Convergence of Caratheodory solutions for primal-dual dynamics in constrained concave optimization*

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Abstract

This paper characterizes the asymptotic convergence properties of the primal-dual dynamics to the solutions of a constrained concave optimization problem using classical notions from stability analysis. We motivate our study by providing an example which rules out the possibility of employing the invariance principle for hybrid automata to analyze the asymptotic convergence. We understand the solutions of the primal-dual dynamics in the Caratheodory sense and establish their existence, uniqueness, and continuity with respect to the initial conditions. We employ the invariance principle for Caratheodory solutions of a discontinuous dynamical system to show that the primal-dual optimizers are globally asymptotically stable under the primal-dual dynamics and that each solution of the dynamics converges to an optimizer.

1 Introduction

The primal-dual dynamics is a popular continuous-time algorithm to determine the primal and dual solutions of constrained convex (or concave) optimization problems. In the case when one deals with inequality constraints, the primal-dual dynamics requires a projection in the dual variables which is not necessary in the case of equality constraints. Such dynamics, first studied in the classic works [1, 12], have found numerous applications in recent times, particularly in network resource allocation problems [8, 5, 9], network optimization [10, 18, 20], and in distributed stabilization and optimization of power networks [16, 24, 17, 23]. These network optimization problems have aggregate objective functions and constraints that can be expressed locally which together allow for a distributed implementation of the primal-dual dynamics, a desirable property for large-scale problems.

The aim of this paper is to provide a rigorous treatment of the convergence analysis of the primal-dual dynamics using classical notions from stability analysis. Since this dynamics has a discontinuous right-hand side, the standard Lyapunov or LaSalle-based stability results for nonlinear systems, see e.g. [11], are not applicable. Due to this roadblock, in [1], the authors follow a direct approach to establish convergence. They approximate the evolution of the distance of the solution of the primal-dual dynamics to an arbitrary primal-dual optimizer using power series expansion. Then, they derive the monotonicity property of this distance along the solutions by analyzing the behavior of each of the terms in the series locally around the optimizer. Various instances of this argument are also combined to provide a global convergence result. Instead, [8] takes an indirect approach and characterizes convergence using classical notions such as invariant sets and LaSalle functions. This work models the primal-dual dynamics as a hybrid automaton, as defined in [15], and employs a LaSalle Invariance Principle for hybrid automata to show the asymptotic convergence of the solutions. Such an approach to establish convergence is appealing because of its conceptual simplicity and the versatility of Lyapunov-like methods in characterizing other properties of the dynamics. However, the hybrid automaton corresponding to the primal-dual dynamics is in general not continuous, thereby not satisfying a key requirement of the invariance principle stated in [15]. The first contribution of this paper is an example that illustrates this point. Our second contribution is an alternative proof strategy that arrives at the same convergence results of [8]. We consider an inequality constrained concave optimization problem described by continuously differentiable functions with locally Lipschitz gradients. Since the primal-dual dynamics has a discontinuous right-hand side, we start by specifying the notion of solution in the Caratheodory sense. The reason for this choice

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is that the equilibria of the primal-dual dynamics exactly correspond to the primal-dual optimizers of the corresponding optimization problem. We show that the primal-dual dynamics is a particular case of a projected dynamical system and, using results from [19], we establish that Caratheodory solutions exist, are unique, and are continuous with respect to the initial condition. Using these properties, we show that the omega-limit set of any solution of the primal-dual dynamics is invariant under the dynamics. Finally, we employ the invariance principle for Caratheodory solutions of discontinuous dynamical systems from [2] to show that the primal-dual optimizers are globally asymptotically stable under the primal-dual dynamics and that each solution of the dynamics converges to an optimizer. For reasons of space, the proofs are omitted and will appear elsewhere.

The paper is organized as follows. Section 2 introduces notation and preliminary concepts on discontinuous dynamical systems. Section 3 defines the primal-dual dynamics and with an example, motivates the need for a convergence analysis using classical stability tools. Section 4 contains the main convergence results. Finally, Section 5 gathers our conclusions and ideas for future work.

