm{tr}(A)$. Similarly, $d_e(t_0)^n$ is proportional to the volume $\mathrm{vol}(E(t_0))$ of the hypercube $E(t_0)$.

C. Event-Triggered Design with Uniform Bound on Data Rate

In this section, we expand on our previous discussion to solve the problem stated in Section II with a uniform bound on the number of bits per transmission. This is particularly relevant in cases where the communication channel imposes a hard bound, say $\bar{p}$, on the number of bits that can be transmitted at each time. Before getting into the technical details, we briefly lay out the rationale behind our design. As a consequence of the hard limit on the channel capacity, a transmission at a time $t_k \in \mathbb{R}_{>0}$ can be caused by any of the following two reasons:

(Ti) the system trajectory hits the limit of the required performance guarantee, i.e., $h_V(t_k) = 0$, as in (14), or
(Tii) even though $h_V(t_k) < 0$, the number of bits required later to keep h_V from going positive would be larger than the channel capacity.

To design an appropriate trigger for (Tii), we need to properly characterize the time it takes h_V to evolve from a negative value to zero. This information will allow us to determine the minimum number of bits to be transmitted so that h_V takes at least a certain pre-designed time to reach zero. Our trigger for (Tii) would then be simply 'transmit if this minimum number of bits reaches the maximum channel capacity'.

1) Bound on the time to reach the limit on performance guarantee: We begin by providing a lower bound on the time it takes the performance-trigger function to reach zero from a negative value. In our discussion, instead of dealing with h_V directly, we find it more convenient analyzing the function

$$b(t) \triangleq \frac{V(x(t))}{V_d(t)}. \tag{24}$$

Both functions capture the same information: in fact, $h_V(t) = 0$ is equivalent to $b(t) = 1$, and $h_V(t) < 0$ is equivalent to

$b(t) < 1$. The following result provides an upper bound on the value of b that is convenient for our purposes.

Lemma IV.6. *(Upper bound on performance ratio). Given $t_k \in \mathbb{R}_{>0}$ such that $b(t_k) \le 1$, then*

$$b(\tau + t_k) \le \tilde{b}(\tau, b(t_k), \epsilon(t_k)),$$

for $\tau \ge 0$, where

$$\epsilon(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}}, \quad \tilde{b}(\tau, b_0, \epsilon_0) \triangleq \frac{f_1(\tau, b_0, \epsilon_0)}{f_2(\tau)}, \quad (25)$$

$$f_1(\tau, b_0, \epsilon_0) \triangleq b_0 + \frac{W\epsilon_0}{w+\theta}(e^{(w+\theta)\tau} - 1) + \frac{c_1 - c_2}{w}(e^{w\tau} - 1) + \frac{c_2}{w + \|A\|_2}(e^{(w+\|A\|_2)\tau} - 1),$$

$$f_2(\tau) \triangleq e^{w\tau},$$

with $w \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - \beta > 0$, $\theta \triangleq \|A\|_2 + \frac{\beta}{2}$ and

$$c_1 \triangleq \frac{2\|P\|_2}{\sqrt{\lambda_m(P)}}\frac{\nu}{\sqrt{V_0}}, \quad c_2 \triangleq \frac{W}{c\|A\|_2}\frac{\nu}{\sqrt{V_0}}.$$

Proof: We start by noting that $w > 0$ follows from (15a). Similarly to the derivation of (12a), we have for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\|x_e(t)\|_2 &= \|e^{A(t-t_k)}\|_2\|x_e(t_k)\|_2 + \frac{\nu}{\|A\|_2}[e^{\|A\|_2(t-t_k)} - 1] \\ &\le \sqrt{n}e^{\|A\|_2(t-t_k)}c\sqrt{V_d(t_k)}\epsilon(t_k) \\ &\quad + \frac{\nu}{\|A\|_2}[e^{\|A\|_2(t-t_k)} - 1],\end{aligned}$$

where we have used $d_e(t_k) = c\epsilon(t_k)\sqrt{V_d(t_k)}$. Substituting this expression in (18), we have for $t \in [t_k, t_{k+1})$

$$\begin{aligned}\dot{V}(t) \le &-\frac{\lambda_m(Q)}{\lambda_M(P)}V(x(t)) + 2\frac{\sqrt{V(x(t))}}{\sqrt{\lambda_m(P)}}\|P\|_2\nu \\ &+ W\sqrt{V(x(t))}e^{\|A\|_2(t-t_k)}\sqrt{V_d(t_k)}\epsilon(t_k) \\ &+ \frac{W}{c}\sqrt{V(x(t))}\frac{\nu}{\|A\|_2}(e^{\|A\|_2(t-t_k)} - 1).\end{aligned}$$

From the definition (24) of b, we compute

$$\dot{b} = \frac{\dot{V}V_d - V\dot{V}_d}{V_d^2} = \frac{\dot{V}}{V_d} + \beta b\frac{(V_d - V_0)}{V_d} \le \frac{\dot{V}}{V_d} + \beta b,$$

where the inequality follows from the fact that V_d is always positive and greater than V_0. Substituting in this equation the upper bound for $\dot{V}$ obtained above, we get

$$\dot{b} \le -wb + \frac{2\|P\|_2}{\sqrt{\lambda_m(P)}}\frac{\nu\sqrt{b}}{\sqrt{V_d}} + W\epsilon(t_k)e^{\theta\tau}\sqrt{b} + \frac{W}{c\|A\|_2}\frac{\nu\sqrt{b}}{\sqrt{V_d}}(e^{\|A\|_2\tau} - 1),$$

where $t = \tau + t_k$. We can further simplify this by noting that our region of interest is when the value of b belongs to $[0, 1]$, in which $\sqrt{b} \le 1$, and that $V_d(t) \ge V_0$ for all time $t \ge t_0$. Thus,

$$\dot{b} \le -wb + W\epsilon(t_k)e^{\theta\tau} + c_1 + c_2(e^{\|A\|_2\tau} - 1).$$

Thus, letting

$$\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tau} \triangleq -w\tilde{b} + W\epsilon(t_k)e^{\theta\tau} + c_1 + c_2(e^{\|A\|_2\tau} - 1), \quad (26)$$

the result follows from the Comparison Lemma [26]. ■

Motivated by Lemma IV.6, we formally define the function

$$\Gamma_1(b_0, \epsilon_0) \triangleq \min\{\tau \ge 0 : \tilde{b}(\tau, b_0, \epsilon_0) = 1, \ \frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tau} \ge 0\}. \quad (27)$$

Thus, $\Gamma_1(b_0, \epsilon_0)$ is a lower bound on the time it takes b to evolve to 1 starting from $b(t_k) = b_0$ and $\epsilon(t_k) = \epsilon_0$. The following result captures some useful properties of this function.

Lemma IV.7. *(Properties of the function Γ_1). The following holds true,*

(i) $\Gamma_1(1, 1) > 0$.

(ii) If $b_1 \ge b_0$ and $\epsilon_1 \ge \epsilon_0$, then $\Gamma_1(b_0, \epsilon_0) \ge \Gamma_1(b_1, \epsilon_1)$. In particular, if $b_0 \in [0, 1]$, then $\Gamma_1(b_0, \epsilon_0) \ge \Gamma_1(1, \epsilon_0)$.

(iii) For $T > 0$, if $b_0 \in [0, 1]$ and

$$\epsilon_0 \le \rho_T(b_0) \triangleq \frac{(w+\theta)(1-b_0)}{W(e^{(w+\theta)T} - 1)} + 1, \quad (28)$$

then $\Gamma_1(b_0, \epsilon_0) \ge \min\{\Gamma_1(1, 1), T\}$.

