Event-triggered stabilization of linear systems under channel blackouts

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Abstract— **This paper addresses the problem of event-triggered control of linear time-invariant systems over time-varying rate limited communication channels. We explicitly account for the possibility of channel blackouts, i.e., intervals of time when the communication channel is unavailable for feedback. Assuming prior knowledge of the channel evolution, we study the data capacity, which is the maximum total number of bits that could be communicated over a given time interval, and provide an efficient real-time algorithm to lower bound it for a deterministic time-slotted model of channel evolution. Equipped with this algorithm we then propose an event-triggering scheme that guarantees Zeno-free, exponential stabilization at a desired convergence rate even in the presence of intermittent channel blackouts.**

I. Introduction

Control under communication constraints is of great theoretical and practical importance and has motivated a vast amount of research. This paper is a contribution to the growing body of results that employ either information-theoretic or opportunistic triggered control to address the problem of stabilization under constrained resources. We seek to combine both approaches to deal with the control of linear time-invariant systems under time-varying channels, including for the possibility of blackouts, i.e., intervals of time during which the channel is completely unavailable.

Literature review: The literature on information-theoretic control focuses on identifying necessary and sufficient conditions on the bit rates that guarantee stabilization under various assumptions on the (often stochastically modeled) communication channels. Comprehensive overviews of this literature on may be found in [1], [2]. Early data rate results [3]–[5] provided tight necessary and sufficient conditions on the data rate of the encoded feedback for asymptotic stabilization in the discrete-time setting. Since then, the problem has been studied under increasingly complex assumptions on the channels, see e.g., [6]–[8]. In the continuous-time setting, the problem has been studied under either periodic sampling or aperiodic sampling with known upper and lower bounds on the sampling period for single input systems in [9], [10], nonlinear feedforward systems in [11], and switched linear systems in [12], which also analyzes the incident convergence rate. In general, this literature has not explored the potential advantages of tuning the sampling period in the periodic case or if state-based aperiodic sampling can provide any gains in efficiency and performance. In this context, [13] explores the stabilization problem under a state based aperiodic transmission policy, with the inter-transmission intervals being integral multiples of a fixed step size. On the other hand, the event-triggered approach, see e.g. [14]–[16] and references therein, exploits the tolerance to measurement errors to design goal-driven opportunistic state-based aperiodic sampling. The literature in this area mainly focuses on minimizing the number of transmissions while largely ignoring quantization, data capacity and other important aspects of communication. Some of the few exceptions include [17], [18], which utilize static logarithmic quantization and [19]–[21] (see also references therein) which use dynamic quantization. All these works guarantee a positive lower bound on the inter-transmission times, while [19]–[21] also provide a uniform bound on the *communication bit rate* (i.e., the number of bits per transmission). However, these references do not address the inverse problem of triggering and quantization given a limit on the communication bit rate. Moreover, the channel is assumed to always be available to the control system and hence event-triggered designs typically do not take into account the possibility of channel blackouts. An important exception to this statement is [22], which uses the deadlines generated by a self-triggered controller to perform a kind of instantaneous or short-term scheduling. However, if the communication latency is time-varying either because of a time-varying channel or because of time-varying packet sizes, which is important in finite precision feedback control, it is difficult to guarantee long-term future schedulability and system performance. Our recent work [23] combines the information-theoretic and event-triggered control approaches to address the problem of event-triggered stabilization of continuous-time linear time-invariant systems under bounded bit rates. The event-triggered formulation allows us to guarantee, in the absence of channel blackouts, a specified rate of convergence in the presence of non-instantaneous communication and possibly time-varying communication rate.