2 Preliminaries

Here we introduce notation and basic concepts about discontinuous and projected dynamical systems.

2.1 Notation This section introduces basic concepts and preliminaries. We start with some notational conventions. We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{Z}_{\geq 1}$ be the set of real, nonnegative real, positive real, and positive integer numbers, respectively. We denote by $\|\cdot\|$ the 2-norm on $\mathbb{R}^n$. The open ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$ is represented by $B_\delta(x)$. Given $x \in \mathbb{R}^n$, x_i denotes the i-th component of x. For $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in \{1, \dots, n\}$. We use the shorthand notation $\mathbf{0}_n = (0, \dots, 0) \in \mathbb{R}^n$. For a real-valued function $V : \mathbb{R}^n \to \mathbb{R}$ and $\alpha > 0$, we denote the sublevel set of V by $V^{-1}(\leq \alpha) = \{x \in \mathbb{R}^n \mid V(x) \leq \alpha\}$. For scalars $a, b \in \mathbb{R}$, the operator $[a]_b^+$ is defined as

$$[a]_b^+ = \begin{cases} a, & \text{if } b > 0, \\ \max\{0, a\}, & \text{if } b = 0. \end{cases}$$

For vectors $a, b \in \mathbb{R}^n$, $[a]_b^+$ denotes the vector whose i-th component is $[a_i]_{b_i}^+$, $i \in \{1, \dots, n\}$. For a set $\mathcal{S} \in \mathbb{R}^n$, its interior, closure, and boundary are denoted by $\text{int}(\mathcal{S})$, $\text{cl}(\mathcal{S})$, and $\text{bd}(\mathcal{S})$, respectively. Given two sets X and Y, a set-valued map $f : X \rightrightarrows Y$ associates to each point in X a subset of Y. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist $\delta_x, L_x > 0$ such that $\|f(y_1) - f(y_2)\| \leq L_x \|y_1 - y_2\|$ for any $y_1, y_2 \in B_{\delta_x}(x)$. If f is locally Lipschitz at every $x \in \mathcal{K} \subset \mathbb{R}^n$, then we simply say that f is locally Lipschitz on $\mathcal{K}$. The map f is Lipschitz on $\mathcal{K} \subset \mathbb{R}^n$ if there exists a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for any $x, y \in \mathcal{K}$. Note that if f is locally Lipschitz on $\mathbb{R}^n$, then it is Lipschitz on every compact set $\mathcal{K} \subset \mathbb{R}^n$.

2.2 Discontinuous dynamical systems Here we present basic concepts on discontinuous dynamical systems following [2, 7]. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the differential equation

$$\dot{x} = f(x). \tag{2.1}$$

A map $\gamma : [0, T) \to \mathbb{R}^n$ is a *(Caratheodory) solution* of (2.1) on the interval $[0, T)$ if it is absolutely continuous on $[0, T)$ and satisfies $\dot{\gamma}(t) = f(\gamma(t))$ almost everywhere in $[0, T)$. A set $\mathcal{S} \subset \mathbb{R}^n$ is *invariant* under (2.1) if every solution starting from any point in $\mathcal{S}$ remains in $\mathcal{S}$. For a solution γ of (2.1) defined on the time interval $[0, \infty)$, the *omega-limit* set $\Omega(\gamma)$ is defined by

$$\Omega(\gamma) = \{y \in \mathbb{R}^n \mid \exists \{t_k\}_{k=1}^\infty \subset [0, \infty) \text{ with } \lim_{k\to\infty} t_k = \infty \text{ and } \lim_{k\to\infty} \gamma(t_k) = y\}.$$

If the solution γ is bounded, then $\Omega(\gamma) \neq \emptyset$ by the Bolzano-Weierstrass theorem [21]. These notions allow us to characterize the asymptotic convergence properties of the solutions of (2.1) via invariance principles. Given a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, the *Lie derivative of* V *along* (2.1) at $x \in \mathbb{R}^n$ is $\mathcal{L}_f V(x) = \nabla V(x)^\top f(x)$. The next result is a simplified version of [2, Proposition 3] which is sufficient for our convergence analysis later.