Proof: To show *(i)*, note that $\tilde{b}(0, 1, 1) = 1$ and

$$\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tau}(0, 1, 1) = -w + W + c_1.$$

Using (15b), we deduce that this value is strictly negative, and therefore $\Gamma_1(1, 1) > 0$. *(ii)* follows from the fact that $\tilde{b}$ is an increasing function of its second and third arguments. To show *(iii)*, observe that

$$\begin{aligned}&\tilde{b}(\tau, b_0, \epsilon_0) - \tilde{b}(\tau, 1, 1) \\ &= e^{-w\tau}\left[(b_0 - 1) + \frac{W(\epsilon_0 - 1)}{w + \theta}(e^{(w+\theta)\tau} - 1)\right] \\ &\le e^{-w\tau}\left[(b_0 - 1) + \frac{1 - b_0}{e^{(w+\theta)T} - 1}(e^{(w+\theta)\tau} - 1)\right]. \quad (29)\end{aligned}$$

Since $b_0 \le 1$, we see that for all $\tau \in [0, \min\{\Gamma_1(1, 1), T\}]$, $\tilde{b}(\tau, b_0, \epsilon_0) \le \tilde{b}(\tau, 1, 1) \le 1$, from which the claim follows. ■

2) Trigger design and analysis: The analysis of Section IV-C1 sets the basis for computing the minimum number of bits that guarantee that the performance specification is met for a certain pre-designed time. Specifically, define the *channel-trigger* function

$$h_{\mathrm{ch}}(t) \triangleq \frac{\epsilon(t)}{\rho_T(b(t))} = \frac{d_e(t)}{c\sqrt{V_d(t)}\rho_T(b(t))}, \quad (30)$$

where $T > 0$ is a fixed design parameter. Lemma IV.7(iii) implies that, if $h_{\mathrm{ch}}(t_k) \le 1$, then $b(t) \le 1$ for at least $t \in [t_k, t_k + \min\{T, \Gamma_1(1, 1)\})$. Building on this observation, our trigger for (Tii) is then transmit if $h_{\mathrm{ch}}(t)/2^{\bar{p}} = 1$, i.e., when 'the number of bits required to have the value of h_{ch} smaller than or equal to 1 just after transmission' is no more than $n\bar{p}$, the upper bound imposed by the channel.

The next result provides an upper bound on the function $h_{\rm ch}$ and is useful later when establishing a uniform lower bound on the inter-transmission times for our design.

Lemma IV.8. *(Upper bound on channel-trigger function). Given $t_k \in \mathbb{R}_{>0}$ such that $b(t_k) \leq 1$, then*

$$h_{\rm ch}(\tau + t_k) \leq \bar{h}_{\rm ch}(\tau, b(t_k), \epsilon(t_k), \epsilon(t_k)),$$

for $\tau \geq 0$, where

$$\bar{h}_{\rm ch}(\tau, b_0, \epsilon_0, \psi_0) \triangleq \frac{\|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}\psi_0}{\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))} + \frac{\nu(e^{\|A\|_2\tau} - 1)}{c\|A\|_2\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))\sqrt{V_0}}. \quad (31)$$

Proof: From its definition, we can bound $h_{\rm ch}$ using (12a), the fact that ρ_T is a decreasing function and Lemma IV.6 as,

$$h_{\rm ch}(\tau + t_k) \leq \frac{\|e^{A\tau}\|_\infty d_e(t_k) + \frac{\nu}{\|A\|_2}(e^{\|A\|_2\tau} - 1)}{c\rho_T(\tilde{b}(\tau, b(t_k), \epsilon(t_k))\sqrt{V_d(\tau + t_k)}}.$$

The result now follows by further simplifying this expression expanding $V_d(\tau+t_k) = V_d(t_k)e^{-\beta\tau}+V_0(1-e^{-\beta\tau})$, observing that $V_0 \geq 0$ and $V_d(t) \geq V_0$ for all $t \geq t_0$, and using the definition of ϵ. ∎

Given Lemma IV.8, we define the function

$$\Gamma_2(b_0, \epsilon_0, \psi_0) \triangleq \min\{\tau \geq 0 : \frac{\bar{h}_{\rm ch}(\tau, b_0, \epsilon_0, \psi_0)}{2^{\bar{p}}} = 1\},$$

which is a lower bound on the time it takes $h_{\rm ch}(\tau+t_k)$ to reach $2^{\bar{p}}$ given $b(t_k) = b_0$ and $\epsilon(t_k) = \epsilon_0$. Note that the argument ψ_0 in the definitions of $\bar{h}_{\rm ch}$ and Γ_2 is redundant for our purposes here, but will play an important role later when discussing the case of non-instantaneous communication.

We are now ready to present the main result of this section.

Theorem IV.9. *(Control under Bounded Channel Capacity). Consider the system (2) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k\in\mathbb{N}_0}$ determined recursively by*

$$t_{k+1} = \min\{t \geq t_k : h_V(t) \geq 0, \; \dot{h}_V(t) \geq 0 \quad \text{OR} \quad \frac{h_{\rm ch}(t)}{2^{\bar{p}}} \geq 1\}, \quad (32)$$

where $n\bar{p}$ is the upper bound on the number of bits that can be sent per transmission and $T > 0$ in the definition (30) of $h_{\rm ch}$ is a design parameter. Assume the encoding scheme is such that (12) is satisfied for all $t \geq t_0$. Further assume that $V(x(t_0)) \leq V_d(t_0)$, $h_{\rm ch}(t_0) \leq 2^{\bar{p}}$ and that (15a)-(15b) hold. Let $\underline{p_k}$ be given by

$$\underline{p_k} \triangleq \left\lceil \log_2\left(\frac{d_e(t_k^-)}{c\rho_T(b(t_k))\sqrt{V_d(t_k)}}\right)\right\rceil, \quad (33)$$

where recall $c = \frac{W\sqrt{\lambda_m(P)}}{2\sqrt{n}\|PBK\|_2}$. Then, the following hold:

(i) $\underline{p_1} \leq \bar{p}$. Further for each $k \in \mathbb{N}$, if $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$, then $\underline{p_{k+1}} \leq \bar{p}$.

(ii) the inter-transmission times $\{T_k = t_{k+1} - t_k\}_{k\in\mathbb{N}}$ have a uniform positive lower bound,

(iii) the origin is exponentially practically stable for the closed-loop system, with $V(x(t)) \leq V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0$ for all $t \geq t_0$.

Proof: Since $V(x(t_0)) \leq V_d(t_0)$ and $h_{\rm ch}(t_0) \leq 2^{\bar{p}}$, the trigger (32) implies that $\underline{p_1} \leq \bar{p}$. Similarly, if for each $k \in \mathbb{N}$, $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$, then (32) implies $\underline{p_{k+1}} \leq \bar{p}$, which proves *(i)*.