Statement of contributions: We continue in the spirit of [23] to address the stabilization problem for linear time-invariant systems over time-varying rate-limited communication channels that may be subject to sporadic blackouts. Our notion of scheduled channel blackouts and stabilization despite their occurrence is a key contribution in the context of event-triggered control. For effective control despite the occurrence of blackouts, we define and use the concept of data capacity, i.e., the maximum number of bits that may be communicated over possibly multiple transmissions during an arbitrary time interval under complete knowledge of the channel evolution. This constitutes our first contribution. The computation of data capacity for general time-varying channels is challenging. Our second contribution is the design

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of an algorithm, for the class of piecewise constant channel functions, to lower bound in real time the data capacity over an arbitrary time interval. Our third and final contribution is the synthesis of event-triggered control schemes that, using prior knowledge of the channel information and the available data capacity, plan the transmissions to guarantee the exponential stabilization of the system at a desired convergence rate, even in the presence of channel blackouts.

Notation: We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{>0}$, and $\mathbb{Z}_{\geq 0}$ denote the set of real, nonnegative real, positive integer, and nonnegative integer numbers, respectively. We let $|S|$ denote the cardinality of the set S. We denote by $\|.\|_2$ and $\|.\|_\infty$ the Euclidean and infinity norm of a vector, respectively, or the corresponding induced norm of a matrix. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we let $\lambda_m(A)$ and $\lambda_M(A)$ denote its smallest and largest eigenvalues, respectively. For any matrix norm $\|.\|$, note that $\|e^{A\tau}\| \leq e^{\|A\|\tau}$. For a number $a \in \mathbb{R}$, we let $[a]_+ \triangleq \max\{0, a\}$. For a function $f : \mathbb{R} \mapsto \mathbb{R}^n$ and any $t \in \mathbb{R}$, we let $f(t^-)$ and $f(t^+)$ denote the limit from the left, $\lim_{s\uparrow t} f(s)$ and the limit from the right, $\lim_{s\downarrow t} f(s)$, respectively.

II. Problem Statement

We start with the description of the system dynamics, then describe the model for the communication channel, and finally state the control objective.

A. System description

We consider a linear time-invariant control system,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the plant and $u \in \mathbb{R}^m$ the control input, while $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrices. Our starting point is the existence of a continuous-time feedback stabilizer of the plant dynamics (1). Formally, we select a control gain matrix $K \in \mathbb{R}^{m\times n}$ such that the matrix $\bar{A} = A + BK$ is Hurwitz. Under this assumption, the continuous-time feedback $u(t) = Kx(t)$ renders the origin of (1) globally exponentially stable.

The plant is equipped with a sensor (the *encoder*) and an actuator (the *decoder*) that are not co-located. The sensor can measure the state exactly and the actuator can exert the input to the plant with infinite precision. However, the sensor may transmit state information to the controller at the actuator only at discrete time instants *of its choice*, using a finite number of bits. We let $\{t_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\geq 0}$ be the sequence of *transmission times* at which the sensor transmits an encoded packet of data, $\{r_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\geq 0}$ the sequence of *reception times* at which the decoder receives a complete packet of data, and $\{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\geq 0}$ the sequence of *update times* at which the decoder updates the controller state. At a transmission time t_k, the sensor sends b_k bits, which encode the plant state. Due to causality, $\tilde{r}_k \geq r_k \geq t_k$, and we denote by

$$\Delta_k \triangleq r_k - t_k, \quad \tilde{\Delta}_k \triangleq \tilde{r}_k - t_k,$$

the k^{th} *communication time* and k^{th} *time-to-update*, respectively. The distinction between the reception times and the update times is a generalization with respect to our previous work [23] and provides greater flexibility in the presence of time-varying channels.

