SADDLE-POINT DYNAMICS: CONDITIONS FOR ASYMPTOTIC STABILITY OF SADDLE POINTS *

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Abstract. This paper considers continuously differentiable functions of two vector variables that have (possibly a continuum of) min-max saddle points. We study the asymptotic convergence properties of the associated saddle-point dynamics (gradient-descent in the first variable and gradient-ascent in the second one). We identify a suite of complementary conditions under which the set of saddle points is asymptotically stable under the saddle-point dynamics. Our first set of results is based on the convexity-concavity of the function defining the saddle-point dynamics to establish the convergence guarantees. For functions that do not enjoy this feature, our second set of results relies on properties of the linearization of the dynamics, the function along the proximal normals to the saddle set, and the linearity of the function in one variable. We also provide global versions of the asymptotic convergence results. Various examples illustrate our discussion.

Key words. saddle-point dynamics, asymptotic convergence, convex-concave functions, proximal calculus, center manifold theory, nonsmooth dynamics

AMS subject classifications. 34A34, 34D05, 34D23, 34D35, 37L10

1. Introduction.

It is well known that the trajectories of the gradient dynamics of a continuously differentiable function with bounded sublevel sets converge asymptotically to its set of critical points, see e.g. [20]. This fact, however, is not true in general for the saddle-point dynamics (gradient descent in one variable and gradient ascent in the other) of a continuously differentiable function of two variables, see e.g. [2, 13]. In this paper, our aim is to investigate conditions under which the above statement is true for the case where the critical points are min-max saddle points and they possibly form a continuum. Our motivation comes from the applications of the saddle-point dynamics (also known as primal-dual dynamics) to find solutions of equality constrained optimization problems and Nash equilibria of zero-sum games.

Literature review. In constrained optimization problems, the pioneering works [2, 25] popularized the use of the primal-dual dynamics to arrive at the saddle points of the Lagrangian. For inequality constrained problems, this dynamics is modified with a projection operator on the dual variables to preserve their nonnegativity, which results in a discontinuous vector field. Recent works have further explored the convergence analysis of such dynamics, both in continuous [17, 9] and in discrete [27] time. The work [16] proposes instead a continuous dynamics whose design builds on first- and second-order information of the Lagrangian. In the context of distributed control and multi-agent systems, an important motivation to study saddle-point dynamics comes from network optimization problems where the objective function is an aggregate of each agents' local objective function and the constraints are given by a set of conditions that are locally computable at the agent level. Because of this structure, the saddle-point dynamics of the Lagrangian for such problems is inherently amenable to distributed implementation. This observation explains the emerging

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body of work that, from this perspective, looks at problems in distributed convex optimization [33, 18, 14], distributed linear programming [30], and applications to power networks [26, 34, 35] and bargaining problems [31]. The work [23] shows an interesting application of the saddle-point dynamics to find a common Lyapunov function for a linear differential inclusion. In game theory, it is natural to study the convergence properties of saddle-point dynamics to find the Nash equilibria of two-person zero-sum games [3, 29]. A majority of these works assume the function whose saddle points are sought to be convex-concave in its arguments. Our focus here instead is on the asymptotic stability of the min-max saddle points under the saddle-point dynamics for a wider class of functions, and without any nonnegativity-preserving projection on individual variables. We explicitly allow for the possibility of a continuum of saddle points, instead of isolated ones, and wherever feasible, on establishing convergence of the trajectories to a point in the set. The issue of asymptotic convergence, even in the case of standard gradient systems, is a delicate one when equilibria are a continuum [1]. In such scenarios, convergence to a point might not be guaranteed, see e.g., the counter example in [28]. Our work here is complementary to [21], which focuses on the characterization of the asymptotic behavior of the saddle-point dynamics when trajectories do not converge to saddle points and instead show oscillatory behaviour.

Statement of contributions. Our starting point is the definition of the saddle-point dynamics for continuously differentiable functions of two (vector) variables, which we term saddle functions. The saddle-point dynamics consists of gradient descent of the saddle function in the first variable and gradient ascent in the second variable. Our objective is to characterize the asymptotic convergence properties of the saddle-point dynamics to the set of min-max saddle points of the saddle function. Assuming this set is nonempty, our contributions can be understood as a catalog of complementary conditions on the saddle function that guarantee that the trajectories of the saddle-point dynamics are proved to converge to the set of saddle points, and possibly to a point in the set. We broadly divide our results in two categories, one in which the saddle function has convexity-concavity properties and the other in which it does not. For the first category, our starting result considers saddle functions that are locally convex-concave on the set of saddle points. We show that asymptotic stability of the set of saddle points is guaranteed if either the convexity or concavity properties are strict, and convergence is pointwise. Furthermore, motivated by equality constrained optimization problems, our second result shows that the same conclusions on convergence hold for functions that depend linearly on one of its arguments if the strictness requirement is dropped. For the third and last result in this category, we relax the convexity-concavity requirement and establish asymptotic convergence for strongly jointly quasiconvex-quasiconcave saddle functions. Moving on to the second category of scenarios, where functions lack convexity-concavity properties, our first condition is based on linearization. We consider piecewise twice continuously differentiable saddle-point dynamics and provide conditions on the eigenvalues of the limit points of Jacobian matrices of the saddle function at the saddle points that ensure local asymptotic stability of a manifold of saddle points. Our convergence analysis is based on a general result of independent interest on the stability of a manifold of equilibria for piecewise smooth vector fields that we state and prove using ideas from center manifold theory. The next two results are motivated by the observation that saddle functions exist in the second category that do not satisfy the linearization hypotheses and yet have convergent dynamics. In one result, we justify convergence by studying the variation of the function and its Hessian along the proximal normal directions to the set of saddle points. Specifically, we assume polynomial bounds for these variations and derive an appropriate relationship between these bounds that ensures asymptotic convergence. In the other result, we assume the saddle function to be linear in one variable and indefinite in another, where the indefinite part satisfies some appropriate regularity conditions. When discussing each of the above scenarios, we extend the conditions to obtain global convergence wherever feasible. Our analysis is based on tools and notions from saddle points, stability analysis of nonlinear systems, proximal normals, and center manifold theory. Various illustrative examples throughout the paper justify the complementary character of the hypotheses in our results.

Organization. Section 2 introduces notation and basic preliminaries. Section 3 presents the saddle-point dynamics and the problem statement. Section 4 deals with saddle functions with convexity-concavity properties. For the case when this property does not hold, Section 5 relies on linearization techniques, proximal normals, and the linearity structure of the saddle function to establish convergence guarantees. Finally, Section 6 summarizes our conclusions and ideas for future work.

2. Preliminaries.

This section introduces basic notation and presents preliminaries on proximal calculus and saddle points.

2.1. Notation.

We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{>0}$ and $\mathbb{Z}_{\geq 1}$ be the set of real, nonnegative real, nonpositive real, positive real, and positive integer numbers, respectively. Given two sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}^n$, we let $\mathcal{A}_1 + \mathcal{A}_2 = \{x + y \mid x \in \mathcal{A}_1, y \in \mathcal{A}_2\}$. We denote by $\|\cdot\|$ the 2-norm on $\mathbb{R}^n$ and also the induced 2-norm on $\mathbb{R}^{n\times n}$. Let $B_\delta(x)$ represent the open ball centered at $x \in \mathbb{R}^n$ of radius $\delta > 0$. Given $x \in \mathbb{R}^n$, x_i denotes the i-th component of x. For vectors $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, the vector $(u; w) \in \mathbb{R}^{n+m}$ denotes their concatenation. For $A \in \mathbb{R}^{n\times n}$, we use $A \succeq 0$, $A \preceq 0$, $A \succ 0$, and $A \prec 0$ to denote the fact that A is positive semidefinite, negative semidefinite, positive definite, and negative definite, respectively. The eigenvalues of A are $\lambda_i(A)$ for $i \in \{1, \dots, n\}$. If A is symmetric, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum and minimum eigenvalues, respectively. The range and null spaces of A are denoted by $\mathrm{range}(A)$, $\mathrm{null}(A)$, respectively. We use the notation $\mathcal{C}^k$ for a function being $k \in \mathbb{Z}_{\geq 1}$ times continuously differentiable. A set $\mathcal{S} \subset \mathbb{R}^n$ is *path connected* if for any two points $a, b \in \mathcal{S}$ there exists a continuous map $\gamma : [0, 1] \to \mathcal{S}$ such that $\gamma(0) = a$ and $\gamma(1) = b$. A set $\mathcal{S}_c \subset \mathcal{S} \subset \mathbb{R}^n$ is an *isolated path connected component* of $\mathcal{S}$ if it is path connected and there exists an open neighborhood $\mathcal{U}$ of $\mathcal{S}_c$ in $\mathbb{R}^n$ such that $\mathcal{U} \cap \mathcal{S} = \mathcal{S}_c$. For a real-valued function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, we denote the partial derivative of F with respect to the first argument by $\nabla_x F$ and with respect to the second argument by $\nabla_z F$. The higher-order derivatives follow the convention $\nabla_{xz} F = \frac{\partial^2 F}{\partial x \partial z}$, $\nabla_{xx} F = \frac{\partial^2 F}{\partial x^2}$, and so on. The restriction of $f : \mathbb{R}^n \to \mathbb{R}^m$ to a subset $\mathcal{S} \subset \mathbb{R}^n$ is denoted by $f_{|\mathcal{S}}$. The Jacobian of a $\mathcal{C}^1$ map $f : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$ is denoted by $Df(x) \in \mathbb{R}^{m\times n}$. For a real-valued function $V : \mathbb{R}^n \to \mathbb{R}$ and $\alpha > 0$, we denote the sublevel set of V by $V^{-1}(\leq \alpha) = \{x \in \mathbb{R}^n \mid V(x) \leq \alpha\}$. Finally, a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *piecewise* $\mathcal{C}^2$ if it is continuous and there exists (1) a finite collection of disjoint open sets $\mathcal{D}_1, \dots, \mathcal{D}_m \subset \mathbb{R}^n$, referred to as *patches*, whose closure covers $\mathbb{R}^n$, that is, $\mathbb{R}^n = \cup_{i=1}^m \mathrm{cl}(\mathcal{D}_i)$ and (2) a finite collection of $\mathcal{C}^2$ functions $\{f_i : \mathcal{D}_i^e \to \mathbb{R}^n\}_{i=1}^m$ where, for each $i \in \{1, \dots, m\}$, $\mathcal{D}_i^e$ is open with $\mathrm{cl}(\mathcal{D}_i) \subset \mathcal{D}_i^e$, such that $f_{|\mathrm{cl}(\mathcal{D}_i)}$ and f_i take the same values over $\mathrm{cl}(\mathcal{D}_i)$.

2.2. Proximal calculus. We present here a few notions on proximal calculus following [11]. Given a closed set $\mathcal{E} \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n \backslash \mathcal{E}$, the distance from x to $\mathcal{E}$ is,

$$d_{\mathcal{E}}(x) = \min_{y \in \mathcal{E}} \|x - y\|. \tag{2.1}$$

We let $\mathrm{proj}_{\mathcal{E}}(x)$ denote the set of points in $\mathcal{E}$ that are closest to x, i.e., $\mathrm{proj}_{\mathcal{E}}(x) = \{y \in \mathcal{E} \mid \|x-y\| = d_{\mathcal{E}}(x)\} \subset \mathcal{E}$. For $y \in \mathrm{proj}_{\mathcal{E}}(x)$, the vector $x-y$ is a *proximal normal direction* to $\mathcal{E}$ at y and any nonnegative multiple $\zeta = t(x-y)$, $t \geq 0$ is called a *proximal normal* (P-normal) to $\mathcal{E}$ at y. The distance function $d_{\mathcal{E}}$ might not be differentiable in general (unless $\mathcal{E}$ is convex), but is globally Lipschitz and regular [11, p. 23]. For a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the *generalized gradient* $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

$$\partial f(x) = \mathrm{co}\{\lim_{i \to \infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \cup \Omega_f\},$$

where co denotes convex hull, $S \subset \mathbb{R}^n$ is any set of measure zero, and Ω_f is the set (of measure zero) of points where f is not differentiable. In the case of the square of the distance function, one can compute [11, p. 99] the generalized gradient as,

$$\partial d^2_{\mathcal{E}}(x) = \mathrm{co}\{2(x-y) \mid y \in \mathrm{proj}_{\mathcal{E}}(x)\}. \tag{2.2}$$

2.3. Saddle points. Here, we provide basic definitions pertaining to the notion of saddle points. A point $(x_*, z_*) \in \mathbb{R}^n \times \mathbb{R}^m$ is a *local min-max saddle point* of a continuously differentiable function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ if there exist open neighborhoods $\mathcal{U}_{x_*} \subset \mathbb{R}^n$ of x_* and $\mathcal{U}_{z_*} \subset \mathbb{R}^m$ of z_* such that

$$F(x_*, z) \leq F(x_*, z_*) \leq F(x, z_*), \tag{2.3}$$

for all $z \in \mathcal{U}_{z_*}$ and $x \in \mathcal{U}_{x_*}$. The point (x_*, z_*) is a *global min-max saddle point* of F if $\mathcal{U}_{x_*} = \mathbb{R}^n$ and $\mathcal{U}_{z_*} = \mathbb{R}^m$. Min-max saddle points are a particular case of the more general notion of saddle points. We focus here on min-max saddle points motivated by problems in constrained optimization and zero-sum games, whose solutions correspond to min-max saddle points. With a slight abuse of terminology, throughout the paper we refer to the local min-max saddle points simply as saddle points. We denote by $\mathrm{Saddle}(F)$ the set of saddle points of F. From (2.3), for $(x_*, z_*) \in \mathrm{Saddle}(F)$, the point $x_* \in \mathbb{R}^n$ (resp. $z_* \in \mathbb{R}^m$) is a local minimizer (resp. local maximizer) of the map $x \mapsto F(x, z_*)$ (resp. $z \mapsto F(x_*, z)$). Each saddle point is a critical point of F, i.e., $\nabla_x F(x_*, z_*) = 0$ and $\nabla_z F(x_*, z_*) = 0$. Additionally, if F is $\mathcal{C}^2$, then $\nabla_{xx} F(x_*, z_*) \preceq 0$ and $\nabla_{zz} F(x_*, z_*) \succeq 0$. Also, if $\nabla_{xx} F(x_*, z_*) \prec 0$ and $\nabla_{zz} F(x_*, z_*) \succ 0$, then the inequalities in (2.3) are strict.