To show *(ii)*, we study each of the two conditions that define (32). Regarding the condition on the performance-trigger function, note that $\Gamma_1(b(t_k), \epsilon(t_k))$ is, by definition, a lower bound on the time it takes the condition to be enabled. Since (32) guarantees that $h_{\rm ch}(t_k^-) \leq 2^{\bar{p}}$ and, as a result, $h_{\rm ch}(t_k) \leq 1$ (with equality holding when $p_k = \underline{p_k}$), we have $\epsilon(t_k) \leq \rho_T(b(t_k))$. Therefore, Lemma IV.7 guarantees that $\Gamma_1(b(t_k), \epsilon(t_k)) \geq \min\{\Gamma_1(1,1), T\} > 0$ for $k \in \mathbb{N}$. Regarding the condition on the channel-trigger function in (32), note that $\Gamma_2(b(t_k), \epsilon(t_k), \epsilon(t_k))$ is, by definition, a lower bound on the time it takes the condition to be enabled. We therefore focus on upper bounding the function $\bar{h}_{\rm ch}$ that defines Γ_2. First, notice that for $b_0 \leq 1$ and $\epsilon_0 \leq \rho_T(b_0)$, (29) implies that $\tilde{b}(\tau, b_0, \epsilon_0) \leq \tilde{b}(\tau, 1, 1)$ for all $\tau \in [0, \min\{\Gamma_1(1,1), T\}]$. The fact that ρ_T is decreasing then implies that the second term in the definition (31) of $\bar{h}_{\rm ch}$ can be bounded by,

$$\frac{\nu(e^{\|A\|_2\tau} - 1)/c}{\|A\|_2\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))\sqrt{V_0}} \leq \phi_2(\tau) \triangleq \frac{\nu(e^{\|A\|_2\tau} - 1)/c}{\|A\|_2\rho_T(\tilde{b}(\tau, 1, 1))\sqrt{V_0}},$$

for $\tau \in [0, \min\{\Gamma_1(1,1), T\}]$. Next, we turn our attention to the first term in the definition (31) of $\bar{h}_{\rm ch}$. Let c_3 be the negative of the coefficient of b_0 in the definition (28) of $\rho_T(b_0)$. Observe that for $b_0 \geq 0$, $\epsilon_0 \geq 0$ and $\tau \in [0, \min\{\Gamma_1(1,1), T\}]$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\psi_0}{\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))} &= \frac{\psi_0 c_3}{\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))^2}[-w\tilde{b} + W\epsilon_0 e^{\theta\tau} + c_1 + c_2(e^{\|A\|_2\tau} - 1)] \\ &\leq \psi_0 c_3[W\epsilon_0 e^{\theta\tau} + c_1 + c_2(e^{\|A\|_2\tau} - 1)],\end{aligned}$$

where we have used (26) and the facts that $\tilde{b}(\tau, b_0, \epsilon_0) \leq \tilde{b}(\tau, 1, 1) \leq 1$ for all $\tau \in [0, \min\{\Gamma_1(1,1), T\}]$ and $\rho_T(y) \geq 1$ for $y \in [0, 1]$. Then, the Comparison Lemma [26] implies that

$$\frac{\psi_0}{\rho_T(\tilde{b}(\tau, b_0, \epsilon_0))} \leq \frac{\psi_0}{\rho_T(b_0)} + \psi_0 c_3\left[\frac{W\epsilon_0}{\theta}(e^{\theta\tau} - 1) + \frac{c_2}{\|A\|_2}(e^{\|A\|_2\tau} - 1) + (c_1 - c_2)\tau\right].$$

Defining now

$$\phi(\tau, \phi_0) \triangleq \|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}\phi_1(\tau, \phi_0) + \phi_2(\tau)$$

with

$$\phi_1(\tau, \phi_0) \triangleq \phi_0 + \phi_0\rho_T(0)c_3\left[\frac{W\rho_T(0)}{\theta}(e^{\theta\tau} - 1) + \frac{c_2}{\|A\|_2}(e^{\|A\|_2\tau} - 1) + (c_1 - c_2)\tau\right].$$

we deduce, for $\epsilon_0 \leq \rho_T(b_0)$ and $\tau \in [0, \min\{\Gamma_1(1,1), T\}]$,

$$\bar{h}_{\rm ch}(\tau, b_0, \epsilon_0, \psi_0) \leq \phi(\tau, \frac{\psi_0}{\rho_T(b_0)}), \quad (34)$$

where we have used $\rho_T(b_0) \le \rho_T(0)$. Note that since we are interested in lower bounding $\Gamma_2(b(t_k), \epsilon(t_k), \epsilon(t_k))$ with $\epsilon(t_k) \le \rho_T(b(t_k))$, we can focus on the case $\psi_0 = \epsilon_0 \le \rho_T(b_0)$, which leads to the bound

$$\bar{h}_{\text{ch}}(\tau, b_0, \epsilon_0, \psi_0) \le \phi(\tau, 1).$$

Thus $\bar{h}_{\text{ch}}$ is bounded by a function that depends only on τ and is equal to 1 at $\tau = 0$. Hence, we deduce the existence of a uniform positive lower bound on the function $\Gamma_2(b_0, \epsilon_0, \psi_0)$ for $b_0 \in [0, 1]$ and $\psi_0 = \epsilon_0 \le \rho_T(b_0)$. Thus $T_k = t_{k+1} - t_k \ge \min\{T, \Gamma_1(b(t_k), \epsilon(t_k)), \Gamma_2(b(t_k), \epsilon(t_k), \epsilon(t_k)\}$, for $k \in \mathbb{N}$ has a uniform positive lower bound, proving *(ii)*. Claim *(iii)* follows by noting that *(i)* and *(ii)* imply $b(t) \le 1$, $t \ge t_0$. ■

The quantity $\underline{p_k}$ in Theorem IV.9 has now a slightly different interpretation than in Theorem IV.1: it essentially corresponds to the minimum number of bits sufficient to ensure that, after transmission, h_V remains negative for the next $\min\{T, \Gamma_1(b(t_k), \epsilon(t_k))$ units of time in the absence of further actions.

V. Event-Triggered Control with Bounded Data Rate and Non-Instantaneous Communication

Here we design event-triggered laws for deciding the transmission times and the number of bits used per transmission when communication is not instantaneous. Such scenarios are common when the model available for the communication channel specifies a capacity in terms of bit rates. In this case, we need to distinguish between the time when the encoder/sensor transmits from the time when the decoder/controller receives a complete packet of data. This corresponds to the setup of Section II in its full generality.

A. Information Consistency Between Encoder and Decoder

Given the difference between transmission and communication times, the first problem we tackle is making sure that the information (the state estimate $\hat{x}$ and the upper bound d_e on the encoding error x_e) used by the encoder and the decoder is consistent. The mechanisms described here rely critically on the assumptions of synchronized clocks assumption and common knowledge of the communication time, cf. Section II. According to the problem statement, the encoder encodes its message at t_k and sends np_k bits which are received completely by the decoder at $r_k \ge t_k$. Algorithms 1 and 2 describe, respectively, how the encoder and the decoder update $\hat{x}$ and d_e synchronously at the time instants r_k.

It is interesting to note that, as described above, the algorithms are also applicable in the case of instantaneous communication. The idea of Step 6 in each algorithm is to propagate $z_{D,k}$ forward in time so that it may be used from time r_k onwards (in the case of instantaneous communication, note that $\hat{x}(r_k) = z_{D,k}$). We next establish that Algorithms 1 and 2 provide consistent signals $t \mapsto \hat{x}(t)$, $t \mapsto d_e(t)$ to the encoder and the decoder.