B. Communication channel

Our model for communication channel is fully determined by the map $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where $R_a = nR$ is the *minimum instantaneous communication-rate* at a given time, and the map $\bar{p} : \mathbb{R}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, where $\bar{b} = n\bar{p}$ is the *maximum packet size* that can be successfully transmitted at a given time. We assume the k^{th} communication time and the k^{th} time-to-update satisfy

$$\tilde{\Delta}_k \geq \Delta_k \geq 0, \tag{2a}$$

$$\Delta_k \leq \Delta(t_k, p_k) \triangleq \frac{p_k}{R(t_k)} = \frac{b_k}{R_a(t_k)}, \tag{2b}$$

where the first condition is that of causal communication and the second is an upper bound on the communication time. Note that the actual instantaneous communication rate at t_k is b_k/Δ_k and we can rewrite (2b) as

$$\frac{b_k}{\Delta_k} = \frac{np_k}{\Delta_k} \geq \frac{np_k}{\Delta(t_k, p_k)} = R_a(t),$$

to realize that $R_a(t)$ is a lower bound on the number of bits communicated per unit time of all the bits transmitted at time t. Thus, for example, if $R_a(t) = \infty$, then the packet sent at t is received instantaneously. The packet size $b_k = np_k$ that can be successfully transmitted starting at t_k is upper bounded as

$$p_k \leq \bar{p}(t_k), \quad p_k \in \mathbb{Z}_{\geq 0} \tag{3a}$$

for all $k \in \mathbb{Z}_{\geq 0}$. We refer to an interval of time during which $\bar{p} = \bar{b} = 0$ as a *(channel) blackout*. We assume that the encoder knows the functions $t \mapsto R(t)$ and $t \mapsto \bar{p}(t)$ a priori or sufficiently in advance.

Since the channel has bounded data capacity and in order to maintain synchronization between the encoder and the decoder, we require that the encoder does not transmit a packet before a previous packet is received by the decoder and the controller updated, i.e.,

$$t_{k+1} \geq \tilde{r}_k, \tag{3b}$$

for all $k \in \mathbb{Z}_{\geq 0}$. We say the *channel is busy* at time t if $t \in [t_k, r_k)$, for some $k \in \mathbb{Z}_{>0}$. Finally, we refer to the sequences of transmission times $\{t_k\} \subset \mathbb{R}_{\geq 0}$, packet sizes $\{b_k\} \subset \mathbb{Z}_{\geq 0}$, and update times $\{\tilde{r}_k\} \subset \mathbb{R}_{\geq 0}$ as *feasible* if (2) and (3) are satisfied for every $k \in \mathbb{Z}_{>0}$.

C. Encoding and decoding

We use dynamic quantization for finite-bit transmissions from the encoder to the decoder and focus exclusively on its zoom-in stage, during which the encoded feedback is used to asymptotically stabilize the system. We assume both the encoder and the decoder have perfect knowledge of the plant system matrices, have synchronized clocks, and synchronously update their states at update times $\{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}}$. For simplicity, we assume that at transmission t_k the sensor (encoder) encodes each dimension of the plant state using p_k bits so that the total number of bits transmitted is $b_k = np_k$.

The state of the encoder/decoder is composed of the controller state $\hat{x} \in \mathbb{R}^n$ and an upper bound $d_e \in \mathbb{R}_{\geq 0}$ on $\|x_e\|_\infty$, where $x_e \triangleq x - \hat{x}$ is the encoding error. Thus, the actual input to the plant is given by $u(t) = K\hat{x}(t)$. During inter-update times, the state of the controller evolves as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) = \bar{A}\hat{x}(t), \quad t \in [\tilde{r}_k, \tilde{r}_{k+1}). \tag{4a}$$

Let the encoding and decoding functions at the k^{th} iteration be represented by $q_{E,k} : \mathbb{R}^n \times \mathbb{R}^n \mapsto G_k$ and $q_{D,k} : G_k \times \mathbb{R}^n \mapsto \mathbb{R}^n$, respectively, where G_k is a finite set of 2^{b_k} symbols. At t_k, the encoder encodes the plant state as $z_{E,k} \triangleq q_{E,k}(x(t_k), \hat{x}(t_k^-))$, where $\hat{x}(t_k^-)$ is the controller state just prior to the encoding time t_k, and sends it to the controller. The decoder can decode this signal as $z_{D,k} \triangleq q_{D,k}(z_{E,k}, \hat{x}(t_k^-))$ at any time during $[r_k, \tilde{r}_k]$. At the update time $\tilde{r}_k$, the sensor and the controller also update $\hat{x}$ using