A function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is *locally convex-concave* at a point $(\tilde{x}, \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^m$ if there exists an open neighborhood $\mathcal{U}$ of $(\tilde{x}, \tilde{z})$ such that for all $(\bar{x}, \bar{z}) \in \mathcal{U}$, the functions $x \mapsto F(x, \bar{z})$ and $z \mapsto F(\bar{x}, z)$ are convex over $\mathcal{U} \cap (\mathbb{R}^n \times \{\bar{z}\})$ and concave over $\mathcal{U} \cap (\{\bar{x}\} \times \mathbb{R}^m)$, respectively. If in addition, either $x \mapsto F(x, \tilde{z})$ is strictly convex in an open neighborhood of $\tilde{x}$, or $z \mapsto F(\tilde{x}, z)$ is strictly concave in an open neighborhood of $\tilde{z}$, then F is *locally strictly convex-concave* at $(\tilde{x}, \tilde{z})$. F is locally (resp. locally strictly) convex-concave on a set $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ if it is so at each point in $\mathcal{S}$. F is *globally convex-concave* if in the local definition $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^m$. Finally, F is *globally strictly convex-concave* if it is globally convex-concave and for any $(\bar{x}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^m$, either $x \mapsto F(x, \bar{z})$ is strictly convex or $z \mapsto F(\bar{x}, z)$ is strictly concave. Note that this notion is different than saying that F is *both* strictly convex and strictly concave.

Next, we define strongly quasiconvex function following [22]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly quasiconvex* with parameter $s > 0$ over a convex set $\mathcal{D} \subset \mathbb{R}^n$ if for all $x, y \in \mathcal{D}$ and all $\lambda \in [0, 1]$ we have,

$$\max\{f(x), f(y)\} - f(\lambda x + (1 - \lambda)y) \geq s\lambda(1 - \lambda)\|x - y\|^2.$$

A function f is *strongly quasiconcave* with parameter $s > 0$ over the set $\mathcal{D}$ if $-f$ is strongly quasiconvex with parameter s over $\mathcal{D}$. A function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is *locally jointly strongly quasiconvex-quasiconcave* at a point $(\tilde{x}, \tilde{z}) \in \mathbb{R}^n \times \mathbb{R}^m$ if there exist $s > 0$ and an open neighborhood $\mathcal{U}$ of $(\tilde{x}, \tilde{z})$ such that for all $(\bar{x}, \bar{z}) \in \mathcal{U}$, the function $x \mapsto F(x, \bar{z})$ is strongly quasiconvex with parameter s over $\mathcal{U} \cap (\mathbb{R}^n \times \{\bar{z}\})$ and the function $z \mapsto F(\bar{x}, z)$ is strongly quasiconvex with parameter s over $\mathcal{U} \cap (\{\bar{x}\} \times \mathbb{R}^m)$. F is locally jointly strongly quasiconvex-quasiconcave on a set $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ if it is so at each point in $\mathcal{S}$. F is *globally jointly strongly quasiconvex-quasiconcave* if in the local definition $\mathcal{U} = \mathbb{R}^n \times \mathbb{R}^m$.

3. Problem statement. Here we formulate the problem of interest in the paper. Given a continuously differentiable function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, which we refer to as *saddle function*, we consider its saddle-point dynamics, i.e., gradient-descent in one argument and gradient-ascent in the other,

$$\dot{x} = -\nabla_x F(x, z), \tag{3.1a}$$
$$\dot{z} = \nabla_z F(x, z). \tag{3.1b}$$

When convenient, we use the shorthand notation $X_{\mathrm{sp}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ to refer to this dynamics. Our aim is to provide conditions on F under which the trajectories of its saddle-point dynamics (3.1) locally asymptotically converge to its set of saddle points, and possibly to a point in the set. We are also interested in identifying conditions to establish global asymptotic convergence. Throughout our study, we assume that the set $\mathrm{Saddle}(F)$ is nonempty. This assumption is valid under mild conditions in the application areas that motivate our study: for the Lagrangian of the constrained optimization problem [6] and the value function for zero-sum games [3]. Our forthcoming discussion is divided in two threads, one for the case of convex-concave functions, cf. Section 4, and one for the case of general functions, cf. Section 5. In each case, we provide illustrative examples to show the applicability of the results.

4. Convergence analysis for convex-concave saddle functions. This section presents conditions for the asymptotic stability of saddle points under the saddle-point dynamics (3.1) that rely on the convexity-concavity properties of the saddle function.

4.1. Stability under strict convexity-concavity. Our first result provides conditions that guarantee the local asymptotic stability of the set of saddle points.

PROPOSITION 4.1. *(Local asymptotic stability of the set of saddle points via convexity-concavity): For $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ continuously differentiable and locally strictly convex-concave on* $\mathrm{Saddle}(F)$*, each isolated path connected component of* $\mathrm{Saddle}(F)$ *is locally asymptotically stable under the saddle-point dynamics X_{sp} and, moreover, the convergence of each trajectory is to a point.*

Proof. Let $\mathcal{S}$ be an isolated path connected component of $\mathrm{Saddle}(F)$ and take $(x_*, z_*) \in \mathcal{S}$. Without loss of generality, we consider the case when $x \mapsto F(x, z_*)$ is locally strictly convex (the proof for the case when $z \mapsto F(x_*, z)$ is locally strictly concave is analogous). Consider the function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$,

$$V(x,z) = \frac{1}{2}\Big(\|x - x_*\|^2 + \|z - z_*\|^2\Big), \tag{4.1}$$

which we note is radially unbounded (and hence has bounded sublevel sets). We refer to V as a LaSalle function because locally, as we show next, its Lie derivative is negative, but not strictly negative. Let $\mathcal{U}$ be the neighborhood of (x_*, z_*) where local convexity-concavity holds. The Lie derivative of V along the dynamics (3.1) at $(x,z) \in \mathcal{U}$ can be written as,

$$\begin{aligned} \mathcal{L}_{X_{\mathrm{sp}}} V(x,z) &= -(x - x_*)^\top \nabla_x F(x,z) + (z - z_*)^\top \nabla_z F(x,z), \\ &\leq F(x_*, z) - F(x,z) + F(x,z) - F(x, z_*), \\ &= F(x_*, z) - F(x_*, z_*) + F(x_*, z_*) - F(x, z_*) \leq 0, \end{aligned} \tag{4.2}$$

where the first inequality follows from the first-order condition for convexity and concavity, and the last inequality follows from the definition of saddle point. As a consequence, for $\alpha > 0$ small enough such that $V^{-1}(\leq \alpha) \subset \mathcal{U}$, we conclude that $V^{-1}(\leq \alpha)$ is positively invariant under X_{sp}. The application of the LaSalle Invariance Principle [24, Theorem 4.4] yields that any trajectory starting from a point in $V^{-1}(\leq \alpha)$ converges to the largest invariant set M contained in $\{(x,z) \in V^{-1}(\leq \alpha) \mid \mathcal{L}_{X_{\mathrm{sp}}} V(x,z) = 0\}$. Let $(x,z) \in M$. From (4.2), $\mathcal{L}_{X_{\mathrm{sp}}} V(x,z) = 0$ implies that $F(x_*, z) = F(x_*, z_*) = F(x, z_*)$. In turn, the local strict convexity of $x \mapsto F(x, z_*)$ implies that $x = x_*$. Since M is positively invariant, the trajectory $t \mapsto (x(t), z(t))$ of X_{sp} starting at (x,z) is contained in M. This implies that along the trajectory, for all $t \geq 0$, (a) $x(t) = x_*$ i.e., $\dot{x}(t) = \nabla_x F(x(t), z(t)) = 0$, and (b) $F(x_*, z(t)) = F(x_*, z_*)$. The later implies

$$0 = \mathcal{L}_{X_{\mathrm{sp}}} F(x_*, z(t)) = X_{\mathrm{sp}}(x_*, z(t)) \cdot (0, \nabla_z F(x_*, z(t))) = \|\nabla_z F(x(t), z(t))\|^2,$$

for all $t \geq 0$. Thus, we get $\nabla_x F(x,z) = 0$ and $\nabla_z F(x,z) = 0$. Further, since $(x,z) \in \mathcal{U}$, local convexity-concavity holds over $\mathcal{U}$, and $\mathcal{S}$ is an isolated component, we obtain $(x,z) \in \mathcal{S}$, which shows $M \subset \mathcal{S}$. Since (x_*, z_*) is arbitrary, the asymptotic convergence property holds in a neighborhood of $\mathcal{S}$. The pointwise convergence follows from the application of Lemma A.3. $\square$

The result above shows that each saddle point is stable and that each path connected component of Saddle(F) is asymptotically stable. Note that each saddle point might not be asymptotically stable. However, if a component consists of a single point, then that point is asymptotically stable. Interestingly, a close look at the proof of Proposition 4.1 reveals that, if the assumptions hold globally, then the asymptotic stability of the set of saddle points is also global, as stated next.

Corollary 4.2. *(Global asymptotic stability of the set of saddle points via convexity-concavity): For $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ continuously differentiable and globally strictly convex-concave,* Saddle(F) *is globally asymptotically stable under the saddle-point dynamics X_{sp} and the convergence of trajectories is to a point.*

Remark 4.3. *(Relationship with results on primal-dual dynamics: I):* Corollary 4.2 is an extension to more general functions and less stringent assumptions of the results stated for Lagrangian functions of constrained convex (or concave) optimization problems in [33, 2, 17] and cost functions of differential games in [29].

In [2, 17], for a concave optimization, the matrix $\nabla_{xx} F$ is assumed to be negative definite at every saddle point and in [33] the set $\text{Saddle}(F)$ is assumed to be a singleton. The work [29] assumes a sufficient condition on the cost functions to guarantee convergence that in the current setup is equivalent to having $\nabla_{xx} F$ and $\nabla_{zz} F$ positive and negative definite, respectively. $\bullet$

4.2. Stability under convexity-linearity or linearity-concavity. Here we study the asymptotic convergence properties of the saddle-point dynamics when the convexity-concavity of the saddle function is not strict but, instead, the function depends linearly on its second argument. The analysis follows analogously for saddle functions that are linear in the first argument and concave in the other. The consideration of this class of functions is motivated by equality constrained optimization problems.