Algorithm 1: Update of encoder variables

At $t = t_0 = r_0$, the encoder initializes

1: $\delta_0 \leftarrow d_e(t_0)$ `{store initial bound on encoding error}`

At $t \in \{t_k\}_{k\in\mathbb{N}}$, the encoder sets

2: $z_k \leftarrow \hat{x}(t_k^-)$ `{store encoder variable}`
3: $z_{E,k} \leftarrow q_{E,k}(x(t_k), z_k)$ `{encode plant state with` p_k `bits}`
4: $\delta_k \leftarrow d_e(t_k^-)/2^{p_k}$ `{compute bound on encoding error}`

At $t \in \{r_k\}_{k\in\mathbb{N}}$, the encoder sets

5: $z_{D,k} \leftarrow q_{D,k}(z_{E,k}, z_k)$ `{decode plant state at` t_k`}`
6: $\hat{x}(r_k) \leftarrow e^{\bar{A}\Delta_k} z_k + e^{A\Delta_k}(z_{D,k} - z_k)$ `{update controller state}`
7: $d_e(r_k) \leftarrow \|e^{A\Delta_k}\|_\infty \delta_k + \frac{\nu}{\|A\|_2}[e^{\|A\|_2\Delta_k} - 1]$ `{update bound on encoding error}`

Algorithm 2: Update of decoder variables

At $t = t_0 = r_0$, the decoder initializes

1: $\delta_0 \leftarrow d_e(t_0)$ `{store initial bound on encoding error}`

At $t \in \{r_k\}_{k\in\mathbb{N}}$, the decoder sets

2: $z_k \leftarrow e^{-\bar{A}\Delta_k}\hat{x}(r_k^-)$ `{compute encoder state at` t_k`}`
3: $z_{E,k}$ `{received from the encoder}`
4: $\delta_k \leftarrow \frac{1}{2^{p_k}}\left(\|e^{A(t_k^- - t_{k-1})}\|_\infty \delta_{k-1} + \frac{\nu}{\|A\|_2}[e^{\|A\|_2(t_k^- - t_{k-1})} - 1]\right)$ `{compute bound on encoding error at` t_k`}`
5: $z_{D,k} \leftarrow q_{D,k}(z_{E,k}, z_k)$ `{decode plant state at` t_k`}`
6: $\hat{x}(r_k) \leftarrow e^{\bar{A}\Delta_k} z_k + e^{A\Delta_k}(z_{D,k} - z_k)$ `{update controller state}`
7: $d_e(r_k) \leftarrow \|e^{A\Delta_k}\|_\infty \delta_k + \frac{\nu}{\|A\|_2}[e^{\|A\|_2\Delta_k} - 1]$ `{update bound on encoding error}`

Lemma V.1. *(Consistency of Algorithms 1 and 2). If initially the encoder and the decoder share identical values for $\hat{x}(t_0)$ and $d_e(t_0)$ then Algorithms 1 and 2 result in consistent $\hat{x}(t)$ and $d_e(t)$ signals for all $t \ge t_0$. Further, $t \mapsto \hat{x}(t)$ evolves according to (4) and $\|x_e(t)\|_\infty \le d_e(t)$ with $d_e(t)$ defined for $t \in [r_k, r_{k+1})$ for $k \in \mathbb{N}_0$ as*

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty \delta_k + \frac{\nu}{\|A\|_2}[e^{\|A\|_2(t-t_k)} - 1], \quad (35a)$$

$$\delta_{k+1} = \frac{1}{2^{p_{k+1}}} d_e(t_{k+1}^-). \quad (35b)$$

Proof: It is sufficient to show that the encoder and the decoder have the same signals after running their respective algorithms at $\{r_k\}_{k\in\mathbb{N}}$. Thus, we will show the equivalence of the corresponding steps of the two algorithms. The encoder and decoder steps will be prefixed by 'E' and 'D' respectively. Steps E1 and D1 are identical initialization of the variable δ_0. Step D2 is simply running (4) backwards in time to obtain $\hat{x}(t_k^-)$. In D3, z_E is simply the message received from the encoder that is encoded in E3. In D4, notice that the terms within the parenthesis add up to $d_e(t_k^-)$. Steps D5 through D7 are exactly identical to steps E5 through E7, respectively with identical data. As a consequence, $\hat{x}(t)$ and $d_e(t)$ values at the encoder and decoder are synchronized for all time $t \ge t_0$. Further, from Steps 6 of the algorithms it is easy to see that $t \mapsto \hat{x}(t)$ evolves according to (4). It is also easy to see that $d_e(t)$ definition in (35) is consistent with the its jump updates in the algorithms. It remains to be shown that $\|x_e(t)\|_\infty \le d_e(t)$ for all $t \ge t_0$.

First, observe that as a consequence of the fact that $x(t) =$

$\hat{x}(t) + x_e(t)$, (4a) and (5b) we have that

$$x(t) = e^{\bar{A}(t-t_k)}\hat{x}(t_k^-) + e^{A(t-t_k)}x_e(t_k^-) + \int_{t_k}^{t} e^{A(t-s)}v(s)\mathrm{d}s.$$

Specifically, letting $z_k = \hat{x}(t_k^-)$ as in Step 2 of the algorithms, consider the solution $y(.)$ that starts at $z_{D,k}$ at t_k and under zero disturbance, i.e.,

$$y(t) = e^{\bar{A}(t-t_k)}z_k + e^{A(t-t_k)}(z_{D,k} - z_k)$$

and specifically from Step 6 of the algorithms, we have $\hat{x}(r_k) = y(r_k)$. Further, given that $\|x(t_k^-) - z_{D,k}\|_\infty \leq \delta_k$, then we have

$$\begin{aligned} x_e(r_k) &= x(r_k) - y(r_k) \\ &= e^{A\Delta_k}(x(t_k^-) - z_{D,k}) + \int_{t_k}^{r_k} e^{A(r_k-s)}v(s)\mathrm{d}s, \end{aligned}$$

which implies that

$$\|x_e(r_k)\|_\infty \leq \|e^{A\Delta_k}\|_\infty \delta_k + \frac{\nu}{\|A\|_2}[e^{\|A\|_2\Delta_k} - 1] = d_e(r_k)$$

which is exactly the quantity in Steps E7 and D7. For $t \in [r_k, r_{k+1})$ for $k \in \mathbb{N}_0$ clearly $\|x_e(t)\|_\infty \leq d_e(t)$, which completes the proof. ■

Note that although d_e is updated by a jump at $\{r_k\}_{k\in\mathbb{N}}$, the reference time in (35a) is still t_k. The reason for this is that using instead the reference time r_k would result in a larger encoding error bound.

B. Trigger design and analysis

The basic underlying idea behind our event-triggered design in the scenario of non-instantaneous communication is to anticipate ahead of time the zero crossings of the performance-trigger function h_V and the channel-trigger function $h_{\mathrm{ch}} - 1$ after transmitting at most $n\bar{p}$ number of bits. Noting the update rule that gives $d_e(r_k)$ in Algorithms 1 and 2 and following arguments analogous to those of Lemma IV.8, we see that

$$h_{\mathrm{ch}}(r_k) \leq \bar{h}_{\mathrm{ch}}\left(\Delta_k, b(t_k^-), \epsilon(t_k^-), \frac{\epsilon(t_k^-)}{2^{p_k}}\right).$$

Unlike in the case of instantaneous communication, we need to distinguish between the third and the fourth argument in $\bar{h}_{\mathrm{ch}}$ because the transmitted bits do not affect the value of ϵ until r_k. If we can ensure that $h_{\mathrm{ch}}(r_k) \leq 1$, then the definition (27) of Γ_1 and Lemma IV.7 guarantee $h_V \leq 0$ until $r_k + \min\{\Gamma_1(1,1), T\}$. To anticipate $h_{\mathrm{ch}}(r_k) \leq 1$, we define

$$\tilde{\Gamma}_2(b_0, \epsilon_0, \psi_0) \triangleq \min\{\tau \geq 0 : \bar{h}_{\mathrm{ch}}(\tau, b_0, \epsilon_0, \psi_0) = 1\}. \quad (36)$$

From (34) we have that for $(2^{\bar{p}}\psi_0) = \epsilon_0 \leq \rho_T(b_0)$, $\tilde{\Gamma}_2(b_0, \epsilon_0, \psi_0) \geq \min\{\Gamma_1(1,1), T, T^*\}$ with

$$T^* \triangleq \min\{\tau \geq 0 : \phi(\tau, 1/(2^{\bar{p}})) = 1\}.$$

Given this discussion, we make the following assumption on the function Δ that describes the communication channel.