$$\begin{aligned}\hat{x}(\tilde{r}_k) &= e^{\bar{A}\tilde{\Delta}_k}\hat{x}(t_k^-) + e^{A\tilde{\Delta}_k}(z_{D,k} - \hat{x}(t_k^-)) \\ &\triangleq q_k(x(t_k), \hat{x}(t_k^-)),\end{aligned} \tag{4b}$$

where $q_k : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ represents the quantization that occurs as a result of the finite-bit coding. We allow the quantization domain, the number of bits and the resulting quantizer, q_k, for each transmission $k \in \mathbb{Z}_{>0}$ to be variable. The evolution of the plant state x and the encoding error x_e on the time interval $[\tilde{r}_k, \tilde{r}_{k+1})$ can be written as

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t), \tag{5a}$$

$$\dot{x}_e(t) = Ax_e(t). \tag{5b}$$

While the encoder knows the encoding error x_e precisely, the decoder can only compute a bound $d_e(t)$ on $\|x_e(t)\|_\infty$ as follows

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty \delta_k, \ t \in [\tilde{r}_k, \tilde{r}_{k+1}), \ k \in \mathbb{Z}_{\geq 0} \tag{6a}$$

$$\delta_{k+1} = \frac{1}{2^{p_{k+1}}} d_e(t_{k+1}). \tag{6b}$$

One can design a pair of algorithms for the encoder and the decoder to implement (4b) in a manner that they maintain consistent $\hat{x}(t)$ and $d_e(t)$ signals for $t \geq t_0$ (see [23] for example). For brevity, we do not present these algorithms here and it suffices to say that $\|x_e(t)\|_\infty \leq d_e(t)$ for all $t \geq t_0$ if $\|x_e(t_0)\|_\infty \leq d_e(t_0)$.

D. Control objective

We measure the performance of the closed-loop system through a Lyapunov function. Given an arbitrary symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$, let P be the unique symmetric positive definite matrix satisfying $P\bar{A} + \bar{A}^T P = -Q$. Define $x \mapsto V(x) = x^T P x$ and let

$$V_d(t) = V_d(t_0)e^{-\beta(t-t_0)}, \tag{7}$$

with $\beta > 0$, be the desired *control performance*. We assume

$$W \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - a\beta > 0, \tag{8}$$

with $a > 1$ an arbitrary constant. Assumption (8) is sufficient to guarantee a convergence rate faster than β for the dynamics (1) under the continuous-time, unquantized feedback $u(t) = Kx(t)$.

Our objective is to design an event-triggered communication and control strategy that ensures the exponential stability of the origin. Formally, we seek to synthesize an event-triggered control strategy that recursively determines the sequences of transmission times $\{t_k\}_{k \in \mathbb{Z}_{>0}}$ and update times $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$, along with a coding scheme for messages and a rule to determine the number of bits $\{b_k\}_{k \in \mathbb{Z}_{>0}}$ to be transmitted, so that $V(x(t)) \leq V_d(t)$, for all $t \geq t_0$.

III. Trigger functions

To achieve the control objective of Section II-D with opportunistic transmissions, we need a performance-trigger function that tells us how close the system state is to violating the convergence requirement. Bounded precision quantization further requires us to keep track (through a channel-trigger function) of the number of bits required at any moment to guarantee performance at least for a certain period of time. Threshold crossings of these two functions form the primary basis of our event-triggering mechanism.