PROPOSITION 2.1. *(Invariance principle for discontinuous Caratheodory systems): Let $\mathcal{S} \in \mathbb{R}^n$ be compact and invariant. Assume that, for each point $x_0 \in \mathcal{S}$, there exists a unique solution of* (2.1) *starting at x_0 and that its omega-limit set is invariant too. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable map such that $\mathcal{L}_f V(x) \leq 0$ for all $x \in \mathcal{S}$. Then, any solution of* (2.1) *starting at $\mathcal{S}$ converges to the largest invariant set in* $\text{cl}(\{x \in \mathcal{S} \mid \mathcal{L}_f V(x) = 0\})$.

2.3 Projected dynamical systems Projected dynamical systems are a particular class of discontinuous dynamical systems. Here, following [19], we gather some basic notions that will be useful later to establish continuity with respect to the initial condition of the solutions of the primal-dual dynamics. Let $\mathcal{K} \subset \mathbb{R}^n$ be a closed convex set. Given a point $y \in \mathbb{R}^n$, the (point) projection of y onto $\mathcal{K}$ is $\text{proj}_{\mathcal{K}}(y) = \text{argmin}_{z\in\mathcal{K}} \|z - y\|$. Note that $\text{proj}_{\mathcal{K}}(y)$ is a singleton and the map $\text{proj}_{\mathcal{K}}$ is Lipschitz on $\mathbb{R}^n$ with constant $L = 1$ [6, Proposition 2.4.1]. Given $x \in \mathcal{K}$ and $v \in \mathbb{R}^n$, the (vector) projection of v at x with respect to $\mathcal{K}$ is

$$\Pi_{\mathcal{K}}(x, v) = \lim_{\delta\to 0^+} \frac{\text{proj}_{\mathcal{K}}(x + \delta v) - x}{\delta}.$$

Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and a closed convex polyhedron $\mathcal{K} \subset \mathbb{R}^n$, the associated projected dynamical system is

$$\dot{x} = \Pi_{\mathcal{K}}(x, f(x)), \quad x(0) \in \mathcal{K}, \tag{2.2}$$

Note that, at any point x in the interior of $\mathcal{K}$, we have $\Pi_{\mathcal{K}}(x, f(x)) = f(x)$. At any boundary point of $\mathcal{K}$, the projection operator restricts the flow of the vector field f such that the solutions of (2.2) remain in $\mathcal{K}$. Therefore, in general, (2.2) is a discontinuous dynamical system. The next result summarizes conditions under which the (Caratheodory) solutions of the projected system (2.2) exist, are unique, and continuous with respect to the initial condition.

PROPOSITION 2.2. *(Existence, uniqueness, and continuity with respect to initial condition [19, Theorem 2.5]): Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz on $\mathcal{K}$. Then,*

(i) (existence and uniqueness): for any $x_0 \in \mathcal{K}$, there exists a unique solution $t \mapsto x(t)$ of the projected system (2.2) with $x(0) = x_0$ defined over the domain $[0, \infty)$,

(ii) (continuity with respect to the initial condition): given a sequence of points $\{x_k\}_{k=1}^\infty \subset \mathcal{K}$ with $\lim_{k\to\infty} x_k = x$, the sequence of solutions $\{t \mapsto \gamma_k(t)\}_{k=1}^\infty$ of (2.2) with $\gamma_k(0) = x_k$ for all k, converge to the solution $t \mapsto \gamma(t)$ of (2.2) with $\gamma(0) = x$ uniformly on every compact set of $[0, \infty)$.