(A) For any $t \in \mathbb{R}_{\geq 0}$, $\Delta(t, 1) \geq 0$. Also, if $s_1 \leq s_2$, then $\Delta(t, s_1) \leq \Delta(t, s_2)$. Given $\bar{p} \in \mathbb{N}$, there exists $T_M \in \mathbb{R}_{\geq 0}$ with $T_M < \min\{\Gamma_1(1,1), T, T^*\}$ such that $\Delta(t, \bar{p}) \leq T_M$ for all $t \geq 0$.

Hence the event-triggering rule in this scenario must anticipate at least T_M units of time ahead the zero crossing of h_V and anticipate $h_{\mathrm{ch}}(r_k) \geq 1$ even after having transmitted the maximum number of bits, $n\bar{p}$, at t_k. In other words, we want to ensure $h_{\mathrm{ch}}(r_k) \leq 1$ so that $h_V < 0$ for at least all $t \in [r_k, t_{k+1})$. The fact that $T_M < \min\{\Gamma_1(1,1), T, T^*\}$ then ensures that $t_{k+1} - r_k > 0$.

Our problem then reduces to checking the zero-crossing of the functions $\Gamma_1 - T_M$, and $\tilde{\Gamma}_2 - T_M$. However, computing the functions Γ_1 and $\tilde{\Gamma}_2$ repeatedly as part of the event-triggering rule would impose an unnecessary computational burden. For this reason, we seek a way to check the conditions without having to explicitly compute Γ_1 and $\tilde{\Gamma}_2$. The following result provides a solution for the case of Γ_1.

Lemma V.2. *(Algebraic Condition to Check if $h_V < 0$ for the next T° units of time). Let $T^\circ > 0$. For any $b_0 \in [0,1]$, $\Gamma_1(b_0, \epsilon_0) > T^\circ$ if and only if $\tilde{b}(T^\circ, b_0, \epsilon_0) < 1$. Further, the corresponding statement with the inequalities reversed and the one in which the inequalities are replaced by equality are true.*

Proof: Given (27) and considering b_0 and ϵ_0 as parameters, it is sufficient to show that the equation $\tilde{b}(\tau, b_0, \epsilon_0) = 1$ has at most one solution in the interval $(0, \infty)$. Recall the functions f_1 and f_2 in the definition of $\tilde{b}$ of Lemma IV.6. Considering b_0 and ϵ_0 as parameters, note that the solutions of the equation $\tilde{b}(\tau, b_0, \epsilon_0) = 1$ are exactly those of $f_1(\tau, b_0, \epsilon_0) = f_2(\tau)$, while $\tilde{b}(\tau, b_0, \epsilon_0) < 1$ iff $f_1(\tau, b_0, \epsilon_0) < f_2(\tau)$.

Since $w > 0$, f_2 is monotonically increasing. Next, note that $\theta = \|A\|_2 + \beta/2 > 0$. Thus, f_1 contains the dominant exponent and hence there is a $\tau_1(\epsilon_0) \geq 0$ such that $\dot{f}_1(\tau, b_0, \epsilon_0) > \dot{f}_2(\tau)$ for all $\tau > \tau_1(\epsilon_0)$ and $\dot{f}_1(\tau, b_0, \epsilon_0) < \dot{f}_2(\tau)$ for all $\tau < \tau_1(\epsilon_0)$. Thus, for each $b_0 \in [0, 1)$, there exists a unique solution for $\tilde{b}(\tau, b_0, \epsilon_0) = 1$. For $b_0 = 1$ and $\tau_1(\epsilon_0) > 0$ there exists a unique solution to the problem. For $b_0 = 1$ and $\tau_1(\epsilon_0) \leq 0$ there exists no solution and $f_1(\tau, b_0, \epsilon_0) > f_2(\tau)$ for all $\tau > 0$. In each scenario the claim of the lemma follows directly. ■

Next, we make a similar observation about Γ_2.

Lemma V.3. *(Algebraic Condition to Check the Sign of $\tilde{\Gamma}_2 - T^\circ$). Let $T^\circ > 0$. For any $b_0 \in [0,1]$ and $\epsilon_0 \in [0,1]$, $\tilde{\Gamma}_2(b_0, \epsilon_0, \psi_0) > T^\circ$ if and only if $\bar{h}_{\mathrm{ch}}(T^\circ, b_0, \epsilon_0, \psi_0) < 1$. Further, the corresponding statement with the inequalities reversed and the one in which the inequalities are replaced by equality are true.*

Proof: Again, considering b_0 and ϵ_0 as parameters, it is sufficient to show that $\bar{h}_{\mathrm{ch}}(\tau, b_0, \epsilon_0, \psi_0) = 1$ has a unique solution. We show the uniqueness through a contradiction argument. Suppose there exists a $\tau^* > \tilde{\Gamma}_2(b_0, \epsilon_0, \psi_0)$ such that $\bar{h}_{\mathrm{ch}}(\tau^*, b_0, \epsilon_0, \psi_0) = 1$. Since $\bar{h}_{\mathrm{ch}}$ is a continuous function, it must then have a local maximum in the time interval $[\tilde{\Gamma}_2(b_0, \epsilon_0), \tau^*]$. Notice from (31) that the numerator of $\bar{h}_{\mathrm{ch}}$ is a monotonously increasing function of time τ. Next, since $\tilde{b} \mapsto \rho_T(\tilde{b})$ is a decreasing function it follows that $\tilde{b}(., b_0, \epsilon_0)$ must have a local maximum in the time interval $[\tilde{\Gamma}_2(b_0, \epsilon_0), \tau^*]$. Thus, considering b_0 and ϵ_0 as parameters, notice that

$$\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tau} = -w\tilde{b} + W\epsilon_0 e^{\theta\tau} + c_1 + c_2(e^{\|A\|_2\tau} - 1),$$

while the second derivative is

$$\frac{\mathrm{d}^2\tilde{b}}{\mathrm{d}\tau^2} = -w\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tau} + W\epsilon_0\theta e^{\theta\tau} + c_2\|A\|_2 e^{\|A\|_2\tau}.$$

Then notice that the second derivative at any critical point of $\tilde{b}$ is positive since the first term vanishes at a critical point of $\tilde{b}$, the second term is positive for any τ because $\theta > 0$ and $c_2\|A\|_2 \geq 0$ by definition. Thus $\tilde{b}$ as a function of τ has no local maximum. Thus, this contradiction proves the result. ■

We are finally ready to present the main result of the section.