A. Performance-trigger function

The *performance-trigger* function is the ratio between the Lyapunov function V and the desired performance V_d,

$$h_{\text{pf}}(t) \triangleq \frac{V(x(t))}{V_d(t)}. \tag{9}$$

Note that the control objective is to maintain $h_{\text{pf}}(t) \leq 1$ at all times. This is why, in general, it is of interest to characterize the open-loop evolution of the performance-trigger function. The next result provides an upper bound on the value of h_{pf} in the future as a function of the information available now.

Lemma 3.1: (Upper bound on open-loop evolution of performance-trigger function [23]). Given $t_k \in \mathbb{R}_{>0}$ such that $h_{\text{pf}}(t_k) \leq 1$, then

$$h_{\text{pf}}(\tau + t_k) \leq \bar{h}_{\text{pf}}(\tau, h_{\text{pf}}(t_k), \epsilon(t_k)),$$

for $\tau \geq 0$, where

$$\epsilon(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}}, \quad \bar{h}_{\text{pf}}(\tau, h_0, \epsilon_0) \triangleq \frac{f_1(\tau, h_0, \epsilon_0)}{f_2(\tau)}, \tag{10}$$

$$f_1(\tau, h_0, \epsilon_0) \triangleq h_0 + \frac{W\epsilon_0}{w+\mu}(e^{(w+\mu)\tau} - 1), \quad f_2(\tau) \triangleq e^{w\tau},$$

$$c \triangleq \frac{W\sqrt{\lambda_m(P)}}{2\sqrt{n}\|PBK\|_2}, \ w \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - \beta > 0, \ \mu \triangleq \|A\|_2 + \frac{\beta}{2}.$$

This result motivates the definition of the function

$$\Gamma_1(h_0, \epsilon_0) \triangleq \min\{\tau \geq 0 : \bar{h}_{\text{pf}}(\tau, h_0, \epsilon_0) = 1, \ \frac{\mathrm{d}\bar{h}_{\text{pf}}}{\mathrm{d}\tau} \geq 0\},$$

as a lower bound on the time it takes h_{pf} to evolve to 1 starting from $h_{\text{pf}}(t_k) = h_0$ with $\epsilon(t_k) = \epsilon_0$.

B. Channel-trigger function

We define, for $h_0 \in [0, 1]$ and a design parameter $T > 0$,

$$\rho_T(h_0) \triangleq \frac{(w+\mu)(1-h_0)}{W(e^{(w+\mu)T} - 1)} + 1, \tag{11}$$

and the *channel-trigger* function as

$$h_{\text{ch}}(t) \triangleq \frac{\epsilon(t)}{\rho_T(h_{\text{pf}}(t))}. \tag{12}$$

The channel-trigger function $h_{\rm ch}$ depends on the bound on the encoding error d_e through ϵ. Note that the channel-trigger function $h_{\rm ch}$ through its dependence on d_e, which evolves as (6), also jumps at the update times $\tilde{r}_k$. In turn, for any time $s_0 \geq t_0$, if $h_{\rm ch}(s_0) \leq 1$, then $h_{\rm pf}(t) \leq 1$ for at least $t \in [s_0, s_0 + \min\{T, \Gamma_1(1,1)\})$ even without any transmissions or receptions. Thus, assuming that the communication delays are smaller than $\min\{T, \Gamma_1(1,1)\}$, a transmission strategy is to ensure that, for each k, $h_{\rm ch}(\tilde{r}_k) \leq 1$ so that $\Gamma_1(h_{\rm pf}(\tilde{r}_k), \epsilon(\tilde{r}_k)) \geq \min\{T, \Gamma_1(1,1)\}$. Thus, we now require an upper bound on the open-loop evolution of $h_{\rm ch}$, which is provided in the following result.