3 Problem statement

This section reviews the primal-dual dynamics for solving constrained optimization problems and justifies the need to rigorously characterize its asymptotic convergence properties. Consider the concave optimization problem on $\mathbb{R}^n$,

$$\text{maximize} \quad f(x), \tag{3.3a}$$
$$\text{subject to} \quad g(x) \le \mathbf{0}_m, \tag{3.3b}$$

where the continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are strictly concave and convex, respectively, and have locally Lipschitz gradients. The Lagrangian of (3.3) is given as

$$L(x, \lambda) = f(x) - \lambda^\top g(x), \tag{3.4}$$

where $\lambda \in \mathbb{R}^m$ is the Lagrange multiplier corresponding to the inequality constraint (3.3b). Note that the Lagrangian is concave in x and convex (in fact linear) in λ. Assume that the Slater's conditions is satisfied for the problem (3.3), that is, there exists $x \in \mathbb{R}^n$ such that $g(x) < \mathbf{0}_m$. Under this assumption, the duality gap between the primal and dual optimizers is zero and a point $(x_*, \lambda_*) \in \mathbb{R}^n \times \mathbb{R}^m_{\ge 0}$ is a primal-dual optimizer of (3.3) if and only if it is a saddle point of L over the domain $\mathbb{R}^n \times \mathbb{R}^m_{\ge 0}$, i.e.,

$$L(x, \lambda) \le L(x_*, \lambda_*) \quad \text{and} \quad L(x_*, \lambda) \ge L(x_*, \lambda_*),$$

for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m_{\ge 0}$. For convenience, we denote the set of saddle points of L (equivalently the primal-dual optimizers) by $\mathsf{X} \times \Lambda \subset \mathbb{R}^n \times \mathbb{R}^m$. Furthermore, (x_*, λ_*) is a primal-dual optimizer if and only if it satisfies the following Karush-Kuhn-Tucker (KKT) conditions (cf. [3, Chapter 5]),

$$\nabla f(x_*) - \sum_{i=1}^m (\lambda_*)_i \nabla g_i(x_*) = 0, \tag{3.5a}$$
$$g(x_*) \le \mathbf{0}_m, \quad \lambda_* \ge \mathbf{0}_m, \quad \lambda_*^\top g(x_*) = 0. \tag{3.5b}$$

Given this characterization of the solutions of the optimization problem, it is natural to consider the *primal-dual dynamics* on $\mathbb{R}^n \times \mathbb{R}^m_{\ge 0}$ to find them

$$\dot{x} = \nabla_x L(x, \lambda) = \nabla f(x) - \sum_{i=1}^m \lambda_i \nabla g_i(x), \tag{3.6a}$$
$$\dot{\lambda} = [-\nabla_\lambda L(x, \lambda)]^+_\lambda = [g(x)]^+_\lambda. \tag{3.6b}$$

When convenient, we use the notation $X_{\text{p-d}} : \mathbb{R}^n \times \mathbb{R}^m_{\ge 0} \to \mathbb{R}^n \times \mathbb{R}^m$ to refer to the dynamics (3.6). Given that the primal-dual dynamics is discontinuous, we consider solutions in the Caratheodory sense. The reason for this is that, with this notion of solution, a point is an equilibrium of (3.6) if and only if it satisfies the KKT conditions (3.5).

Our objective is to establish that the solutions of (3.6) exist and asymptotically converge to a solution of the concave optimization problem (3.3) using classical notions and tools from stability analysis.

Our motivation for this aim comes from the conceptual simplicity and versatility of Lyapunov-like methods and their amenability for performing robustness analysis and studying generalizations of the dynamics. One way of tackling this problem, see e.g., [8], is to interpret the dynamics as a state-dependent switched system, formulate the latter as a hybrid automaton as defined in [15], and then use the invariance principle for hybrid automata to characterize its asymptotic convergence properties. However, this route is not valid in general because one of the key assumptions required by the invariance principle for hybrid automata is not satisfied by the primal-dual dynamics. The next example justifies this claim.