Theorem V.4. *(Bounded Data and Communication Rate). Consider the system* (2) *under the feedback law* $u = K\hat{x}$*, with* $t \mapsto \hat{x}(t)$ *evolving according to* (4) *and the sequence* $\{t_k\}_{k\in\mathbb{N}_0}$ *determined recursively by*

$$t_{k+1} = \min\{t \geq r_k : \tilde{b}(T_M, b(t), \epsilon(t)) \geq 1 \text{ OR } \bar{h}_{\text{ch}}(T_M, b(t), \epsilon(t), (\epsilon(t)/2^{\bar{p}})) \geq 1\}, \quad (37)$$

where $n\bar{p}$ *is the upper bound on the number of bits that can be sent per transmission,* $T > 0$ *in the definition* (31) *of* $\bar{h}_{\text{ch}}$ *is a design parameter, and* T_M *is as given in Assumption* (A)*. Let* $\{r_k\}_{k\in\mathbb{N}_0}$ *be given as* $r_0 = t_0$ *and* $r_k = t_k + \Delta_k$ *for* $k \in \mathbb{N}$*. Assume the encoding scheme is such that* (35) *is satisfied for all* $t \geq t_0$*. Further assume that* $V(x(t_0)) \leq V_d(t_0)$*,* $\bar{h}_{\text{ch}}(T_M, b(t_0), \epsilon(t_0), (\epsilon(t_0)/2^{\bar{p}})) \leq 1$ *and that* (15a)*-*(15b) *hold. Let* $\underline{p_k}$ *be given by*

$$\underline{p_k} \triangleq \min\{p \in \mathbb{N} : \bar{h}_{\text{ch}}(T_M, b(t_k), \epsilon(t_k), \tfrac{\epsilon(t_k)}{2^p}) \leq 1\}. \quad (38)$$

Then, the following hold:

(i) $\underline{p_1} \leq \bar{p}$*. Further for each* $k \in \mathbb{N}$*, if* $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$*, then* $\underline{p_{k+1}} \leq \bar{p}$*.*

(ii) the inter-transmission times $\{T_k = t_{k+1} - t_k\}_{k\in\mathbb{N}}$ *and inter-reception times* $\{R_k \triangleq r_{k+1} - r_k\}_{k\in\mathbb{N}}$ *have a uniform positive lower bound,*

(iii) the origin is exponentially practically stable for the closed-loop system, with $V(x(t)) \leq V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0$ *for all* $t \geq t_0$*.*

Proof: Since $V(x(t_0)) \leq V_d(t_0)$ and $\bar{h}_{\text{ch}}(T_M, b(t_0), \epsilon(t_0), (\epsilon(t_0)/2^{\bar{p}})) \leq 1$, the trigger (37) implies that $\underline{p_1} \leq \bar{p}$. Similarly, if for each $k \in \mathbb{N}$, $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$, then (37) implies $\underline{p_{k+1}} \leq \bar{p}$, which proves *(i)*.

Regarding *(ii)*, note that Assumption (A) implies that $r_k - t_k \geq 0$ for $k \in \mathbb{N}$. Therefore, it is enough to show that there exists a uniform lower bound on $t_{k+1} - r_k$. Notice that (38) implies that

$$\bar{h}_{\text{ch}}(T_M, b(t_k^-), \epsilon(t_k^-), (\epsilon(t_k^-)/2^{p_k})) \leq 1,$$

which in turn implies, as a consequence of the fact that $\Delta_k \leq T_M$ and Lemma V.3, that $\tilde{\Gamma}_2(b(t_k^-), \epsilon(t_k^-), (\epsilon(t_k^-)/2^{p_k})) - \Delta_k \geq 0$. Invoking Lemma V.3 once more, we see that

$$h_{\text{ch}}(r_k) \leq \bar{h}_{\text{ch}}(\Delta_k, b(t_k^-), \epsilon(t_k^-), (\epsilon(t_k^-)/2^{p_k})) \leq 1.$$

In other words, $\epsilon(r_k) \leq \rho_T(b(r_k))$. Now, let us pick $\tilde{T} \in (0, T)$ and notice that Lemma IV.7 guarantees that for all $\epsilon_0 \leq \rho_{\tilde{T}}(b_0)$, $\Gamma_1(b_0, \epsilon_0) \geq \min\{\Gamma_1(1,1), \tilde{T}\}$. Since $\tilde{T} \in (0, T)$, there exists a constant $\varpi \in (0,1)$ such that $\epsilon(r_k) \leq \varpi\rho_{\tilde{T}}(b(r_k))$. Thus, again for all $\epsilon_0 \leq \rho_{\tilde{T}}(b_0)$, we have that $\tilde{\Gamma}_2(b_0, \epsilon_0, \psi_0) \geq \min\{\Gamma_1(1,1), \tilde{T}, T^\bullet\}$, with

$$T^\bullet \triangleq \min\{\tau \geq 0 : \phi(\tau, \varpi/(2^{\bar{p}})) = 1\}.$$

Since $T_M < T$ by Assumption (A), there exists a choice of $\tilde{T} \in (T_M, T)$ such that $T_M < \min\{\Gamma_1(1,1), \tilde{T}, T^\bullet\}$. Thus, by Lemma V.3, we have that for all $\epsilon_0 \leq \rho_{\tilde{T}}(b_0)$, $\bar{h}_{\text{ch}}(T_M, b_0, \epsilon_0, (\epsilon_0/2^{\bar{p}})) < 1$. As a consequence, for $k \in \mathbb{N}_0$, $t_{k+1} - r_k$ is uniformly lower bounded by the time it takes $\dfrac{\epsilon(t)}{\rho_{\tilde{T}}(b(t))}$ to evolve from ϖ to 1, which in turn can be shown to have a uniform positive lower bound following arguments analogous to those in the proof of Theorem IV.9.

Regarding *(iii)*, note that from the triggering rule (37), we see that $\tilde{b}(T_M, b(t_k), \epsilon(t_k)) \leq 1$, which from Lemma V.2 implies that $\Gamma_1(b(t_k), \epsilon(t_k)) \geq T_M$. In other words, $V(x(t)) \leq V_d(t)$ (i.e., $b(t) \leq 1$) for *at least* all $t \in [t_k, r_k]$ for any $k \in \mathbb{N}_0$. Since $\bar{h}_{\text{ch}}(T_M, b(t_0), \epsilon(t_0), (\epsilon(t_0)/2^{\bar{p}})) \leq 1$ it means that $\epsilon(r_0) \leq \rho_T(b(r_0))$. Further, we have already seen that for any $k \in \mathbb{N}$, $\epsilon(r_k) \leq \rho_T(b(r_k))$. Therefore, for any $k \in \mathbb{N}_0$, $\Gamma_1(b(r_k), \epsilon(r_k)) \geq \Gamma_1(1,1) > 0$. This means $V(x(t)) \leq V_d(t)$ (i.e., $b(t) \leq 1$) for *at least* all $t \in [r_k, t_{k+1}]$. Putting these two facts together with *(ii)* concludes the proof. ■

Despite its appearance, note that the event-triggering rule (37) in Theorem V.4 is a generalization of the rule (32) in Theorem IV.9. In fact, when communication is instantaneous, $T_M = 0$, and we have $\tilde{b}(T_M, b(t), \epsilon(t)) = b(t)$ and $\bar{h}_{\text{ch}}(T_M, b(t), \epsilon(t), (\epsilon(t)/2^{\bar{p}})) = h_{\text{ch}}(t)/(2^{\bar{p}})$. Finally, note that the condition $b(t) \geq 1$ is equivalent to $h_V(t) \geq 0$. The next result upper bounds $\underline{p_k}$ in terms of the history of the number of bits transmitted.

Corollary V.5. *(Upper bound on* $\underline{p_k}$ *in terms of the history of the number of bits transmitted). Under the assumptions of Theorem V.4, the following holds for any* $k \in \mathbb{N}$*,*

$$\underline{p_k} \leq \log_2\left(\frac{e^{\theta T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M)}\right) + 1 + \log_2\left(\frac{e^{\theta(t_k - t_0)}}{\prod_{j=1}^{k-1} 2^{p_j}}\epsilon(t_0) + \sum_{i=0}^{k-1}\prod_{j=i+1}^{k-1}\frac{e^{\theta T_j}}{2^{p_j}}\alpha(T_i)\right),$$

with $\alpha(\tau) \triangleq \dfrac{\nu(e^{\|A\|_2\tau} - 1)}{c\|A\|_2\sqrt{V_0}}$.