PROPOSITION 4.4. *(Local asymptotic stability of the set of saddle points via convexity-linearity): For a continuously differentiable function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, if*

(i) F is locally convex-concave on $\text{Saddle}(F)$ *and linear in z,*

(ii) for each $(x_*, z_*) \in \text{Saddle}(F)$, there exists a neighborhood $\mathcal{U}_{x_*} \subset \mathbb{R}^n$ of x_* where, if $F(x, z_*) = F(x_*, z_*)$ with $x \in \mathcal{U}_{x_*}$, then $(x, z_*) \in \text{Saddle}(F)$,

then each isolated path connected component of $\text{Saddle}(F)$ *is locally asymptotically stable under the saddle-point dynamics X_{sp} and, moreover, the convergence of trajectories is to a point.*

Proof. Given an isolated path connected component $\mathcal{S}$ of $\text{Saddle}(F)$, Lemma A.1 implies that $F_{|\mathcal{S}}$ is constant. Our proof proceeds along similar lines as those of Proposition 4.1. With the same notation, given $(x_*, z_*) \in \mathcal{S}$, the arguments follow verbatim until the identification of the largest invariant set M contained in $\{(x,z) \in V^{-1}(\leq \alpha) \mid \mathcal{L}_{X_{\text{sp}}} V(x,z) = 0\}$. Let $(x,z) \in M$. From (4.2), $\mathcal{L}_{X_{\text{sp}}} V(x,z) = 0$ implies $F(x_*, z) = F(x_*, z_*) = F(x, z_*)$. By assumption (ii), this means $(x, z_*) \in \mathcal{S}$, and by assumption (i), the linearity property gives $\nabla_z F(x,z) = \nabla_z F(x, z_*) = 0$. Therefore $\nabla_z F_{|M} = 0$. For $(x,z) \in M$, the trajectory $t \mapsto (x(t), z(t))$ of X_{sp} starting at (x,z) is contained in M. Consequently, $z(t) = z$ for all $t \in [0, \infty)$ and $\dot{x}(t) = -\nabla_x F(x(t), z)$ corresponds to the gradient dynamics of the (locally) convex function $y \mapsto F(y, z)$. Therefore, $x(t)$ converges to a minimizer x' of this function, i.e., $\nabla_x F(x', z) = 0$. Since $\nabla_z F_{|M} = 0$, the continuity of $\nabla_z F$ implies that $\nabla_z F(x', z) = 0$, and hence $(x', z) \in \mathcal{S}$. By continuity of F, it follows that $F(x(t), z) \to F(x', z) = F(x_*, z_*)$, where for the equality we use the fact that $F_{|\mathcal{S}}$ is constant. On the other hand, note that $0 = \mathcal{L}_{X_{\text{sp}}} V(x(t), z) = -(x(t) - x_*)^\top \nabla_x F(x(t), z) \leq F(x_*, z) - F(x(t), z)$ implies

$$F(x(t), z) \leq F(x_*, z) = F(x_*, z_*),$$

for all $t \in [0, \infty)$. Therefore, the monotonically nonincreasing sequence $\{F(x(t), z)\}$ converges to $F(x_*, z_*)$, which is also an upper bound on the whole sequence. This can only be possible if $F(x(t), z) = F(x_*, z_*)$ for all $t \in [0, \infty)$. This further implies $\nabla_x F(x(t), z) = 0$ for all $t \in [0, \infty)$, and hence, $(x, z) \in \mathcal{S}$. Consequently, $M \subset \mathcal{S}$. Since (x_*, z_*) has been chosen arbitrarily, the convergence property holds in a neighborhood of $\mathcal{S}$. The pointwise convergence follows now from the application of Lemma A.3. $\square$

The assumption (ii) in the above result is a generalization of the local strict convexity condition for the function $F(\cdot, z_*)$. That is, (ii) allows other points in the neighborhood of x_* to have the same value of the function $F(\cdot, z_*)$ as that at x_*, as long as they are saddle points (whereas, under local strict convexity, x_* is the local unique minimizer of $F(\cdot, z_*)$). The next result extends the conclusions of Proposition 4.4 globally when the assumptions hold globally.

COROLLARY 4.5. *(Global asymptotic stability of the set of saddle points via convexity-linearity): For a $\mathcal{C}^1$ function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, if*

(i) F is globally convex-concave and linear in z,

(ii) for each $(x_*, z_*) \in \mathrm{Saddle}(F)$, if $F(x, z_*) = F(x_*, z_*)$, then $(x, z_*) \in \mathrm{Saddle}(F)$,

then $\mathrm{Saddle}(F)$ is globally asymptotically stable under the saddle-point dynamics X_{sp} and, moreover, convergence of trajectories is to a point.

EXAMPLE 4.6. *(Saddle-point dynamics for convex optimization):* Consider the following convex optimization problem on $\mathbb{R}^3$,

$$\text{minimize} \quad (x_1 + x_2 + x_3)^2, \tag{4.3a}$$

$$\text{subject to} \quad x_1 = x_2. \tag{4.3b}$$

The set of solutions of this optimization is $\{x \in \mathbb{R}^3 \mid 2x_1 + x_3 = 0, x_2 = x_1\}$, with Lagrangian

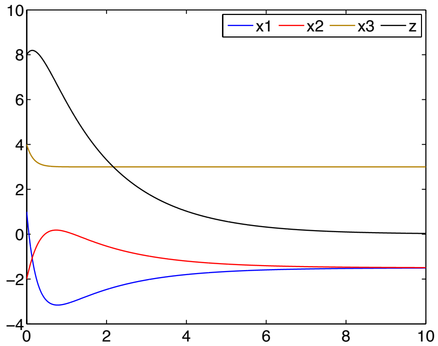
$$L(x, z) = (x_1 + x_2 + x_3)^2 + z(x_1 - x_2), \tag{4.4}$$

where $z \in \mathbb{R}$ is the Lagrange multiplier. The set of saddle points of L (which correspond to the set of primal-dual solutions to (4.3)) are $\mathrm{Saddle}(L) = \{(x, z) \in \mathbb{R}^3 \times \mathbb{R} \mid 2x_1 + x_3 = 0, x_1 = x_2, \text{ and } z = 0\}$. However, L is not strictly convex-concave and hence, it does not satisfy the hypotheses of Corollary 4.2. While L is globally convex-concave and linear in z, it does not satisfy assumption (ii) of Corollary 4.5. Therefore, to identify a dynamics that renders $\mathrm{Saddle}(L)$ asymptotically stable, we form the augmented Lagrangian

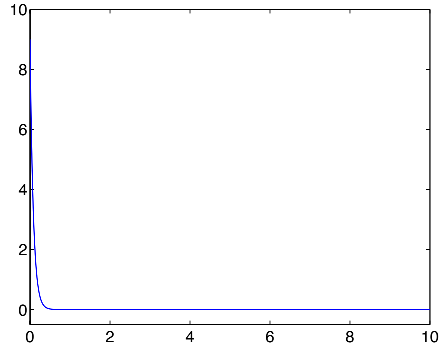
$$\tilde{L}(x, z) = L(x, z) + (x_1 - x_2)^2, \tag{4.5}$$

that has the same set of saddle points as L. Note that $\tilde{L}$ is not strictly convex-concave but it is globally convex-concave (this can be seen by computing its Hessian) and is linear in z. Moreover, given any $(x_*, z_*) \in \mathrm{Saddle}(L)$, we have $\tilde{L}(x_*, z_*) = 0$, and if $\tilde{L}(x, z_*) = \tilde{L}(x_*, z_*) = 0$, then $(x, z_*) \in \mathrm{Saddle}(L)$. By Corollary 4.5, the trajectories of the saddle-point dynamics of $\tilde{L}$ converge to a point in $\mathcal{S}$ and hence, solve the optimization problem (4.3). Figure 4.1 illustrates this fact. Note that the point of convergence depends on the initial condition. •

REMARK 4.7. *(Relationship with results on primal-dual dynamics: II):* The work [17, Section 4] considers concave optimization problems under inequality constraints where the objective function is not strictly concave but analyzes the convergence properties of a different dynamics. Specifically, the paper studies a discontinuous dynamics based on the saddle-point information of an augmented Lagrangian combined with a projection operator that restricts the dual variables to the nonnegative orthant. We have verified that, for the formulation of the concave optimization problem in [17] but with equality constraints, the augmented Lagrangian satisfies the hypotheses of Corollary 4.5, implying that the dynamics X_{sp} renders the primal-dual optima of the problem asymptotically stable. •



(a) (x, z)



(b) $(x_1 + x_2 + x_3)^2$

FIG. 4.1. *(a) Trajectory of the saddle-point dynamics of the augmented Lagrangian* $\tilde{L}$ *in* (4.5) *for the optimization problem* (4.3)*. The initial condition is* $(x, z) = (1, -2, 4, 8)$*. The trajectory converges to* $(-1.5, -1.5, 3, 0) \in \mathrm{Saddle}(L)$*. (b) Evolution of the objective function of the optimization* (4.3) *along the trajectory. The value converges to the minimum,* 0.

4.3. Stability under strong quasiconvexity-quasiconcavity. Motivated by the aim of further relaxing the conditions for asymptotic convergence, we conclude this section by weakening the convexity-concavity requirement on the saddle function. The next result shows that strong quasiconvexity-quasiconcavity is sufficient to ensure convergence of the saddle-point dynamics.

PROPOSITION 4.8. *(Local asymptotic stability of the set of saddle points via strong quasiconvexity-quasiconcavity): Let* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *be* $\mathcal{C}^2$ *and the map* $(x, z) \mapsto \nabla_{xz} F(x, z)$ *be locally Lipschitz. Assume that* F *is locally jointly strongly quasiconvex-quasiconcave on* $\mathrm{Saddle}(F)$*. Then, each isolated path connected component of* $\mathrm{Saddle}(F)$ *is locally asymptotically stable under the saddle-point dynamics* X_{sp} *and, moreover, the convergence of trajectories is to a point. Further, if* F *is globally jointly strongly quasiconvex-quasiconcave and* $\nabla_{xz} F$ *is constant over* $\mathbb{R}^n \times \mathbb{R}^m$*, then* $\mathrm{Saddle}(F)$ *is globally asymptotically stable under* X_{sp} *and the convergence of trajectories is to a point.*

Proof. Let $(x_*, z_*) \in \mathcal{S}$, where $\mathcal{S}$ is an isolated path connected component of $\mathrm{Saddle}(F)$, and consider the function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ defined in (4.1). Let $\mathcal{U}$ be the neighborhood of (x_*, z_*) where the local joint strong quasiconvexity-quasiconcavity holds. The Lie derivative of V along the saddle-point dynamics at $(x, z) \in \mathcal{U}$ can be written as,

$$\begin{aligned}
\mathcal{L}_{X_{\mathrm{sp}}} V(x, z) &= -(x - x_*)^\top \nabla_x F(x, z) + (z - z_*)^\top \nabla_z F(x, z), \\
&= -(x - x_*)^\top \nabla_x F(x, z_*) + (z - z_*)^\top \nabla_z F(x_*, z) + M_1 + M_2,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
M_1 &= -(x - x_*)^\top (\nabla_x F(x, z) - \nabla_x F(x, z_*)), \\
M_2 &= (z - z_*)^\top (\nabla_z F(x, z) - \nabla_z F(x_*, z)).
\end{aligned}$$

Writing

$$\nabla_x F(x,z) - \nabla_x F(x,z_*) = \int_0^1 \nabla_{zx} F(x, z_* + t(z-z_*))(z-z_*)dt,$$
$$\nabla_z F(x,z) - \nabla_z F(x_*,z) = \int_0^1 \nabla_{xz} F(x_* + t(x-x_*), z)(x-x_*)dt,$$

we get

$$\begin{aligned} M_1 + M_2 = (z-z_*)^\top \Big(\int_0^1 \big(&\nabla_{xz} F(x_* + t(x-x_*), z) \\ &- \nabla_{xz} F(x, z_* + t(z-z_*))\big) dt\Big)(x-x_*) \\ \le \|z-z_*\| (L\|x-x_*\| &+ L\|z-z_*\|)\|x-x_*\|, \end{aligned} \tag{4.7}$$

where in the inequality, we have used the fact that $\nabla_{xz}F$ is locally Lipschitz with some constant $L > 0$. From the first-order property of a strong quasiconvex function, cf. Lemma A.2, there exist constants $s_1, s_2 > 0$ such that

$$-(x-x_*)^\top \nabla_x F(x,z_*) \le -s_1 \|x-x_*\|^2, \tag{4.8a}$$
$$(z-z_*)^\top \nabla_z F(x_*,z) \le -s_2 \|z-z_*\|^2, \tag{4.8b}$$

for all $(x,z) \in \mathcal{U}$. Substituting (4.7) and (4.8) into the expression for the Lie derivative (4.6), we obtain

$$\mathcal{L}_{X_{\rm sp}} V(x,z) \le -s_1\|x-x_*\|^2 - s_2\|z-z_*\|^2 + L\|x-x_*\|^2\|z-z_*\| + L\|x-x_*\|\|z-z_*\|^2.$$

To conclude the proof, note that if $\|z - z_*\| < \frac{s_1}{L}$ and $\|x - x_*\| < \frac{s_2}{L}$, then $\mathcal{L}_{X_{\rm sp}} V(x,z) < 0$, which implies local asymptotic stability. The pointwise convergence follows from Lemma A.3. The global asymptotic stability can be reasoned using similar arguments as above using the fact that here $M_1 + M_2 = 0$ because $\nabla_{xz}F$ is constant. □

In the following, we present an example where the above result is employed to explain local asymptotic convergence. In this case, none of the results from Section 4.1 and 4.2 apply, thereby justifying the importance of the above result.