Lemma 3.2: (Upper bound on the channel-trigger function at the update times $\tilde{r}_k$). If $t_k \in \mathbb{R}_{>0}$ is such that $h_{\rm pf}(t_k) \in [0,1]$, then

$$h_{\rm ch}(\tilde{r}_k) \leq \bar{h}_{\rm ch}(\tilde{r}_k - t_k, h_{\rm pf}(t_k), \epsilon(t_k), p_k), \quad (13)$$

where $b_k = np_k$ bits are transmitted at t_k and

$$\bar{h}_{\rm ch}(\tau, h_0, \epsilon_0, p) \triangleq \frac{\|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}\epsilon_0}{\rho_T(\bar{h}_{\rm pf}(\tau, h_0, \epsilon_0))} \cdot \frac{1}{2^p}. \quad (14)$$

Note that for $t, t+\tau \in [\tilde{r}_k, t_{k+1})$, for any $k \in \mathbb{Z}_{\geq 0}$, we have $h_{\rm ch}(t+\tau) \leq \bar{h}_{\rm ch}(\tau, h_{\rm pf}(t), \epsilon(t), 0)$. Now, analogous to Γ_1, we define

$$\Gamma_2(b_0, \epsilon_0, p) \triangleq \min\{\tau \geq 0 : \bar{h}_{\rm ch}(\tau, b_0, \epsilon_0, p) = 1\}, \quad (15)$$

which essentially is an upper bound on the communication delay $\tilde{r}_k - t_k$, for which we can still guarantee $h_{\rm ch}(\tilde{r}_k) \leq 1$. Given the interpretation of Γ_2, one of the conditions in our event-triggering rule would be to check if Γ_2 is less than a maximum communication delay.

Lemma 3.3: (Lower bound on Γ_2). If $\epsilon_0 \in [0, \rho_T(h_0)]$ then $\Gamma_2(h_0, \epsilon_0, p) \geq T^*(p)$ with

$$T^*(p) \triangleq \min\{\tau \geq 0 : g(\tau, p) = 1\},$$
$$g(\tau, p) \triangleq \frac{\|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}}{2^p} \cdot \frac{e^{(w+\mu)T} - 1}{e^{(w+\mu)T} - e^{(w+\mu)\tau}}.$$

IV. Characterization of the data capacity

Our study of data capacity here is motivated by the need of the encoder to know how much data can be transmitted successfully before a channel blackout.

A. Data capacity

We denote the data capacity during the time interval $[\tau_1, \tau_2]$ by $\mathcal{D}(\tau_1, \tau_2)$ and define it as the maximum data that can be communicated, bits that are transmitted and also received, during the time interval under *all* possible communication delays, i.e.,

$$\mathcal{D}(\tau_1, \tau_2) \triangleq \max_{\substack{\{t_k\}, \{p_k\} \\ \text{s.t. (3) holds}}} n \sum_{k=\underline{k}_{\tau_1}}^{\overline{k}_{\tau_2}} p_k, \quad (16)$$

where $\underline{k}_{\tau_1} = \min\{k : t_k \geq \tau_1\}$ and $\overline{k}_{\tau_2} = \max\{k : t_k + \tilde{\Delta}_k \leq \tau_2\}$. Note that a greedy approach does not necessarily maximize the communicated data. In general, the precise computation of $\mathcal{D}(\tau_1, \tau_2)$ involves solving an integer program with non-convex feasibility constraints. Given the difficulty of solving this problem, we seek a class of channel functions R and $\bar{b}$ that are meaningful and yet simple enough to efficiently compute a lower bound for the data capacity. To this end, we make the following observation.

Lemma 4.1: (Data capacity under constant communication rate). Suppose $\forall t \in [\tau_1, \tau_2]$ (i) $R(t) = R \geq 0$ and (ii) $\bar{p}(t) \geq 1$ (no blackouts). Then, $\mathcal{D}(\tau_1, \tau_2) = n\lfloor R(\tau_2 - \tau_1)\rfloor$.