Proof: Using (25) and (35) recursively along with the fact that $V_d(t) \geq V_0$ for all $t \geq t_0$ and the definition $\theta = \|A\|_2 + \dfrac{\beta}{2}$ gives for $k \in \mathbb{N}$

$$\begin{aligned}\epsilon(t_k^-) &\leq e^{\theta T_{k-1}}\frac{\epsilon(t_{k-1}^-)}{2^{p_{k-1}}} + \alpha(T_{k-1}) \\ &\leq \frac{e^{\theta T_{k-1}}}{2^{p_{k-1}}}\left[\frac{e^{\theta T_{k-2}}}{2^{p_{k-2}}}\epsilon(t_{k-2}^-) + \alpha(T_{k-2})\right] + \alpha(T_{k-1}) \\ &\leq \frac{e^{\theta(t_k - t_0)}}{\prod_{j=1}^{k-1} 2^{p_j}}\epsilon(t_0) + \sum_{i=0}^{k-1}\prod_{j=i+1}^{k-1}\frac{e^{\theta T_j}}{2^{p_j}}\alpha(T_i). \quad (39)\end{aligned}$$

Next, observe that, for each $k \in \mathbb{N}$, $\epsilon(t_k^-) \geq \rho_T(b(t_k^-))$. If this were not the case, then $\Gamma_1(b(t_k^-), \epsilon(t_k^-)) \geq \min\{\Gamma_1(1,1), T\}$ by Lemma IV.7, and on the other hand

$\tilde{\Gamma}_2(b(t_k^-), \epsilon(t_k^-), \epsilon(t_k^-)/2^{\bar{p}}) \geq \min\{\Gamma_1(1,1), T, T^*\} > T_M$. These two conditions together would imply, by Lemmas V.2 and V.3, that neither of the conditions in the trigger (37) is satisfied at $t = t_k^-$, which is a contradiction.

Now, since Theorem V.4 guarantees $b(t) \leq 1$ for all $t \geq t_0$ and since $\rho_T(y) \geq 1$ for all $y \in [0,1]$, we have $\epsilon(t_k^-) \geq 1$. Next, the trigger (37) and Theorem V.4(i) ensure that $\bar{h}_{\text{ch}}(T_M, b(t_k^-), \epsilon(t_k^-), (\epsilon(t_k^-)/2^{\bar{p}})) \leq 1$, i.e.,

$$e^{\theta T_M}\frac{\epsilon(t_k^-)}{2^{\bar{p}}} + \alpha(T_M) \leq \rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)).$$

Rearranging the terms, we have

$$\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M) \geq \frac{e^{\theta T_M}}{2^{\bar{p}}}\epsilon(t_k^-) \geq \frac{e^{\theta T_M}}{2^{\bar{p}}} > 0,$$

where we have used $\epsilon(t_k^-) \geq 1$. Now (38) implies that

$$e^{\theta T_M}\frac{\epsilon(t_k^-)}{2^{(\underline{p_k}-1)}} + \alpha(T_M) \geq \rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)),$$

which in turn implies that

$$2^{\underline{p_k}} \leq \frac{2e^{\theta T_M}\epsilon(t_k^-)}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M)}.$$

In other words,

$$\underline{p_k} \leq \log_2(\epsilon(t_k^-)) + \log_2\left(\frac{2e^{\theta T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M)}\right).$$

Substituting (39) (and multiplying by n to give us the number of bits) yields the result. ■

Although this result does not explicitly give a data rate as in Corollary IV.5, it provides an implicit characterization of it. This becomes more clear in the absence of disturbances.

Corollary V.6. *(Upper bound on sufficient bit rate in the absence of disturbances). Under the assumptions of Theorem V.4 and no disturbance, the following holds for any $k \in \mathbb{N}$,*

$$n\Big(\underline{p_k} + \sum_{i=1}^{k-1} p_i\Big) \leq n\Bigg[\log_2\left(\frac{e^{\theta T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-))}\right) + 1 + \theta\log_2(e)(t_k - t_0) + \log_2(\epsilon(t_0))\Bigg].$$

Proof: In the no disturbance case, $\nu = 0$ and the second term of (35a) vanishes, which justifies $\alpha(\tau) \equiv 0$ in Corollary V.5 even in the case $V_0 = 0$. As a result, we have

$$\underline{p_k} \leq \log_2\left(\frac{e^{\theta T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-))}\right) + 1 + \log_2\left(\frac{e^{\theta(t_k - t_0)}}{\prod_{j=1}^{k-1} 2^{p_j}}\epsilon(t_0)\right).$$

Multiplying by n and rearranging the terms yields the sufficient bit rate in the statement. ■

Notice that the effect of non-instantaneous communication, through T_M, in Corollary V.6 only has a transient effect on the sufficient bit rate. When $T_M = 0$, the first term is non-positive (recall $\rho_T \geq 1$) and we recover the result of Corollary IV.5.

VI. Simulations

In this section, we illustrate our results in simulation for three scenarios: instantaneous communication with no disturbance and non-instantaneous communication with and without disturbance. We begin by describing the problem data. Consider the system on $\mathbb{R}^2$ given by (2) with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} 2 & -8 \end{bmatrix}.$$

The plant matrix A has eigenvalues at 2 and 3, while the control gain matrix K places the eigenvalues of the matrix $\bar{A} = A + BK$ at -1 and -2. We select the matrix $Q = \mathrm{I}_2$, for which the solution to the Lyapunov equation (6) is

$$P = \begin{bmatrix} 2.2500 & -0.9167 \\ -0.9167 & 0.5833 \end{bmatrix}.$$

The desired control performance is specified by

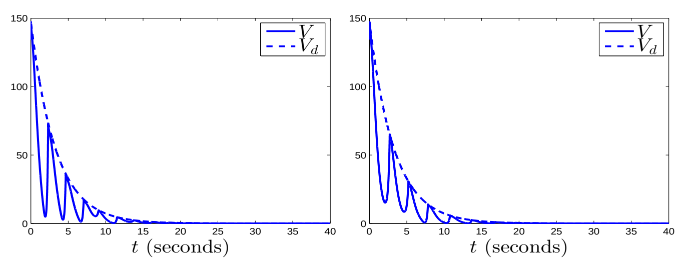
$$V_d(t_0) = 1.1V(x(t_0)), \quad \beta = 0.8\frac{\lambda_m(Q)}{\lambda_M(P)},$$

and V_0 chosen according to (15b) in each scenario. We set $a = 1.2$ in (15a), so that $W > 0$, and assume, without loss of generality, $t_0 = 0$. We choose the design parameter $T = 0.5 \times \Gamma_1(1,1)$. The initial condition is $x(t_0) = (6, -4)$, and the encoder and decoder use the information

$$\hat{x}(t_0) = (0,0), \quad d_e(t_0) = 2\|x(t_0) - \hat{x}(t_0)\|_\infty.$$

Finally, in each scenario, the number of bits transmitted at each transmission time, np_k, is $n\underline{p_k}$, the minimum number of bits as prescribed by (33) and (38), respectively.

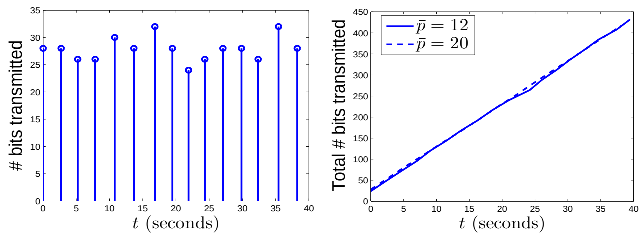
Instantaneous communication and no disturbance: in this scenario, we let $\nu = V_0 = 0$, for which we obtain $\Gamma_1(1,1) = 0.5699$. We present simulations for two cases, $\bar{p} = 12$ and $\bar{p} = 20$, where $n\bar{p} = 2\bar{p}$ is the uniform upper bound on the number of bits per transmission imposed by the communication channel.