Example 4.9. *(Convergence for locally jointly strongly quasiconvex-quasiconcave function):* Consider $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by,

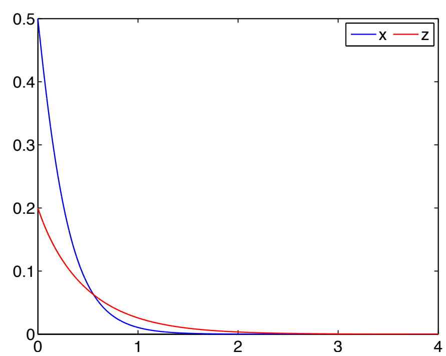
$$F(x,z) = (2 - e^{-x^2})(1 + e^{-z^2}). \tag{4.9}$$

Note that F is $\mathcal{C}^2$ and $\nabla_{xz}F(x,z) = -4xze^{-x^2}e^{-z^2}$ is locally Lipschitz. To see this, note that the function $x \mapsto xe^{-x^2}$ is bounded and is locally Lipschitz (as its derivative is bounded). Further, the product of two bounded and locally Lipschitz functions is locally Lipschitz [32, Theorem 4.6.3] and so, $(x,z) \mapsto \nabla_{xz}F(x,z)$ is locally Lipschitz. The set of saddle points of F is $\text{Saddle}(F) = \{0\}$. Next, we show that $x \mapsto f(x) = c_1 - c_2 e^{-x^2}$, $c_2 > 0$, is locally strongly quasiconvex at 0. Fix $\delta > 0$ and let $x, y \in B_\delta(0)$ such that $f(y) \le f(x)$. Then, $|y| \le |x|$ and

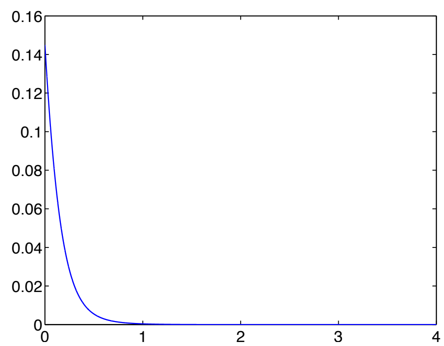
$$\begin{aligned} &\max\{f(x), f(y)\} - f(\lambda x + (1-\lambda)y) - s\lambda(1-\lambda)(x-y)^2 \\ &\qquad = c_2(-e^{-x^2} + e^{-(\lambda x + (1-\lambda)y)^2}) - s\lambda(1-\lambda)(x-y)^2 \end{aligned}$$

$$
\begin{aligned}
&= c_2 e^{-x^2}(-1 + e^{x^2-(\lambda x+(1-\lambda)y)^2}) - s\lambda(1-\lambda)(x-y)^2 \\
&\geq c_2 e^{-x^2}(x^2 - (\lambda x + (1-\lambda)y)^2) - s\lambda(1-\lambda)(x-y)^2 \\
&= (1-\lambda)(x-y)\Big(c_2 e^{-x^2}(x+y) + \lambda(x-y)(c_2 e^{-x^2} - s)\Big) \geq 0,
\end{aligned}
$$

for $s \leq c_2 e^{-\delta^2}$, given the fact that $|y| \leq |x|$. Therefore, f is locally strongly quasiconvex and so $-f$ is locally strongly quasiconcave. Using these facts, we deduce that F is locally jointly strongly quasiconvex-quasiconcave. Thus, the hypotheses of Proposition 4.8 are met, implying local asymptotic stability of Saddle(F) under the saddle-point dynamics. Figure 4.2 illustrates this fact in simulation. Note that F does not satisfy the conditions outlined in results of Section 4.1 and 4.2. •



(a) (x, z)



(b) $(x^2 + z^2)/2$

FIG. 4.2. *(a) Trajectory of the saddle-point dynamics for F given in* (4.9)*. The initial condition is* $(x, z) = (0.5, 0.2)$*. The trajectory converges to the saddle point* $(0, 0)$*. (b) Evolution of the function V along the trajectory.*

5. Convergence analysis for general saddle functions. We study here the convergence properties of the saddle-point dynamics associated to functions that are not convex-concave. Our first result explores conditions for local asymptotic stability based on the linearization of the dynamics and properties of the eigenstructure of the Jacobian matrices. In particular, we assume that X_{sp} is piecewise $\mathcal{C}^2$ and that the set of limit points of the Jacobian of X_{sp} at any saddle point have a common kernel and negative real parts for the nonzero eigenvalues. The proof is a direct consequence of Proposition A.5.

PROPOSITION 5.1. *(Local asymptotic stability of manifold of saddle points via linearization – piecewise $\mathcal{C}^3$ saddle function): Given $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, let $\mathcal{S} \subset$ Saddle(F) be a p-dimensional submanifold of saddle points. Assume that F is $\mathcal{C}^1$ with locally Lipschitz gradient on a neighborhood of $\mathcal{S}$ and that the vector field X_{sp} is piecewise $\mathcal{C}^2$. Assume that at each $(x_*, z_*) \in \mathcal{S}$, the set of matrices $\mathcal{A}_* \subset \mathbb{R}^{n+m\times n+m}$ defined as*

$$
\mathcal{A}_* = \{\lim_{k\to\infty} DX_{sp}(x_k, z_k) \mid (x_k, z_k) \to (x, z), (x_k, z_k) \in \mathbb{R}^{n+m} \setminus \Omega_{X_{sp}}\},
$$

where $\Omega_{X_{sp}}$ is the set of points where X_{sp} is not differentiable, satisfies the following:

(i) there exists an orthogonal matrix $Q \in \mathbb{R}^{n+m \times n+m}$ such that

$$Q^\top A Q = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A} \end{bmatrix}, \tag{5.1}$$

for all $A \in \mathcal{A}_*$, where $\tilde{A} \in \mathbb{R}^{n+m-p \times n+m-p}$,

(ii) the nonzero eigenvalues of the matrices in $\mathcal{A}_*$ have negative real parts,

(iii) there exists a positive definite matrix $P \in \mathbb{R}^{n+m-p \times n+m-p}$ such that

$$\tilde{A}^\top P + P\tilde{A} \prec 0,$$

for all $\tilde{A}$ obtained by applying transformation (5.1) on each $A \in \mathcal{A}_*$.

Then, $\mathcal{S}$ is locally asymptotically stable under (A.7) and the trajectories converge to a point in $\mathcal{S}$.

When F is sufficiently smooth, we can refine the above result as follows.

COROLLARY 5.2. *(Local asymptotic stability of manifold of saddle points via linearization – $\mathcal{C}^3$ saddle function): Given $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, let $\mathcal{S} \subset \mathrm{Saddle}(F)$ be a p-dimensional manifold of saddle points. Assume F is $\mathcal{C}^3$ on a neighborhood of $\mathcal{S}$ and that the Jacobian of X_{sp} at each point in $\mathcal{S}$ has no eigenvalues in the imaginary axis other than 0, which is semisimple with multiplicity p. Then, $\mathcal{S}$ is locally asymptotically stable under the saddle-point dynamics X_{sp} and the trajectories converge to a point.*

Proof. Since F is $\mathcal{C}^3$, the map X_{sp} is $\mathcal{C}^2$ and so, the limit point of Jacobian matrices at a saddle point $(x_*, z_*) \in \mathcal{S}$ is the Jacobian at that point itself, that is,

$$DX_{\mathrm{sp}} = \begin{bmatrix} -\nabla_{xx}F & -\nabla_{xz}F \\ \nabla_{zx}F & \nabla_{zz}F \end{bmatrix}_{(x_*,z_*)}.$$

From the definition of saddle point, we have $\nabla_{xx}F(x_*, z_*) \succeq 0$ and $\nabla_{zz}F(x_*, z_*) \preceq 0$. In turn, we obtain $DX_{\mathrm{sp}} + DX_{\mathrm{sp}}^\top \preceq 0$, and since $\mathrm{Re}(\lambda_i(DX_{\mathrm{sp}})) \leq \lambda_{\max}(\frac{1}{2}(DX_{\mathrm{sp}} + DX_{\mathrm{sp}}^\top))$ [4, Fact 5.10.28], we deduce that $\mathrm{Re}(\lambda_i(DX_{\mathrm{sp}})) \leq 0$. The statement now follows from Proposition 5.1 using the fact that the properties of the eigenvalues of DX_{sp} shown here imply existence of an orthonormal transformation leading to a form of DX_{sp} that satisfies assumptions *(i)-(iii)* of Proposition 5.1. ∎

Next, we provide a sufficient condition under which the Jacobian of X_{sp} for a saddle function F that is linear in its second argument satisfies the hypothesis of Corollary 5.2 regarding the lack of eigenvalues on the imaginary axis other than 0.

LEMMA 5.3. *(Sufficient condition for absence of imaginary eigenvalues of the Jacobian of X_{sp}): Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be $\mathcal{C}^2$ and linear in the second argument. Then, the Jacobian of X_{sp} at any saddle point (x_*, z_*) of F has no eigenvalues on the imaginary axis except for 0 if* $\mathrm{range}(\nabla_{zx}F(x_*, z_*)) \cap \mathrm{null}(\nabla_{xx}F(x_*, z_*)) = \{0\}$.

Proof. The Jacobian of X_{sp} at a saddle point (x_*, z_*) for a saddle function F that is linear in z is given as

$$DX_{\mathrm{sp}} = \begin{bmatrix} A & B \\ -B^\top & 0 \end{bmatrix},$$

where $A = -\nabla_{xx}F(x_*, z_*)$ and $B = -\nabla_{zx}F(x_*, z_*)$. We reason by contradiction. Let $i\lambda$, $\lambda \neq 0$ be an imaginary eigenvalue of DX_{sp} with the corresponding eigenvector $a + ib$. Let $a = (a_1; a_2)$ and $b = (b_1; b_2)$ where $a_1, b_1 \in \mathbb{R}^n$ and $a_2, b_2 \in \mathbb{R}^m$. Then the real and imaginary parts of the condition $DX_{\mathrm{sp}}(a+ib) = (i\lambda)(a+ib)$ yield

$$Aa_1 + Ba_2 = -\lambda b_1, \qquad -B^\top a_1 = -\lambda b_2, \tag{5.2}$$

$$Ab_1 + Bb_2 = \lambda a_1, \qquad -B^\top b_1 = \lambda a_2. \tag{5.3}$$

Pre-multiplying the first equation of (5.2) with $a_1^\top$ gives $a_1^\top A a_1 + a_1^\top B a_2 = -\lambda a_1^\top b_1$. Using the second equation of (5.2), we get $a_1^\top A a_1 = -\lambda(a_1^\top b_1 + a_2^\top b_2)$. A similar procedure for the set of equations in (5.3) gives $b_1^\top A b_1 = \lambda(a_1^\top b_1 + a_2^\top b_2)$. These conditions imply that $a_1^\top A a_1 = -b_1^\top A b_1$. Since A is negative semi-definite, we obtain $a_1, b_1 \in \mathrm{null}(A)$. Note that $a_1, b_1 \neq 0$, because otherwise it would mean that $a = b = 0$. Further, using this fact in the first equations of (5.2) and (5.3), respectively, we get

$$Ba_2 = -\lambda b_1, \qquad Bb_2 = \lambda a_1.$$

That is, $a_1, b_1 \in \mathrm{range}(B)$, a contradiction. □

The following example illustrates an application of the above results to a nonconvex constrained optimization problem.

Example 5.4. *(Saddle-point dynamics for nonconvex optimization):* Consider the following constrained optimization on $\mathbb{R}^3$,

$$\text{minimize} \quad (\|x\| - 1)^2, \tag{5.4a}$$

$$\text{subject to} \quad x_3 = 0.5, \tag{5.4b}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The optimizers are $\{x \in \mathbb{R}^3 \mid x_3 = 0.5, x_1^2 + x_2^2 = 0.75\}$. The Lagrangian $L : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is given by

$$L(x, z) = (\|x\| - 1)^2 + z(x_3 - 0.5),$$

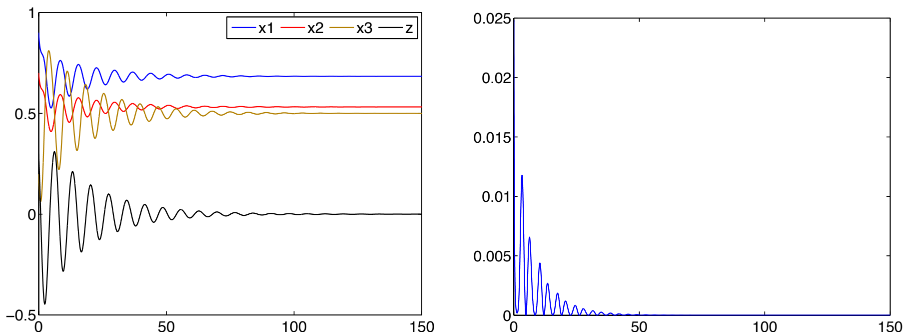
and its set of saddle points is the one-dimensional manifold $\mathrm{Saddle}(L) = \{(x, z) \in \mathbb{R}^3 \times \mathbb{R} \mid x_3 = 0.5,\ x_1^2 + x_2^2 = 0.75,\ z = 0\}$. The saddle-point dynamics of L takes the form

$$\dot{x} = -2\Big(1 - \frac{1}{\|x\|}\Big)x - [0, 0, z]^\top, \tag{5.5a}$$

$$\dot{z} = x_3 - 0.5. \tag{5.5b}$$

Note that $\mathrm{Saddle}(L)$ is nonconvex and that L is nonconvex in its first argument on any neighborhood of any saddle point. Therefore, results that rely on the convexity-concavity properties of L are not applicable to establish the asymptotic convergence of (5.5) to the set of saddle points. This can, however, be established through Corollary 5.2 by observing that the Jacobian of X_{sp} at any point of $\mathrm{Saddle}(L)$ has 0 as an eigenvalue with multiplicity one and the rest of the eigenvalues are not on the imaginary axis. To show this, consider $(x_*, z_*) \in \mathrm{Saddle}(L)$. Note that $DX_{\mathrm{sp}}(x_*, z_*) = \begin{bmatrix} -2x_*^\top x_* & -e_3 \\ e_3^\top & 0 \end{bmatrix}$, where $e_3 = [0, 0, 1]^\top$. One can deduce from this that $v \in \mathrm{null}(DX_{\mathrm{sp}}(x_*, z_*))$ if and only if $x_*^\top [v_1, v_2, v_3]^\top = 0$, $v_3 = 0$, and $v_4 = 0$. These three conditions define a one-dimensional space and so 0 is an eigenvalue of $DX_{\mathrm{sp}}(x_*, z_*)$ with multiplicity 1. To show that the rest of eigenvalues do not lie on the imaginary axis, we show that the hypotheses of Lemma 5.3 are met. At any saddle point (x_*, z_*), we have $\nabla_{zx} L(x_*, z_*) = e_3$ and $\nabla_{xx} L(x_*, z_*) = 2x_*^\top x_*$. If