Motivated by this result, we assume that the channel function R is piecewise constant so that the problem of finding a reasonable lower bound on $\mathcal{D}(\tau_1, \tau_2)$ is tractable while also ensuring that the overall problem is meaningful. According to (2b), R is a lower bound on the instantaneous communication rate and it is reasonable to assume it is piecewise constant. Also, note that $\bar{p}$ takes integer values and hence is piecewise constant. Specifically, we assume that

$$R(t) = R_j, \quad \forall t \in (\theta_j, \theta_{j+1}] \quad (17a)$$
$$\bar{p}(t) = \bar{\pi}_j, \quad \forall t \in (\theta_j, \theta_{j+1}] \quad (17b)$$

where $\{\theta_j\}_{j=0}^\infty$ is a strictly increasing sequence of time instants and $\bar{\pi}_j \in \mathbb{Z}_{\geq 0}$ for each j. We also denote $T_j \triangleq \theta_{j+1} - \theta_j$ as the length of the $j^{\rm th}$ time slot $I_j \triangleq (\theta_j, \theta_{j+1}]$. Again note that identical $\{\theta_j\}$ sequences for R and $\bar{p}$ is not a restriction because one can always refine the sequence $\{\theta_j\}$. We assume, without loss of generality, that $\tau_1 = \theta_{j_0}$ and $\tau_2 = \theta_{j_f}$, for some $j_0, j_f \in \mathbb{Z}_{\geq 0}$.

B. Formulation as an allocation problem

Here we show that, for piecewise constant channel functions, we can think of the computation of $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ as an allocation problem: that of allocating the number of bits $\{n\phi_j\}$, with $\phi_j \in \mathbb{Z}_{\geq 0}$, to be transmitted in the time slots $\{I_j\}$ for $j \in \mathcal{N}_{j_0}^{j_f} \triangleq \{j_0, \ldots, j_f - 1\}$. For convenience, we let $\phi_{j_0}^{j_f} \triangleq (\phi_{j_0}, \ldots, \phi_{j_f-1})$. Given $\phi_{j_0}^{j_f}$, the sequences $\{t_k\}$ and $\{p_k\}$ are determined so that transmissions start at the earliest possible time in I_j and the channel is not idle until all the allocated bits ϕ_j are received, i.e., $t_{k+1} = \tilde{r}_k = r_k = \Delta(t_k, p_k)$ during I_j and $\{p_k\}$ during I_j is any sequence that respects the channel upper bound $\bar{\pi}_j$ and adds up to ϕ_j. Our forthcoming discussion focuses on expressing the constraints in the optimization problem in terms of the ϕ variables. In the sequel, a standing constraint is that $\phi_j \in \mathbb{Z}_{\geq 0}$ for each j, unless we mention otherwise.

Maximum bits that may be transmitted: According to Lemma 4.1, in the time slot I_j, $n\lfloor R_j T_j \rfloor$ bits could be transmitted and received within $\lfloor R_j T_j \rfloor / R_j \leq T_j$ units of time. In addition, $n\bar{\pi}_j$ more bits could be transmitted during the closed interval $[\ \lfloor R_j T_j \rfloor, \theta_{j+1}]$, though these bits are received only in subsequent time slots. Thus, for $j \in \mathcal{N}_{j_0}^{j_f}$

$$n\phi_j \leq \begin{cases} nR_jT_j + n\bar{\pi}_j, & \text{if } \bar{\pi}_j > 0 \\ 0, & \text{if } \bar{\pi}_j = 0 \end{cases} \quad (18)$$

where in the first case we have used the fact that $\phi_j \in \mathbb{Z}_{\geq 0}$ to avoid the use of the floor function.

Reduced channel availability in a time slot due to prior transmissions: As noted above, if $\phi_j > \lfloor R_j T_j \rfloor$, then these bits take up some of the time in I_{j+1} and possibly even subsequent slots. Thus, effectively the time available in I_{j+1} and consequently the upper bound on ϕ_{j+1} is reduced. Moreover, in general, the number of bits transmitted in I_j has an effect on the number that could be transmitted in all subsequent intervals either directly or indirectly. Thus, for each $j_1, j \in \mathcal{N}_{j_0}^{j_f}$, we introduce

$$\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \triangleq \Big(T_j - \sum_{i=j_1}^{j-1}\big(\frac{\phi_i}{R_i} - T_i\big)\Big) = \theta_{j+1} - \theta_{j_1} - \sum_{i=j_1}^{j-1}\frac{\phi_i}{R_i}.$$

As we shall see in the next result, these functions determine the available time in slot I_j given $\phi_{j_0}^{j_f}$.