Example 3.1. *(The hybrid automaton corresponding to the primal-dual dynamics is not continuous):* Consider the concave optimization problem (3.3) on $\mathbb{R}$ with $f(x) = -(x-5)^2$ and $g(x) = x^2 - 1$, whose set of primal-dual optimizers is $\mathsf{X} \times \Lambda = \{(1,4)\}$. The associated primal-dual dynamics takes the form

$$\dot{x} = -2(x-5) - 2x\lambda, \tag{3.7a}$$

$$\dot{\lambda} = [x^2 - 1]^+_\lambda. \tag{3.7b}$$

We next formulate this dynamics as a hybrid automaton as defined in [15, Definition II.1]. The idea to build the hybrid automaton is to divide the state space $\mathbb{R} \times \mathbb{R}_{\geq 0}$ into two domains over which the vector field (3.7) is continuous. To this end, we define two modes represented by the discrete variable q, taking values in $\mathbf{Q} = \{1,2\}$. The value $q = 1$ represents the mode where the projection in (3.7b) is active and $q = 2$ represents the mode where it is not. Formally, the projection is *active* at (x,λ) if $[g(x)]^+_\lambda \neq g(x)$, i.e, $\lambda = 0$ and $g(x) < 0$. The hybrid automaton is then given by the collection $H = (Q, X, f, \text{Init}, D, E, G, R)$, where $Q = \{q\}$ is the set of discrete variables, taking values in $\mathbf{Q}$; $X = \{x, \lambda\}$ is the set of continuous variables, taking values in $\mathbf{X} = \mathbb{R} \times \mathbb{R}_{\geq 0}$; the vector field $f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ is

$$f(1,(x,\lambda)) = \begin{bmatrix} -2(x-5) - 2x\lambda \\ 0 \end{bmatrix},$$

$$f(2,(x,\lambda)) = \begin{bmatrix} -2(x-5) - 2x\lambda \\ x^2 - 1 \end{bmatrix};$$

$\text{Init} = \mathbf{X}$ is the set of initial conditions; $D : \mathbf{Q} \rightrightarrows \mathbf{X}$ specifies the domain of each discrete mode,

$$D(1) = (-1,1) \times \{0\}, \quad D(2) = \mathbf{X} \setminus D(1),$$

i.e., the dynamics is defined by the vector field $(x,\lambda) \to f(1,(x,\lambda))$ over $D(1)$ and by $(x,\lambda) \to f(2,(x,\lambda))$ over $D(2)$; $E = \{(1,2),(2,1)\}$ is the set of edges specifying the transitions between modes; the guard map $G : \mathbf{Q} \rightrightarrows \mathbf{X}$ specifies when a solution can jump from one mode to the other,

$$G(1,2) = \{(1,0),(-1,0)\}, \quad G(2,1) = (-1,1) \times \{0\},$$

i.e., $G(q,q')$ is the set of points where a solution jumps from mode q to mode q'; and, finally, the reset map $R : \mathbf{Q} \times \mathbf{X} \rightrightarrows \mathbf{X}$ specifies that the state is preserved after a jump from one mode to another,

$$R((1,2),(x,\lambda)) = R((2,1),(x,\lambda)) = \{(x,\lambda)\}.$$

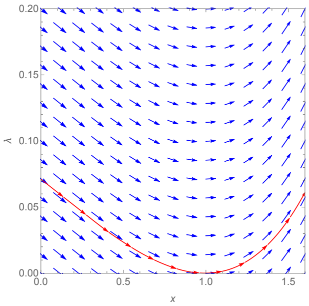

Figure 1: An illustration depicting the vector field (3.7) in the range $(x,\lambda) \in [0,1.6] \times [0,0.2]$. As shown (with a red streamline), there exists a solution of (3.7) that starts at a point $(x(0),\lambda(0))$ with $x(0) < 1$ and $\lambda(0) > 0$ such that it remains in the domain $\lambda > 0$ at all times except at one time instant t when $(x(t),\lambda(t)) = (1,0)$.