$v \in \operatorname{range}(\nabla_{zx} L(x_*, z_*)) \cap \operatorname{null}(\nabla_{xx} L(x_*, z_*))$ then $v = [0, 0, \lambda]^\top$, $\lambda \in \mathbb{R}$, and $x_*^\top v = 0$. Since $(x_*)_3 = 0.5$, we get $\lambda = 0$ and hence, the hypotheses of Lemma 5.3 are satisfied. Figure 5.1 illustrates in simulation the convergence of the trajectories to a saddle point. The point of convergence depends on the initial condition. •



(a) (x, z) (b) $(x_1 + x_2 + x_3)^2$

Fig. 5.1. *(a) Trajectory of the saddle-point dynamics* (5.5) *for the Lagrangian of the constrained optimization problem* (5.4). *The initial condition is* $(x, z) = (0.9, 0.7, 0.2, 0.3)$. *The trajectory converges to* $(0.68, 0.53, 0.50, 0) \in \operatorname{Saddle}(L)$. *(b) Evolution of the objective function of the optimization* (5.4) *along the trajectory. The value converges to the minimum,* 0.

There are functions that do not satisfy the hypotheses of Proposition 5.1 whose saddle-point dynamics still seems to enjoy local asymptotic convergence properties. As an example, consider the function $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$,

$$F(x, z) = (\|x\| - 1)^4 - z^2 \|x\|^2, \tag{5.6}$$

whose set of saddle points is the one-dimensional manifold $\operatorname{Saddle}(F) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} \mid \|x\| = 1, z = 0\}$. The Jacobian of the saddle-point dynamics at any $(x, z) \in \operatorname{Saddle}(F)$ has -2 as an eigenvalue and 0 as the other eigenvalue, with multiplicity 2, which is greater than the dimension of $\mathcal{S}$ (and therefore Proposition 5.1 cannot be applied). Simulations show that the trajectories of the saddle-point dynamics asymptotically approach $\operatorname{Saddle}(S)$ if the initial condition is close enough to this set. Our next result allows us to formally establish this fact by studying the behavior of the function along the proximal normals to $\operatorname{Saddle}(F)$.

Proposition 5.5. *(Asymptotic stability of manifold of saddle points via proximal normals): Let* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *be* $\mathcal{C}^2$ *and* $\mathcal{S} \subset \operatorname{Saddle}(F)$ *be a closed set. Assume there exist constants* $\lambda_M, k_1, k_2, \alpha_1, \beta_1 > 0$ *and* $L_x, L_z, \alpha_2, \beta_2 \geq 0$ *such that the following hold*

(i) either $L_x = 0$ *or* $\alpha_1 \leq \alpha_2 + 1$,
(ii) either $L_z = 0$ *or* $\beta_1 \leq \beta_2 + 1$,
(iii) for every $(x_*, z_*) \in \mathcal{S}$ *and every proximal normal* $\eta = (\eta_x, \eta_z) \in \mathbb{R}^n \times \mathbb{R}^m$ *to* $\mathcal{S}$ *at* (x_*, z_*) *with* $\|\eta\| = 1$, *the functions*

$$[0, \lambda_M) \ni \lambda \mapsto F(x_* + \lambda \eta_x, z_*),$$
$$[0, \lambda_M) \ni \lambda \mapsto F(x_*, z_* + \lambda \eta_z),$$

are convex and concave, respectively, with

$$F(x_* + \lambda\eta_x, z_*) - F(x_*, z_*) \geq k_1 \|\lambda\eta_x\|^{\alpha_1}, \tag{5.7a}$$

$$F(x_*, z_* + \lambda\eta_z) - F(x_*, z_*) \leq -k_2 \|\lambda\eta_z\|^{\beta_1}, \tag{5.7b}$$

and, for all $\lambda \in [0, \lambda_M)$ *and all* $t \in [0, 1]$,

$$\begin{aligned} \|\nabla_{xz} F(x_* + t\lambda\eta_x, z_* + \lambda\eta_z) - \nabla_{xz} F(x_* + \lambda\eta_x, z_* + t\lambda\eta_z)\| \\ \leq L_x \|\lambda\eta_x\|^{\alpha_2} + L_z \|\lambda\eta_z\|^{\beta_2}. \end{aligned} \tag{5.8}$$

Then, $\mathcal{S}$ *is locally asymptotically stable under the saddle-point dynamics* X_{sp}*. Moreover, the convergence of the trajectories is to a point if every point of* $\mathcal{S}$ *is stable. The convergence is global if, for every* $\lambda_M \in \mathbb{R}_{\geq 0}$*, there exist* $k_1, k_2, \alpha_1, \beta_1 > 0$ *such that the above hypotheses (i)-(iii) are satisfied by these constants along with* $L_x = L_z = 0$.

Proof. Our proof is based on showing that there exists $\bar{\lambda} \in (0, \lambda_M]$ such that the distance function $d_{\mathcal{S}}$ decreases monotonically and converges to zero along the trajectories of X_{sp} that start in $\mathcal{S} + B_{\bar{\lambda}}(0)$. From (2.2),

$$\partial d_{\mathcal{S}}^2(x, z) = \text{co}\{2(x - x_*; z - z_*) \mid (x_*, z_*) \in \text{proj}_{\mathcal{S}}(x, z)\}.$$

Following [12], we compute the set-valued Lie derivative of $d_{\mathcal{S}}^2$ along X_{sp}, denoted $\mathcal{L}_{X_{\text{sp}}} d_{\mathcal{S}}^2 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}$, as

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}} d_{\mathcal{S}}^2(x, z) = \text{co}\{&-2(x - x_*)^\top \nabla_x F(x, z) + \\ &2(z - z_*)^\top \nabla_z F(x, z) \mid (x_*, z_*) \in \text{proj}_{\mathcal{S}}(x, z)\}. \end{aligned}$$

Since $d_{\mathcal{S}}^2$ is globally Lipschitz and regular, cf. Section 2.2, the evolution of the function $d_{\mathcal{S}}^2$ along any trajectory $t \mapsto (x(t), z(t))$ of (3.1) is differentiable at almost all $t \in \mathbb{R}_{\geq 0}$, and furthermore, cf. [12, Proposition 10],

$$\frac{d}{dt}(d_{\mathcal{S}}^2(x(t), z(t)) \in \mathcal{L}_{X_{\text{sp}}} d_{\mathcal{S}}^2(x(t), z(t))$$

for almost all $t \in \mathbb{R}_{\geq 0}$. Therefore, our goal is to show that $\max \mathcal{L}_{X_{\text{sp}}} d_{\mathcal{S}}^2(x, z) < 0$ for all $(x, z) \in (\mathcal{S} + B_{\bar{\lambda}}(0)) \setminus \mathcal{S}$ for some $\bar{\lambda} \in (0, \lambda_M]$. Let $(x, z) \in \mathcal{S} + B_{\lambda_M}(0)$ and take $(x_*, z_*) \in \text{proj}_{\mathcal{S}}(x, z)$. By definition, there exists a proximal normal $\eta = (\eta_x, \eta_z)$ to $\mathcal{S}$ at (x_*, z_*) with $\|\eta\| = 1$ and $x = x_* + \lambda\eta_x$, $z = z_* + \lambda\eta_z$, and $\lambda \in [0, \lambda_M)$. Let $2\xi \in \mathcal{L}_{X_{\text{sp}}} d_{\mathcal{S}}^2(x, z)$ denote

$$\xi = -(x - x_*)^\top \nabla_x F(x, z) + (z - z_*)^\top \nabla_z F(x, z). \tag{5.9}$$

Writing

$$\begin{aligned} \nabla_x F(x, z) &= \nabla_x F(x, z_*) + \int_0^1 \nabla_{zx} F(x, z_* + t(z - z_*))(z - z_*) dt, \\ \nabla_z F(x, z) &= \nabla_z F(x_*, z) + \int_0^1 \nabla_{xz} F(x_* + t(x - x_*), z)(x - x_*) dt, \end{aligned}$$

and substituting in (5.9) we get

$$\xi = -(x - x_*)^\top \nabla_x F(x, z_*) + (z - z_*)^\top \nabla_z F(x_*, z) + (z - z_*)^\top M(x - x_*), \tag{5.10}$$

where $M = \int_0^1 (\nabla_{xz} F(x_* + t(x - x_*), z) - \nabla_{xz} F(x, z_* + t(z - z_*)))dt$. Using the convexity and concavity along the proximal normal and applying the bounds (5.7), we obtain

$$-(x - x_*)^\top \nabla_x F(x, z_*) \leq F(x_*, z_*) - F(x, z_*) \leq -k_1 \|\lambda \eta_x\|^{\alpha_1}, \tag{5.11a}$$

$$(z - z_*)^\top \nabla_z F(x_*, z) \leq F(x_*, z) - F(x_*, z_*) \leq -k_2 \|\lambda \eta_z\|^{\beta_1}. \tag{5.11b}$$

On the other hand, using (5.8), we bound M by

$$\|M\| \leq L_x \|\lambda \eta_x\|^{\alpha_2} + L_z \|\lambda \eta_z\|^{\beta_2}. \tag{5.12}$$

Using (5.11) and (5.12) in (5.10), and rearranging the terms yields

$$\xi \leq \big(-k_1 \|\lambda \eta_x\|^{\alpha_1} + L_x \|\lambda \eta_x\|^{\alpha_2 + 1} \|\lambda \eta_z\|\big) + \big(-k_2 \|\lambda \eta_z\|^{\beta_1} + L_z \|\lambda \eta_z\|^{\beta_2 + 1} \|\lambda \eta_x\|\big).$$

If $L_x = 0$, then the first parenthesis is negative whenever $\lambda \eta_x \neq 0$ (i.e., $x \neq x_*$). If $L_x \neq 0$ and $\alpha_1 \leq \alpha_2 + 1$, then for $\|\lambda \eta_x\| < 1$ and $\|\lambda \eta_z\| < \min(1, k_1/L_x)$, the first parenthesis is negative whenever $\lambda \eta_x \neq 0$. Analogously, the second parenthesis is negative for $z \neq z_*$ if either $L_z = 0$ or $\beta_1 \leq \beta_2 + 1$ with $\|\lambda \eta_z\| < 1$ and $\|\lambda \eta_x\| < \min(1, k_2/L_z)$. Thus, if $\lambda < \min\{1, k_1/L_x, k_2/L_z\}$ (excluding from the min operation the elements that are not well defined due to the denominator being zero), then hypotheses (i)-(ii) imply that $\xi < 0$ whenever $(x, z) \neq (x_*, z_*)$. Moreover, since $(x_*, z_*) \in \mathrm{proj}_{\mathcal{S}}(x, z)$ was chosen arbitrarily, we conclude that $\max \mathcal{L}_{X_{\mathrm{sp}}} d_{\mathcal{S}}^2(x, z) < 0$ for all $(x, z) \in \mathcal{S} + B_{\bar{\lambda}}(0)$ where $\bar{\lambda} \in (0, \lambda_M]$ satisfies $\bar{\lambda} < \min\{1, k_1/L_x, k_2/L_z\}$. This proves the local asymptotic stability. Finally, convergence to a point follows from Lemma A.3 and global convergence follows from the analysis done above. □

Intuitively, the hypotheses of Proposition 5.5 imply that along the proximal normal to the saddle set, the convexity (resp. concavity) in the x-coordinate (resp. z-coordinate) is 'stronger' than the influence of the x- and z-dynamics on each other, represented by the off-diagonal Hessian terms. When this coupling is absent (i.e., $\nabla_{xz} F \equiv 0$), the x- and z-dynamics are independent of each other and they function as individually aiming to minimize (resp. maximize) a function of one variable, thereby, reaching a saddle point. Note that the assumptions of Proposition 5.5 do not imply that F is locally convex-concave. As an example, the function in (5.6) is not convex-concave in any neighborhood of any saddle point but we show next that it satisfies the assumptions of Proposition 5.5, establishing local asymptotic convergence of the respective saddle-point dynamics.