(a) (b)

Fig. 1. Instantaneous communication and no disturbance: evolution of V_d and V under the event-triggered design (32) for (a) $\bar{p} = 12$ and (b) $\bar{p} = 20$.

Figure 1 shows the evolution of V and V_d in both cases. As established in Theorem IV.9, the desired convergence rate is guaranteed in each case. In the case of $\bar{p} = 12$, it turns out that $\underline{p_k} = \bar{p}$ for each $k \in \mathbb{N}$. On the other hand, in the case when $\bar{p} = 20$, the performance of V with respect to V_d plays a more relevant role in determining the transmission times in (32). In fact, in the presented simulation, $\underline{p_k} < \bar{p}$ on all transmissions, as depicted in Figure 2(a). Figure 2(b) shows the interpolated plot of the total number of bits transmitted for both cases, $\bar{p} = 12$ and $\bar{p} = 20$. Although in reality



(a) (b)

Fig. 2. For the event-triggered implementations shown in Figure 1, (a) shows the number of bits on each transmission in the case $\bar{p} = 20$ and (b) shows the interpolated plot of the total number of bits transmitted for the cases $\bar{p} = 12$ and $\bar{p} = 20$.

the total number of bits transmitted as a function of time is piecewise constant, the interpolated plots enable a more insightful comparison. In the case $\bar{p} = 20$, after having transmitted more bits initially than for $\bar{p} = 12$, the gap in the cumulative bit counts diminishes eventually. Finally, during the time interval $[0, 40]$, the number of transmissions, average inter-transmission time, and minimum inter-transmission time in the case of $\bar{p} = 12$ are 18, 2.3211 and 1.8248, respectively. In the case of $\bar{p} = 20$, these quantities are 15, 2.7310 and 2.4384, respectively.

Non-instantaneous communication and non-zero disturbance: in this scenario, we let $\nu = 0.01$ and, following (15b) with $\sigma = 0.9$, we set $V_0 = 5.3942$, for which we obtain $\Gamma_1(1, 1) = 0.0172$. The actual disturbance signal employed in the simulation is

$$v_1(t) = \nu \sin(0.5t), \quad v_2(t) = \nu \cos(0.5t).$$

We present a simulation for the case $\bar{p} = 20$. We choose $T_M = 0.5 \times \min\{\Gamma_1(1, 1), T, T^*\} = 1.2699 \times 10^{-4}$ and the communication time $\Delta_k = r_k - t_k = T_M$ for all $k \in \mathbb{N}$ (consequently, note that $R_k = T_k$ for $k \in \mathbb{N}_0$). Figure 3

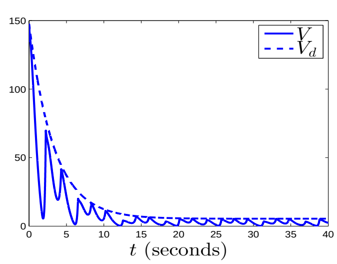
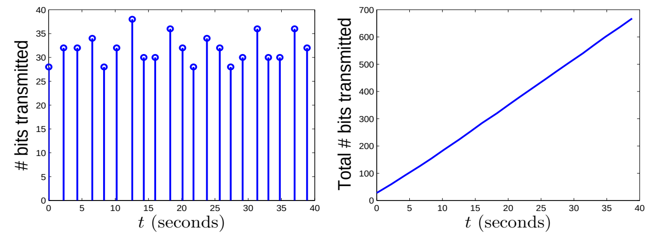


Fig. 3. Non-instantaneous communication and non-zero disturbance: evolution of V_d and V under the event-triggered design (37) for $\bar{p} = 20$.

shows the evolution of V and V_d, which is in accordance with Theorem V.4. Figure 4 displays the evolution of the number of bits transmitted. During the time interval $[0, 40]$, the number of transmissions is 21, with an average inter-transmission interval of 1.9048, and a minimum inter-transmission interval of 1.6479.

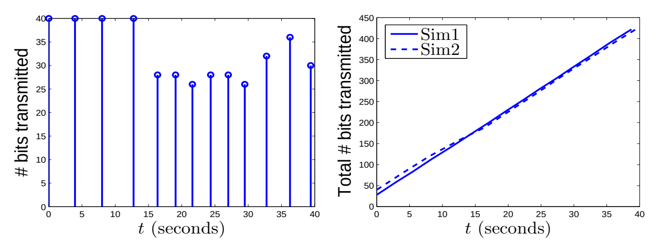
Non-instantaneous communication and no disturbance: in this scenario, we let $\nu = V_0 = 0$, $\bar{p} = 20$ and $T_M = 1.2699 \times 10^{-4}$. The values of $\Gamma_1(1, 1)$ and T are as in the case of instantaneous communication with no disturbance. We choose the communication time as $\Delta_k = r_k - t_k = T_M$ for all $k \in \mathbb{N}$.



(a) (b)

Fig. 4. For the event-triggered implementation shown in Figure 3 with $\bar{p} = 20$, (a) shows the number of bits on each transmission and (b) shows the interpolated plot of the total number of bits transmitted.

To illustrate Corollary V.6, we compare the results of two simulations: in "Sim1" we choose $p_k = \underline{p_k}$ for all $k \in \mathbb{N}$ while in "Sim2" we choose $p_k = \bar{p}$ for $k \in \{1, 2, 3, 4\}$ and $p_k = \underline{p_k}$ for all $k \in [5, \infty) \cap \mathbb{N}$. Figure 5(a) shows the number of bits on each transmission for "Sim2" while Figure 5(b) compares the interpolated total number of bits transmitted in "Sim1" and "Sim2". Notice that until 5^{th} transmission time of "Sim2", the cumulative bit count for "Sim2" exceeds that of "Sim1" but the gap is immediately closed at that time and thereafter remains slightly lower than that of "Sim1". This demonstrate the ability of the event-trigger design to transmit fewer bits if more bits than prescribed were transmitted in the past. We also see that the data rate, as interpreted in Corollary V.6, remains approximately fixed irrespective of the past history of transmitted bit count as long as the constraints of Theorem V.4 are respected. We did not observe a similar behavior in the scenario with disturbance.



(a) (b)

Fig. 5. For non-instantaneous communication without disturbance and $\bar{p} = 20$, (a) shows the number of bits on each transmission for "Sim2" (b) shows a comparison plot of the interpolated total number of bits transmitted in "Sim1" and "Sim2".

VII. Conclusions

We have studied the problem of exponential practical stabilization of linear-time invariant systems, in the presence of disturbance, and under bounded communication bit rate. Our event-triggered design opportunistically determines the times for communication as well as the numbers of bits to be transmitted at each time. Given a uniform bound on the norm of the disturbance and a prescribed rate of convergence, the control strategy proposed here asymptotically confines the plant to a compact set, guarantees a uniform positive lower bound on inter-transmission and inter-reception communication times, and ensures that the number of bits transmitted at each transmission is uniformly upper bounded. These guarantees are valid for instantaneous transmissions with finite precision data as well as for non-instantaneous transmissions with bounded communication rate. The combination of elements from event-triggered control and information theory has also enabled us to guarantee an arbitrarily prescribed convergence rate (something not typically ensured in the information-theoretic approach) and characterize necessary and sufficient conditions on the number of bits required for stabilization under opportunistic transmissions (an issue mostly overlooked in event-triggered control). Future work will further explore the characterization of data rates under disturbances, the suppression of the synchronization requirement between the encoder and the decoder to maintain a synchronized quantization domain, the extension of the results to stochastic time-varying communication channels, and, more generally, the understanding of the trade-offs between system performance and timeliness and size of transmissions.

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