We are now ready to show that the hybrid automaton is not continuous in the sense defined by [15, Definition III.3]. This notion plays a key role in the study of omega-limit sets and their stability, and is in fact a basic assumption of the invariance principle developed in [15, Theorem IV.1]. Roughly speaking, H is continuous if two solutions starting close to one another remain close to one another. Therefore, to disprove the continuity of H, it is enough to show that there exist two solutions that start arbitrarily close and yet experience mode transitions at time instances that are not arbitrarily close.

Select an initial condition $(x(0),\lambda(0)) \in (0,1) \times (0,\infty)$ that gives rise to a solution of (3.7) that remains in the set $(0,1) \times (0,\infty)$ for a finite time interval $(0,t)$, $t > 0$, satisfies $(x(t),\lambda(t)) = (1,0)$, and stays in the set $(1,\infty) \times (0,\infty)$ for some finite time interval (t,T), $T > t$. The existence of such a solution becomes clear by plotting the vector field (3.7), see Figure 1. Note that by construction, this is also a solution of the hybrid automaton H. This solution starts and remains in domain $D(2)$ for the time interval $[0,T]$ and so it does not encounter any jumps in its discrete mode. Further, by observing the vector field, we deduce that in every neighborhood of $(x(0),\lambda(0))$, there exists a point $(\tilde{x},\tilde{\lambda})$ such that a solution of (3.7) (that is also a solution of H) starting at this point reaches the set $(0,1)\times\{0\}$ in finite time $t_1 > 0$, remains in $(0,1)\times\{0\}$ for a finite time interval $[t_1,t_2]$, and then enters the set $(1,\infty)\times(0,\infty)$ upon reaching the point $(1,0)$. Indeed, this is true for $\tilde{x} < x(0)$ and $\tilde{\lambda} < \lambda(0)$. Such a solution of H starts in $D(2)$, enters $D(1)$ in finite time t_1, and returns to $D(2)$ at time t_2. Thus, the discrete variable representing the mode of the solution switches from 2 to 1 and back to 2, whereas the solution starting at $(x(0),\lambda(0))$ never switches mode. This shows that the hybrid automaton is not continuous. •

Interestingly, even though the hybrid automaton H described in Example 3.1 is not continuous, one can infer from Figure 1 that two solutions of (3.7) remain close to each other if they start close enough. This suggests that continuity with respect to the initial condition might hold provided this notion is formalized the way it is done for traditional nonlinear systems (and not as done for hybrid automata where both discrete and continuous states have to be aligned). The next section shows that this in fact is the case. This, along with the existence and uniqueness of solutions, allows us to analyze the asymptotic convergence properties of the primal-dual dynamics.

4 Convergence analysis of primal-dual dynamics

In this section, we show that the solutions of the primal-dual dynamics (3.6) asymptotically converge to a solution of the constrained optimization problem (3.3). Our proof strategy is to employ the invariance principle for Caratheodory solutions of discontinuous dynamical systems stated in Proposition 2.1. To this end, we verify that all its hypotheses hold.

We start by stating a useful monotonicity property of the primal-dual dynamics with respect to the set of primal-dual optimizers $\mathsf{X}\times\Lambda$. This property can be found in [1, 8].

Lemma 4.1. *(Monotonicity of the primal-dual dynamics with respect to primal-dual optimizers): Let $(x_*,\lambda_*)\in\mathsf{X}\times\Lambda$ and define $V:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}_{\geq 0}$,*

$$V(x,\lambda) = \frac{1}{2}\big(\|x-x_*\|^2 + \|\lambda-\lambda_*\|^2\big). \tag{4.8}$$

Then $\mathcal{L}_{X_{p\text{-}d}}V(x,\lambda)\leq 0$ for all $(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$.