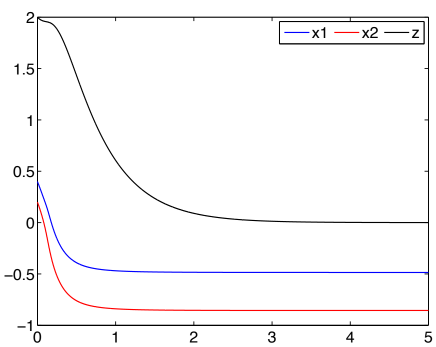
Example 5.6. *(Convergence analysis via proximal normals):* Consider the function F defined in (5.6). Consider a saddle point $(x_*, z_*) = (\cos\theta, \sin\theta, 0) \in \mathrm{Saddle}(F)$, where $\theta \in [0, 2\pi)$. Let

$$\eta = (\eta_x, \eta_z) = ((a_1 \cos\theta, a_1 \sin\theta), a_2),$$

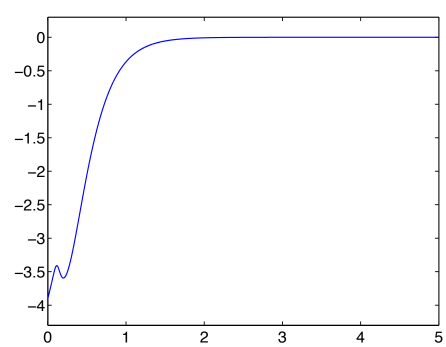
with $a_1, a_2 \in \mathbb{R}$ and $a_1^2 + a_2^2 = 1$, be a proximal normal to $\mathrm{Saddle}(F)$ at (x_*, z_*). Note that the function $\lambda \mapsto F(x_* + \lambda \eta_x, z_*) = (\lambda a_1)^4$ is convex, satisfying (5.7a) with $k_1 = 1$ and $\alpha_1 = 4$. The function $\lambda \mapsto F(x_*, z_* + \lambda \eta_z) = -(\lambda a_2)^2$ is concave, satisfying (5.7b) with $k_2 = 1$, $\beta_1 = 2$. Also, given any $\lambda_M > 0$ and for all $t \in [0,1]$, we can write

$$\begin{aligned}
&\|\nabla_{xz} F(x_* + t\lambda\eta_x, z_* + \lambda\eta_z) - \nabla_{xz} F(x_* + \lambda\eta_x, z_* + t\lambda\eta_z)\| \\
&\quad = \| -4(\lambda a_2)(1 + t\lambda a_1) \left(\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right) + 4(t\lambda a_2)(1 + \lambda a_1) \left(\begin{smallmatrix}\cos\theta \\ \sin\theta\end{smallmatrix}\right) \|, \\
&\quad \leq \|4(\lambda a_2)(1 + t\lambda a_1) - 4(t\lambda a_2)(1 + \lambda a_1)\|, \\
&\quad \leq 8(1 + \lambda a_1)(\lambda a_2) \leq L_z(\lambda a_2),
\end{aligned}$$

for $\lambda \leq \lambda_M$, where $L_z = 8(1 + \lambda_M a_1)$. This implies that $L_x = 0$, $L_z \neq 0$ and $\beta_2 = 1$. Therefore, hypotheses (i)-(iii) of Proposition 5.5 are satisfied and this establishes asymptotic convergence of the saddle-point dynamics. Figure 5.2 illustrates this fact. Note that since $L_z \neq 0$, we cannot guarantee global convergence. •



(a) (x, z)



(b) F

Fig. 5.2. *(a) Trajectory of the saddle-point dynamics for the function defined by* (5.6)*. The initial condition is* $(x, z) = (0.1, 0.2, 4)$*. The trajectory converges to* $(0.49, 0.86, 0) \in \mathrm{Saddle}(F)$*. (b) Evolution of the function* F *along the trajectory. The value converges to* 0*, the value that the function takes on its saddle set.*

Interestingly, Propositions 5.1 and 5.5 complement each other. The function (5.6) satisfies the hypotheses of Proposition 5.5 but not those of Proposition 5.1. Conversely, the Lagrangian of the constrained optimization (5.4) satisfies the hypotheses of Proposition 5.1 but not those of Proposition 5.5.

In the next result, we consider yet another scenario where the saddle function might not be convex-concave in its arguments but the saddle-point dynamics converges to the set of equilibrium points. As a motivation, consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F(x, z) = xz^2$. The set of saddle points of F are $\mathrm{Saddle}(F) = \mathbb{R}_{\leq 0} \times \{0\}$. One can show that, at the saddle point $(0, 0)$, neither the hypotheses of Proposition 5.1 nor those of Proposition 5.5 are satisfied. Yet, simulations show that the trajectories of the dynamics converge to the saddle points from almost all initial conditions in $\mathbb{R}^2$, see Figure 5.3 below. This asymptotic behavior can be characterized through the following result which generalizes [23, Theorem 3].

Proposition 5.7. *(Global asymptotic stability of equilibria of saddle-point dynamics for saddle functions linear in one argument): For* $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$*, assume the following form* $F(x, z) = g(z)^\top x$*, where* $g : \mathbb{R}^m \to \mathbb{R}^n$ *is* $\mathcal{C}^1$*. Assume that there exists* $(x_*, z_*) \in \mathrm{Saddle}(F)$ *such that*

(i) $F(x_*, z_*) \geq F(x_*, z)$ *for all* $z \in \mathbb{R}^m$*,*
(ii) for any $z \in \mathbb{R}^m$*, the condition* $g(z)^\top x_* = 0$ *implies* $g(z) = 0$*,*

(iii) any trajectory of X_{sp} is bounded.

Then, all trajectories of the saddle-point dynamics X_{sp} converge asymptotically to the set of equilibrium points of X_{sp}.

Proof. Consider the function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$V(x, z) = -x_*^\top x.$$

The Lie derivative of V along the saddle-point dynamics X_{sp} is

$$\mathcal{L}_{X_{\text{sp}}} V(x, z) = x_*^\top \nabla_x F(x, z) = x_*^\top g(z) = F(x_*, z) \leq F(x_*, z_*) = 0, \tag{5.13}$$

where in the inequality we have used assumption (i), and $F(x_*, z_*) = 0$ is implied by the definition of the saddle point, that is, $\nabla_x F(x_*, z_*) = g(z_*) = 0$. Now consider any trajectory $t \mapsto (x(t), z(t))$, $(x(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ of X_{sp}. Since the trajectory is bounded by assumption (iii), the application of the LaSalle Invariance Principle [24, Theorem 4.4] yields that the trajectory converges to the largest invariant set $\mathcal{M}$ contained in $\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathcal{L}_{X_{\text{sp}}} V(x, z) = 0\}$, which from (5.13) is equal to the set $\{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid F(x_*, z) = 0\}$. Let $(x, z) \in \mathcal{M}$. Then, we have $F(x_*, z) = g(z)^\top x_* = 0$ and by hypotheses (ii) we get $g(z) = 0$. Therefore, if $(x, z) \in \mathcal{M}$ then $g(z) = 0$. Consider the trajectory $t \mapsto (x(t), z(t))$ of X_{sp} with $(x(0), z(0)) = (x, z)$ which is contained in $\mathcal{M}$. Then, along the trajectory we have

$$\dot{x}(t) = -\nabla_x F(x(t), z(t)) = -g(z(t)) = 0$$

Further, note that along this trajectory we have $g(z(t)) = 0$ for all $t \geq 0$. Thus, $\frac{d}{dt} g(z(t)) = 0$ for all $t \geq 0$, which implies that

$$\frac{d}{dt} g(z(t)) = Dg(z(t))\dot{z}(t) = Dg(z(t))Dg(z(t))^\top x = 0.$$

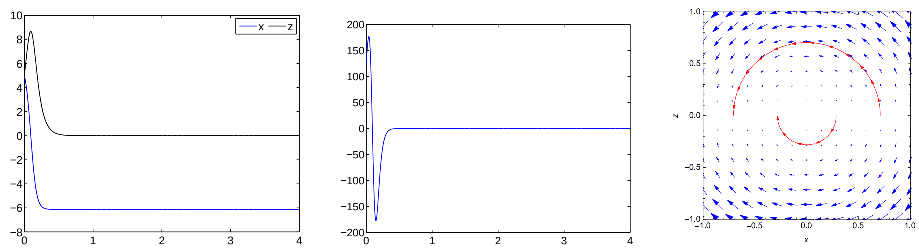
From the above expression we deduce that $\dot{z}(t) = Dg(z(t))^\top x = 0$. This can be seen from the fact that $Dg(z(t))Dg(z(t))^\top x = 0$ implies $x^\top Dg(z(t))Dg(z(t))^\top x = (Dg(z(t))^\top x)^2 = 0$. From the above reasoning, we conclude that (x, z) is an equilibrium point of X_{sp}. □

The proof of Proposition 5.7 hints at the fact that hypothesis (ii) can be omitted if information about other saddle points of F is known. Specifically, consider the case where n saddle points $(x_*^{(1)}, z_*^{(1)}), \dots, (x_*^{(n)}, z_*^{(n)})$ of F exist, each satisfying hypothesis (i) of Proposition 5.7 and such that the vectors $x_*^{(1)}, \dots, x_*^{(n)}$ are linearly independent. In this scenario, for those points $z \in \mathbb{R}^m$ such that $g(z)^\top x_*^{(i)} = 0$ for all $i \in \{1, \dots, n\}$ (as would be obtained in the proof), the linear independence of $x_*^{(i)}$'s already implies that $g(z) = 0$, making hypothesis (ii) unnecessary.

Corollary 5.8. *(Almost global asymptotic stability of saddle points for saddle functions linear in one argument): If, in addition to the hypotheses of Proposition 5.7, the set of equilibria of X_{sp} other than those belonging to* Saddle(F) *are unstable, then the trajectories of X_{sp} converge asymptotically to* Saddle(F) *from almost all initial conditions (all but the unstable equilibria). Moreover, if each point in* Saddle(F) *is stable under X_{sp}, then* Saddle(F) *is almost globally asymptotically stable under the saddle-point dynamics X_{sp} and the trajectories converge to a point in* Saddle(F).

Next, we illustrate how the above result can be applied to the motivating example given before Proposition 5.7 to infer almost global convergence of the trajectories.

Example 5.9. *(Convergence for saddle functions linear in one argument):* Consider again $F(x,z) = xz^2$ with $\text{Saddle}(F) = \{(x,z) \in \mathbb{R} \times \mathbb{R} \mid x \leq 0 \text{ and } z = 0\}$. Pick $(x_*, z_*) = (-1, 0)$. One can verify that this saddle point satisfies the hypotheses (i) and (ii) of Proposition 5.7. Moreover, along any trajectory of the saddle-point dynamics for F, the function $x^2 + \frac{z^2}{2}$ is preserved, which implies that all trajectories are bounded. One can also see that the equilibria of the saddle-point dynamics that are not saddle points, that is the set $\mathbb{R}_{>0} \times \{0\}$, are unstable. Therefore, from Corollary 5.8, we conclude that the trajectories of the saddle-point dynamics asymptotically converge to the set of saddle points from almost all initial conditions. Figure 5.3 illustrates these observations. •



(a) (x,z) (b) F (c) Vector field X_{sp}

Fig. 5.3. *(a) Trajectory of the saddle-point dynamics for the function $F(x,z) = xz^2$. The initial condition is $(x,z) = (5,5)$. The trajectory converges to $(-6.13, 0) \in \text{Saddle}(F)$. (b) Evolution of the function F along the trajectory. The value converges to 0, the value that the function takes on its saddle set. (c) The vector field X_{sp}, depicting that the set of saddle points are attractive while the other equilibrium points $\mathbb{R}_{>0} \times \{0\}$ are unstable.*

6. Conclusions. We have studied the asymptotic stability of the saddle-point dynamics associated to a continuously differentiable function. We have identified a set of complementary conditions under which the trajectories of the dynamics are proved to converge to the set of saddle points of the saddle function and, wherever feasible, we have also established global stability guarantees and convergence to a point in the set. Our first class of convergence results is based on the convexity-concavity properties of the saddle function. In the absence of these properties, our second class of results explore, respectively, the existence of convergence guarantees using linearization techniques, the properties of the saddle function along proximal normals to the set of saddle points, and the linearity properties of the saddle function in one variable. For the linearization result, borrowing ideas from center manifold theory, we have established a general stability result of a manifold of equilibria for a piecewise twice continuously differentiable vector field. Several examples throughout the paper highlight the connections among the results and illustrate their applicability, in particular, for finding the primal-dual solutions of constrained optimization problems. Future work will study the robustness properties of the dynamics against disturbances, investigate the characterization of the rate of convergence, generalize the results to the case of nonsmooth functions (where the associated saddle-point dynamics takes the form of a differential inclusion involving the generalized gradient of the function), and explore the application to optimization problems with inequality constraints. We also plan to build on our results to synthesize distributed algorithmic solutions for various networked optimization problems in power networks.

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Appendix. This section contains some auxiliary results for our convergence analysis in Sections 4 and 5. Our first result establishes the constant value of the saddle function over its set of (local) saddle points.

Lemma A.1. *(Constant function value over saddle points): For $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ continuously differentiable, let $\mathcal{S} \subset \operatorname{Saddle}(F)$ be a path connected set. If F is locally convex-concave on $\mathcal{S}$, then $F_{|\mathcal{S}}$ is constant.*

Proof. We start by considering the case when $\mathcal{S}$ is compact. Given $(x, z) \in \mathcal{S}$, let $\delta(x, z) > 0$ be such that $B_{\delta(x,z)}(x, z) \subset (\mathcal{U}_x \times \mathcal{U}_z) \cap \mathcal{U}$, where $\mathcal{U}_x$ and $\mathcal{U}_z$ are neighborhoods where the saddle property (2.3) holds and $\mathcal{U}$ is the neighborhood of (x, z) where local convexity-concavity holds (cf. Section 2.3). This defines a covering of $\mathcal{S}$ by open sets as

$$\mathcal{S} \subset \cup_{(x,z)\in\mathcal{S}} B_{\delta(x,z)}(x, z).$$

Since $\mathcal{S}$ is compact, there exist a finite number of points $(x_1, z_1), (x_2, z_2), \dots, (x_n, z_n)$ in $\mathcal{S}$ such that $\cup_{i=1}^n B_{\delta(x_i,z_i)}(x_i, z_i)$ covers $\mathcal{S}$. For convenience, denote $B_{\delta(x_i,z_i)}(x_i, z_i)$ by B_i. Next, we show that $F_{|\mathcal{S}\cap B_i}$ is constant for all $i \in \{1, \dots, n\}$. To see this, let $(\bar{x}, \bar{z}) \in \mathcal{S} \cap B_i$. From (2.3), we have

$$F(x_i, \bar{z}) \le F(x_i, z_i) \le F(\bar{x}, z_i). \tag{A.1}$$

From the convexity of $x \mapsto F(x, \bar{z})$ over $\mathcal{U} \cap (\mathbb{R}^n \times \{\bar{z}\})$, (cf. definition of local convexity-concavity in Section 2.3), and the fact that $\nabla_x F(\bar{x}, \bar{z}) = 0$, we obtain $F(x_i, \bar{z}) \ge F(\bar{x}, \bar{z}) + (x_i - \bar{x})^\top \nabla_x F(\bar{x}, \bar{z}) = F(\bar{x}, \bar{z})$. Similarly, using the concavity of $z \mapsto F(\bar{x}, z)$, we get $F(\bar{x}, z_i) \le F(\bar{x}, \bar{z})$. These inequalities together with (A.1) yield

$$F(x_i, z_i) \le F(\bar{x}, z_i) \le F(\bar{x}, \bar{z}) \le F(x_i, \bar{z}) \le F(x_i, z_i).$$

That is, $F(\bar{x}, \bar{z}) = F(x_i, z_i)$ and hence $F_{|\mathcal{S}\cap B_i}$ is constant. Using this reasoning, if $\mathcal{S} \cap B_i \cap B_j \neq \emptyset$ for any $i, j \in \{1, \dots, n\}$, then $F_{|\mathcal{S}\cap(B_i\cup B_j)}$ is constant. Using that $\mathcal{S}$ is path connected, the fact [15, p. 117] states that, for any two points $(x_l, z_l), (x_m, z_m) \in \mathcal{S}$, there exist distinct members $i_1, i_2, \dots, i_k$ of the set $\{1, \dots, n\}$ such that $(x_l, z_l) \in \mathcal{S} \cap B_{i_1}$, $(x_m, z_m) \in \mathcal{S} \cap B_{i_k}$ and $\mathcal{S} \cap B_{i_t} \cap B_{i_{t+1}} \neq \emptyset$ for all $t \in \{1, \dots, k-1\}$. Hence, we conclude that $F_{|\mathcal{S}}$ is constant. Finally, in the case when $\mathcal{S}$ is not compact, pick any two points $(x_l, z_l), (x_m, z_m) \in \mathcal{S}$ and let $\gamma : [0, 1] \to \mathcal{S}$ be a continuous map with $\gamma(0) = (x_l, z_l)$ and $\gamma(1) = (x_m, z_m)$ denoting the path between these points. The image $\gamma([0, 1]) \subset \mathcal{S}$ is closed and bounded, hence compact, and therefore, $F_{|\gamma([0,1])}$ is constant. Since the two points are arbitrary, we conclude that $F_{|\mathcal{S}}$ is constant. ∎

The difficulty in Lemma A.1 arises due to the local nature of the saddle points (the result is instead straightforward for global saddle points). The next result provides a first-order condition for strongly quasiconvex functions.

Lemma A.2. *(First-order property of a strongly quasiconvex function): Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function that is strongly quasiconvex on a convex set $\mathcal{D} \subset \mathbb{R}^n$. Then, there exists a constant $s > 0$ such that*

$$f(x) \leq f(y) \Rightarrow \nabla f(y)^\top (x - y) \leq -s\|x - y\|^2, \tag{A.2}$$

for any $x, y \in \mathcal{D}$.

Proof. Consider $x, y \in \mathcal{D}$ such that $f(x) \leq f(y)$. From strong quasiconvexity we have $f(y) \geq f(\lambda x + (1 - \lambda)y) + s\lambda(1 - \lambda)\|x - y\|^2$, for any $\lambda \in [0, 1]$. Rearranging,

$$f(\lambda x + (1 - \lambda)y) - f(y) \leq -s\lambda(1 - \lambda)\|x - y\|^2. \tag{A.3}$$

On the other hand, the Taylor's approximation of f at y yields the following equality at point $y + \lambda(x - y)$, which is equal to $\lambda x + (1 - \lambda)y$, as

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) - f(y) &= \nabla f(y)^\top (\lambda x + (1 - \lambda)y - y) + g(\lambda x + (1 - \lambda)y - y) \\ &= \lambda \nabla f(y)^\top (x - y) + g(\lambda(x - y)), \end{aligned} \tag{A.4}$$

for some function g with the property $\lim_{\lambda \to 0} \frac{g(\lambda(x-y))}{\lambda} = 0$. Using (A.4) in (A.3), dividing by λ, and taking the limit $\lambda \to 0$ yields the result. □

The next result is helpful when dealing with dynamical systems that have non-isolated equilibria to establish the asymptotic convergence of the trajectories to a point, rather than to a set.

Lemma A.3. *(Asymptotic convergence to a point [5, Corollary 5.2]): Consider the nonlinear system*

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{A.5}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. Let $\mathcal{W} \subset \mathbb{R}^n$ be a compact set that is positively invariant under (A.5) *and let $\mathcal{E} \subset \mathcal{W}$ be a set of stable equilibria. If a trajectory $t \mapsto x(t)$ of* (A.5) *with $x_0 \in \mathcal{W}$ satisfies* $\lim_{t \to \infty} d_{\mathcal{E}}(x(t)) = 0$*, then the trajectory converges to a point in $\mathcal{E}$.*

Finally, we establish the asymptotic stability of a manifold of equilibria through linearization techniques. We start with a useful intermediary result.

Lemma A.4. *(Limit points of Jacobian of a piecewise $\mathcal{C}^2$ function): Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be piecewise $\mathcal{C}^2$. Then, for every $x \in \mathbb{R}^n$, there exists a finite index set $\mathcal{I}_x \subset \mathbb{Z}_{\geq 1}$ and a set of matrices $\{A_{x,i} \in \mathbb{R}^{n \times n}\}_{i \in \mathcal{I}_x}$ such that*

$$\{A_{x,i} \mid i \in \mathcal{I}_x\} = \{\lim_{k \to \infty} Df(x_k) \mid x_k \to x, x_k \in \mathbb{R}^n \setminus \Omega_f\}, \tag{A.6}$$

where Ω_f is the set of points where f is not differentiable.

Proof. Since f is piecewise $\mathcal{C}^2$, cf. Section 2.1, let $\mathcal{D}_1, \dots, \mathcal{D}_m \subset \mathbb{R}^n$ be the finite collection of disjoint open sets such that f is $\mathcal{C}^2$ on $\mathcal{D}_i$ for each $i \in \{1, \dots, m\}$ and $\mathbb{R}^n = \cup_{i=1}^m \mathrm{cl}(\mathcal{D}_i)$. Let $x \in \mathbb{R}^n$ and define $\mathcal{I}_x = \{i \in \{1, \dots, m\} \mid x \in \mathrm{cl}(\mathcal{D}_i)\}$ and $A_{x,i} = \{\lim_{k \to \infty} Df(x_k) \mid x_k \to x, x_k \in \mathcal{D}_i\}$. Note that $A_{x,i}$ is uniquely defined for each i as, by definition, $f_{|\mathrm{cl}(\mathcal{D}_i)}$ is $\mathcal{C}^2$. To show that (A.6) holds for the above defined matrices, first note that the set $\{A_{x,i} \mid i \in \mathcal{I}_x\}$ is included in the right hand side of (A.6) by definition. In order to show the other inclusion, consider any sequence

$\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^n \setminus \Omega_f$ with $x_k \to x$. One can partition this sequence into subsequences, each contained in one of the sets $\mathcal{D}_i$, $i \in \mathcal{I}_x$ and each converging to x. Therefore, the limit $\lim_{k\to\infty} Df(x_k)$ is contained in the set $\{A_{x,i}\}_{i\in\mathcal{I}_x}$, proving the other inclusion and hence yielding (A.6). Note that, in the nonsmooth analysis literature [10, Chapter 2], the convex hull of the set of matrices $\{A_{x,i}\}_{i\in\mathcal{I}_x}$ is referred to as the generalized Jacobian of f at x. ☐

The following statement is an extension of [19, Exercise 6] to vector fields that are only piecewise twice continuously differentiable. Its proof is inspired, but cannot be directly implied from, center manifold theory [7].

Proposition A.5. *(Asymptotic stability of a manifold of equilibrium points for piecewise $\mathcal{C}^2$ vector fields): Consider the system*

$$\dot{x} = f(x), \tag{A.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is piecewise $\mathcal{C}^2$ and locally Lipschitz in a neighborhood of a p-dimensional submanifold of equilibrium points $\mathcal{E} \subset \mathbb{R}^n$ of (A.7)*. Assume that at each $x_* \in \mathcal{E}$, the set of matrices $\{A_{x_*,i}\}_{i\in\mathcal{I}_{x_*}}$ from Lemma A.4 satisfy:*

(i) there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that, for all $i \in \mathcal{I}_{x_*}$,

$$Q^\top A_{x_*,i} Q = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{x_*,i} \end{bmatrix},$$

where $\tilde{A}_{x_*,i} \in \mathbb{R}^{n-p\times n-p}$,

(ii) the eigenvalues of the matrices $\{\tilde{A}_{x_*,i}\}_{i\in\mathcal{I}_{x_*}}$ have negative real parts,

(iii) there exists a positive definite matrix $P \in \mathbb{R}^{n-p\times n-p}$ such that

$$\tilde{A}_{x_*,i}^\top P + P\tilde{A}_{x_*,i} \prec 0, \qquad \textit{for all } i \in \mathcal{I}_{(x_*,z_*)}.$$

Then, $\mathcal{E}$ is locally asymptotically stable under (A.7) *and the trajectories converge to a point in $\mathcal{E}$.*

Proof. Our strategy to prove the result is to linearize the vector field f on each of the patches around any equilibrium point and employ a common Lyapunov function and a common upper bound on the growth of the second-order term to establish the convergence of the trajectories. This approach is an extension of the proof of [24, Theorem 8.2], where the vector field f is assumed to be $\mathcal{C}^2$ everywhere. Let $x_* \in \mathcal{E}$. For convenience, translate x_* to the origin of (A.7). We divide the proof in its various parts to make it easier to follow the technical arguments.

Step I: linearization of the vector field on patches around the equilibrium point. From Lemma A.4, define $\mathcal{I}_0 = \{i \in \{1,\dots,m\} \mid 0 \in \mathrm{cl}(\mathcal{D}_i)\}$ and matrices $\{A_{0,i}\}_{i\in\mathcal{I}_0}$ as the limit points of the Jacobian matrices. From the definition of piecewise $\mathcal{C}^2$ function, there exist $\mathcal{C}^2$ functions $\{f_i : \mathcal{D}_i^e \to \mathbb{R}^n\}_{i\in\mathcal{I}_0}$ with $\mathcal{D}_i^e$ open such that with $\mathrm{cl}(\mathcal{D}_i) \subset \mathcal{D}_i^e$ and the maps $f_{|\mathrm{cl}(\mathcal{D}_i)}$ and f_i take the same value over the set $\mathrm{cl}(\mathcal{D}_i)$. Note that $0 \in \mathcal{D}_i^e$ for every $i \in \mathcal{I}_0$. By definition of the matrices $\{A_{0,i}\}_{i\in\mathcal{I}_0}$, we deduce that $Df_i(0) = A_{0,i}$ for each $i \in \mathcal{I}_0$. Therefore, there exists a neighborhood $\mathcal{N}_0 \subset \mathbb{R}^n$ of the origin and a set of $\mathcal{C}^2$ functions $\{g_i : \mathbb{R}^n \to \mathbb{R}^n\}_{i\in\mathcal{I}_0}$ such that, for all $i \in \mathcal{I}_0$, $f_i(x) = A_{0,i}x + g_i(x)$, for all $x \in \mathcal{N}_0 \cap \mathcal{D}_i^e$, where

$$g_i(0) = 0 \quad \text{and} \quad \frac{\partial g_i}{\partial x}(0) = 0. \tag{A.8}$$

Without loss of generality, select $\mathcal{N}_0$ such that $\mathcal{N}_0 \cap \mathcal{D}_i$ is empty for every $i \notin \mathcal{I}_0$. That is, $\cup_{i \in \mathcal{I}_0}(\mathcal{N}_0 \cap \mathrm{cl}(\mathcal{D}_i))$ contains a neighborhood of the origin. With the above construction, the vector field f in a neighborhood around the origin is written as

$$f(x) = f_i(x) = A_{0,x}x + g_i(x), \text{ for all } x \in \mathcal{N}_0 \cap \mathrm{cl}(\mathcal{D}_i), i \in \mathcal{I}_0, \tag{A.9}$$

where for each $i \in \mathcal{I}_0$, g_i satisfies (A.8).