The next step is to show the existence, uniqueness, and continuity of the solutions of $X_{\text{p-d}}$ starting from $\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$. The proof strategy consists of expressing the primal-dual dynamics $X_{\text{p-d}}$ as a projected dynamical system and then using Proposition 2.2. A minor technical hurdle in this process is ensuring the Lipschitz property of the vector field, which can be tackled by using the monotonicity property of the primal-dual dynamics stated in Lemma 4.1.

Lemma 4.2. *(Existence, uniqueness, and continuity of solutions of the primal-dual dynamics): Starting from any point $(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$, a unique solution $t\mapsto\gamma(t)$ of the primal-dual dynamics $X_{p\text{-}d}$ exists and remains in $(\mathbb{R}^n\times\mathbb{R}^m_{\geq 0})\cap V^{-1}(\leq V(x,\lambda))$. Moreover, if a sequence of points $\{(x_k,\lambda_k)\}_{k=1}^{\infty}\subset\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$ converge to (x,λ) as $k\to\infty$, then the sequence of solutions $\{t\mapsto\gamma_k(t)\}_{k=1}^{\infty}$ of $X_{p\text{-}d}$ starting at these points converge uniformly to the solution $t\mapsto\gamma(t)$ on every compact set of $[0,\infty)$.*

The next result uses the continuity property with respect to the initial condition of the primal-dual dynamics to show that the omega-limit set of any solution is invariant. This ensures that all hypotheses of the invariance principle, Proposition 2.1, are met.

Lemma 4.3. *(Omega-limit set of solution of primal-dual dynamics is invariant): The omega-limit set of any solution of the primal-dual dynamics starting from any point in $\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$ is invariant under (3.6).*

Next, we state our main result, the asymptotic convergence of the solutions of the primal-dual dynamics to a solution of the optimization problem. The proof first uses the above stated results to show that for any $\delta > 0$, the set $\mathcal{S} = V^{-1}(\leq\delta)\cap(\mathbb{R}^n\times\mathbb{R}^m_{\geq 0})$ is invariant under $X_{\text{p-d}}$. The proof then concludes by showing that the largest invariant set contained in $\{(x,\lambda)\in\mathcal{S}\mid\mathcal{L}_{X_{\text{p-d}}}V(x,\lambda)=0\}$ is the set $\mathsf{X}\times\Lambda$.

Theorem 4.1. *(Convergence of the primal-dual dynamics to a primal-dual optimizer): The set of primal-dual solutions of (3.3) is globally asymptotically stable on $\mathbb{R}^n\times\mathbb{R}^m_{\geq 0}$ under the primal-dual dynamics (3.6), and the convergence of each solution is to a point.*

Remark 4.1. *(Alternative proof strategy via evolution variational inequalities):* We briefly describe here an alternative proof strategy to the one we have used here to establish the asymptotic convergence of the primal-dual dynamics. The Caratheodory solutions of the primal-dual dynamics can also be seen as solutions of an evolution variational inequality (EVI) problem [4]. Then, one can show that the resulting EVI problem has a unique solution starting from each point in $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$, which moreover remains in $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$. With this in place, the LaSalle Invariance Principle [4, Theorem 4] for the solutions of the EVI problem can be applied to conclude the convergence to the set of primal-dual optimizers. •

Remark 4.2. *(Primal-dual dynamics with gains):* In power network optimization problems [24, 17, 23] and network congestion control problems [14, 22] it is common to see generalizations of the primal-dual dynamics involving gain matrices. Formally, these dynamics take the form

$$\dot{x} = K_1 \nabla_x L(x,\lambda), \tag{4.9a}$$

$$\dot{\lambda} = K_2[-\nabla_\lambda L(x,\lambda)]^+_\lambda, \tag{4.9b}$$

where $K_1 \in \mathbb{R}^{n\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ are diagonal, positive definite matrices. In such cases, the analysis performed here can be replicated following the same steps but using instead the Lyapunov function