Step II: change of coordinates. Subsequently, from hypothesis (i), there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$, defining an orthonormal transformation denoted by $\mathcal{T}_Q : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (u, v)$, that yields the new form of (A.9) as

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{0,i} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \tilde{g}_{i,1}(u,v) \\ \tilde{g}_{i,2}(u,v) \end{bmatrix}, \text{ for all } (u,v) \in \mathcal{T}_Q(\mathcal{N}_0 \cap \mathrm{cl}(\mathcal{D}_i)),\ i \in \mathcal{I}_0, \tag{A.10}$$

where for each $i \in \mathcal{I}_0$, the matrix $\tilde{A}_{0,i}$ has eigenvalues with negative real parts (cf. hypothesis (ii)) and for each $i \in \mathcal{I}_0$ and $k \in \{1, 2\}$ we have

$$\tilde{g}_{i,k}(0,0) = 0, \quad \frac{\partial \tilde{g}_{i,k}}{\partial u}(0,0) = 0, \quad \text{and} \quad \frac{\partial \tilde{g}_{i,k}}{\partial v}(0,0) = 0. \tag{A.11}$$

With a slight abuse of notation, denote the manifold of equilibrium points in the transformed coordinates by $\mathcal{E}$ itself, i.e., $\mathcal{E} = \mathcal{T}_Q(\mathcal{E})$. From (A.10), we deduce that the tangent and the normal spaces to the equilibrium manifold $\mathcal{E}$ at the origin are $\{(u,v) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \mid v = 0\}$ and $\{(u,v) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \mid u = 0\}$, respectively. Due to this fact and since $\mathcal{E}$ is a submanifold of $\mathbb{R}^n$, there exists a smooth function $h : \mathbb{R}^p \to \mathbb{R}^{n-p}$ and a neighborhood $\mathcal{U} \subset \mathcal{T}_Q(\mathcal{N}_0) \subset \mathbb{R}^n$ of the origin such that for any $(u,v) \in \mathcal{U}$, $v = h(u)$ if and only if $(u,v) \in \mathcal{E} \cap \mathcal{U}$. Moreover,

$$h(0) = 0 \text{ and } \frac{\partial h}{\partial u}(0) = 0. \tag{A.12}$$

Now, consider the coordinate $w = v - h(u)$ to quantify the distance of a point (u,v) from the set $\mathcal{E}$ in the neighborhood $\mathcal{U}$. To conclude the proof, we focus on showing that there exists a neighborhood of the origin such that along a trajectory of (A.10) initialized in this neighborhood, we have $w(t) \to 0$ and $(u(t), h(u(t))) \in \mathcal{U}$ at all $t \geq 0$. In (u,w)-coordinates, over the set $\mathcal{U}$, the system (A.10) reads as

$$\begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_{0,i} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} \bar{g}_{i,1}(u,w) \\ \bar{g}_{i,2}(u,w) \end{bmatrix}, \text{ for } (u, w + h(u)) \in \mathcal{U} \cap \mathcal{T}_Q(\mathrm{cl}(\mathcal{D}_i)),\ i \in \mathcal{I}_0, \tag{A.13}$$

where $\bar{g}_{i,1}(u,w) = \tilde{g}_{i,1}(u, w + h(u))$ and $\bar{g}_{i,2}(u,w) = \tilde{A}_{0,i}h(u) + \tilde{g}_{i,2}(u, w + h(u)) - \frac{\partial h}{\partial u}(u)(\tilde{g}_{i,1}(u, w + h(u)))$. Further, the equilibrium points $\mathcal{E} \cap \mathcal{U}$ in these coordinates are represented by the set of points $(u, 0)$, where u satisfies $(u, h(u)) \in \mathcal{E} \cap \mathcal{U}$. These facts, along with the conditions on the first-order derivatives of $\tilde{g}_{i,1}$, $\tilde{g}_{i,2}$ in (A.11) and that of h in (A.12) yield

$$\bar{g}_{i,k}(u, 0) = 0 \text{ and } \frac{\partial \bar{g}_{i,k}}{\partial w}(0,0) = 0, \tag{A.14}$$

for all $i \in \mathcal{I}_0$ and $k \in \{1, 2\}$. Note that the functions $\bar{g}_{i,1}$ and $\bar{g}_{i,2}$ are $\mathcal{C}^2$. This implies that, for small enough $\epsilon > 0$, we have $\|\bar{g}_{i,k}(u,w)\| \leq M_{i,k}\|w\|$, for $k \in \{1,2\}$, $i \in \mathcal{I}_0$, and $(u,w) \in B_\epsilon(0)$, where the constants $\{M_{i,k}\}_{i\in\mathcal{I}_0,k\in\{1,2\}} \subset \mathbb{R}_{>0}$ can be made arbitrarily small by selecting smaller ϵ. Defining $M_\epsilon = \max\{M_{i,k} \mid i \in \mathcal{I}_0, k \in \{1,2\}\}$,

$$\|\bar{g}_{i,k}(u,w)\| \leq M_\epsilon\|w\|, \quad \text{for } k \in \{1,2\} \text{ and } i \in \mathcal{I}_0. \tag{A.15}$$

Step III: Lyapunov analysis. With the bounds above, we proceed to carry out the Lyapunov analysis for (A.13). Using the matrix P from assumption (iii), define the candidate Lyapunov function $V : \mathbb{R}^{n-p} \to \mathbb{R}_{\geq 0}$ for (A.13) as $V(w) = w^\top P w$ whose Lie derivative along (A.13) is

$$\begin{aligned} \mathcal{L}_{\,(\mathrm{A.13})}V(w) = w^\top(\tilde{A}_{0,i}^\top P + P\tilde{A}_{0,i})w + 2w^\top P\bar{g}_{i,2}(u,w),& \\ \text{for } (u, w + h(u)) \in \mathcal{U} \cap \mathcal{T}_Q(\mathrm{cl}(\mathcal{D}_i)),\ i \in \mathcal{I}_0.& \end{aligned}$$

By assumption (iii), there exists $\lambda > 0$ such that $w^\top(\tilde{A}_{0,i}^\top P + P\tilde{A}_{0,i})w \leq -\lambda\|w\|^2$. Pick ϵ such that $(u,w) \in B_\epsilon(0)$ implies $(u, h(u) + w) \in \mathcal{U}$. Then, the above Lie derivative can be upper bounded as

$$\mathcal{L}_{\,(\mathrm{A.13})}V(w) \leq -\lambda\|w\|^2 + 2M_\epsilon\|P\|\|w\|^2 = -\beta_1\|w\|^2, \quad \text{for } (u,w) \in B_\epsilon(0),$$

where $\beta_1 = \lambda - 2M_\epsilon$. Let ϵ small enough so that $\beta_1 > 0$ and therefore $\mathcal{L}_{\,(\mathrm{A.13})}V(w) \leq -\beta_1\|w\|^2 < 0$ for $w \neq 0$. Now, assume that there exists a trajectory $t \mapsto (u(t), w(t))$ of (A.13) that satisfies $(u(t), w(t)) \in B_\epsilon(0)$ for all $t \geq 0$. Then, using the following inequalities

$$\lambda_{\min}(P)\|w\|^2 \leq w^\top P w \leq \lambda_{\max}(P)\|w\|^2,$$

we get $V(w(t)) \leq e^{-\beta_1 t/\lambda_{\max}(P)}V(w(0))$ along this trajectory. Employing the same inequalities again, we get

$$\|w(t)\| \leq K\|w(0)\|e^{-\beta_2 t}, \tag{A.16}$$

where $K = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$ and $\beta_2 = \frac{\beta_1}{2\lambda_{\max}(P)} > 0$. This proves that $w(t) \to 0$ exponentially for the considered trajectory. Finally, we show that there exists $\delta > 0$ such that all trajectories of (A.13) with initial condition $(u(0), w(0)) \in B_\delta(0)$ satisfy $(u(t), w(t)) \in B_\epsilon(0)$ for all $t \geq 0$ and hence, converge to $\mathcal{E}$. From (A.13), (A.15) and (A.16), we have

$$\|u(t)\| \leq \|u(0)\| + \int_0^t M_\epsilon K e^{-\beta_2 s}\|w(0)\|ds, \leq \|u(0)\| + \frac{M_\epsilon K}{\beta_2}\|w(0)\|. \tag{A.17}$$

By choosing ϵ small enough, M_ϵ can be made arbitrarily small and β_2 can be bounded away from the origin. With this, from (A.16) and (A.17), one can select a small enough $\delta > 0$ such that $(u(0), w(0)) \in B_\delta(0)$ imply $(u(t), w(t)) \in B_\epsilon(0)$ for all $t \geq 0$ and $w(t) \to 0$. From this, we deduce that the trajectories staring in $B_\delta(0)$ converge to the set $\mathcal{E}$ and the origin is stable. Since x_* was selected arbitrarily, we conclude local asymptotic stability of the set $\mathcal{E}$. Convergence to a point follows from the application of Lemma A.3. ☐

The next example illustrates the application of the above result to conclude local convergence of trajectories to a point in the manifold of equilibria.

EXAMPLE A.6. *(Asymptotic stability of a manifold of equilibria for piecewise $\mathcal{C}^2$ vector fields):* Consider the system $\dot{x} = f(x)$, where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by
(A.18)

$$f(x) = \begin{cases} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (x_1 - x_3)^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \text{if } x_1 - x_3 \geq 0, \\ \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (x_1 - x_3)^2(1 - x_1 + x_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \text{if } x_1 - x_3 < 0. \end{cases}$$

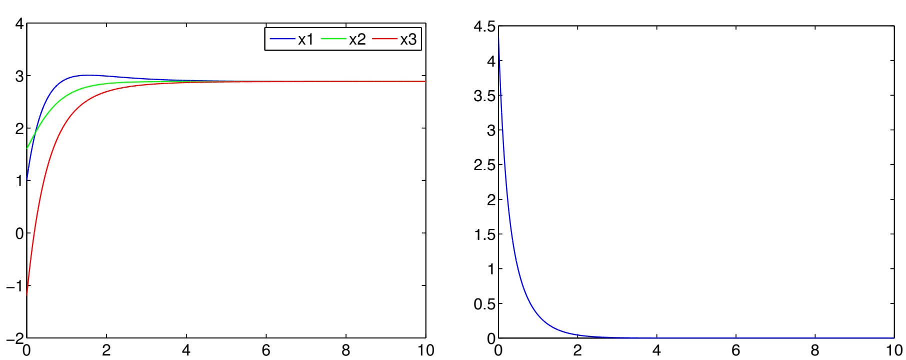
The set of equilibria of f is the one-dimensional manifold $\mathcal{E} = \{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$. Consider the regions $\mathcal{D}_1 = \{x \in \mathbb{R}^2 \mid x_1 - x_3 > 0\}$ and $\mathcal{D}_2 = \{x \in \mathbb{R}^2 \mid x_1 - x_3 < 0\}$. Note that f is locally Lipschitz on $\mathbb{R}^3$ and $\mathcal{C}^2$ on $\mathcal{D}_1$ and $\mathcal{D}_2$. At any equilibrium point $x_* \in \mathcal{E}$, the limit point of the generalized Jacobian belongs to $\{A_1, A_2\}$, where

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

With the orthogonal matrix $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ we get,

$$Q^\top A_1 Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 3 \\ 0 & 3 & -9 \end{bmatrix}, \qquad Q^\top A_2 Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -18 \end{bmatrix}.$$

The nonzero 2×2-submatrices obtained in the above equation have eigenvalues with negative real parts and have the identity matrix as a common Lyapunov function. Therefore, from Proposition A.5, we conclude that $\mathcal{E}$ is locally asymptotically stable under $\dot{x} = f(x)$, as illustrated in Figure 6.1. •



(a) Trajectory

(b) Distance to equilibrium set

FIG. 6.1. *(a) Trajectory of the vector field f defined in* (A.18). *The initial condition is* $x = (1, 1.6, -1.2)$. *The trajectory converges to the equilibrium point* $(2.88, 2.88, 2.88)$. *(b) Evolution of the distance to the equilibrium set $\mathcal{E}$ of the trajectory.*