$$V'(x,\lambda) = \frac{1}{2}((x-x_*)^\top K_1^{-1}(x-x_*) + (\lambda-\lambda_*)^\top K_2^{-1}(\lambda-\lambda_*)),$$

to establish the required monotonicity and convergence properties of (4.9). •

Remark 4.3. *(Partial primal-dual dynamics):* In certain power network optimization problems [17, 13], the Lagrangian might not be strictly concave (or strictly convex) in the primal variable. In those cases, a possible way of finding the optimizers is to employ a partial primal-dual dynamics obtained from a reduced Lagrangian. Specifically, for problem (3.3), assume the state is partitioned into two components, $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^r$ and $x_2 \in \mathbb{R}^{n-r}$, with $r \in \mathbb{Z}_{\geq 1}$, and consider the reduced Lagrangian

$$\tilde{L}(x_1,\lambda) = \max_{x_2\in\mathbb{R}^{n-r}} L((x_1,x_2),\lambda) = L((x_1, x_2^*(x_1,\lambda)),\lambda),$$

where $x_2^*(x_1,\lambda)$ is a maximizer of the function $x_2 \mapsto L((x_1,x_2),\lambda)$ for fixed x_1 and λ. Assume the following holds

(i) $(x_1^*, \lambda^*) \in \mathbb{R}^r \times \mathbb{R}^m_{\geq 0}$ is a saddle point of $\tilde{L}$ over the domain $\mathbb{R}^r \times \mathbb{R}^m_{\geq 0}$ only if $(x_1^*, x_2^*(x_1^*,\lambda^*),\lambda^*)$ is a saddle point of the Lagrangian L over the domain $\mathbb{R}^n \times \mathbb{R}^m_{\geq 0}$,

(ii) the map $x_1 \mapsto \tilde{L}(x_1,\lambda)$ is strictly concave or the map $\lambda \mapsto \tilde{L}(x_1,\lambda)$ is strictly convex.

Then, any solution $t \mapsto (x_1(t),\lambda(t))$ of the primal-dual dynamics for the reduced Lagrangian $\tilde{L}$, starting from $\mathbb{R}^r \times \mathbb{R}^m_{\geq 0}$, will converge to the saddle points of $\tilde{L}$. This solution augmented with the map $t \mapsto x_2^*(x_1(t),\lambda(t))$, that gives the maximizer of the function $x_2 \mapsto L((x_1(t),x_2),\lambda(t))$ at each time t, results into the trajectory $t \mapsto (x_1(t), x_2^*(x_1(t),\lambda(t)),\lambda(t))$ that converges asymptotically to the primal-dual optimizers of (3.3). A common scenario in which assumptions (i) and (ii) mentioned above hold (see, e.g., [17, 13]) is when the Lagrangian L is separable in the primal variables, taking the form

$$L((x_1,x_2),\lambda) = L_1(x_1,\lambda) + L_2(x_2,\lambda),$$

where L_1 and L_2 are concave (resp. convex) in the primal (resp. dual) variable, and either the map $x_1 \mapsto L_1(x_1,\lambda)$ is strictly concave or the map $\lambda \mapsto L_1(x_1,\lambda)$ is strictly convex. •

5 Conclusions

For the primal-dual dynamics corresponding to a constrained concave optimization problem, we have considered its Caratheodory solutions and established their asymptotic convergence to the primal-dual optimizers of the problem. Our technical treatment used the results from projected dynamical systems to show the existence, uniqueness, and continuity of solutions and then applied the invariance principle for discontinuous Caratheodory solutions to prove their asymptotic convergence. Leveraging on the technical approach presented in this paper, in future we wish to rigorously characterize the robustness properties of the primal-dual dynamics against unmodeled dynamics, disturbances, and noise. Further, motivated by applications to power networks, we also aim to explore the design of discontinuous dynamics that can find the solutions to semidefinite programs and quadratically constrained quadratic programs.

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