Lemma 4.2: (Available time in slot I_j). Let $\bar{T}_j(\phi_{j_0}^{j_f})$ be the time available in the slot I_j given the allocation $\phi_{j_0}^{j_f}$. Then,

$$\bar{T}_j(\phi_{j_0}^{j_f}) = \Big[\min_{j_1 \in \mathcal{N}_{j_0}^{j_f}} \{\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}), T_j\}\Big]_+.$$

As a consequence of Lemma 4.2, for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1]$, consider the constraints

$$n\phi_j \leq \begin{cases} nR_j\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) + n\bar{\pi}_j, & \text{if } \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{19a}$$

which we obtain using the same reasoning as in (18) with T_j replaced by $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f})$. Note that if $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \geq T_j$, then the constraint (19a) is weaker than (18) and hence inactive. For $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \in (0, T_j)$, the constraint reflects the reduced available time in the time slot I_j and if $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \leq 0$, for some $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1]$, then it corresponds to the case when the channel is busy for the whole time slot I_j ($\bar{T}_j(\phi_{j_0}^{j_f}) = 0$). Thus (19a) reflects the effect of reduced available time during the slot I_j due to prior transmissions.

Counting only the bits transmitted and received during $[\theta_{j_0}, \theta_{j_f}]$: Finally, since in the computation of $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$, we are interested in the maximum number of bits that can be communicated (transmitted and received), we require that any bits transmitted during the slot I_j are received before θ_{j_f}, i.e., for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j]$

$$\frac{\phi_j}{R_j} \leq \begin{cases} \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) + \theta_{j_f} - \theta_{j+1}, & \text{if } \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{19b}$$

Then, the data capacity is given as

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) = \max_{\substack{\phi_j \in \mathbb{Z}_{\geq 0}, \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (18), (19) hold}}} n \sum_{j=j_0}^{j_f-1} \phi_j. \tag{20}$$

Ignoring the fact that this is an integer program, the constraints (19) still make the problem combinatorial.

C. Efficient approximation of data capacity

The following result is the basis for the construction of a sub-optimal and efficient solution to the problem (20).

Lemma 4.3: (Bound on "channel variation"). If there exists $J \in \mathbb{Z}_{\geq 0}$ such that

$$\frac{\bar{\pi}_j}{R_j} < \sum_{i=j+1}^{i=j+1+J} T_i, \quad \forall j \in \mathcal{N}_{j_0}^{j_f}, \tag{21}$$

then, for any $j \in \mathcal{N}_{j_0}^{j_f}$, any bits transmitted in time slot I_j would be received strictly before the end of the slot I_{j+1+J}.

Lemma 4.3 relates the three sequences of parameters, $\{R_j\}$, $\{\bar{\pi}_j\}$ and $\{T_j\}$, that define the channel state at any given time. The parameter J may be interpreted as a uniform upper bound on the number of consecutive time slots that may be fully occupied due to a prior transmission.

1) Guaranteed channel availability in each time slot: We address here the case of $J = 0$, which is of special interest.

Lemma 4.4: (Data capacity in the case of $J = 0$). Suppose the channel is such that $J = 0$ for all $j \in \mathcal{N}_{j_0}^{j_f}$. Then, the constraints (19a) reduce to

$$n\phi_j + nR_j \sum_{i=j_1}^{j-1} \frac{\phi_i}{R_i} \leq nR_j(\theta_{j+1} - \theta_{j_1}) + n\bar{\pi}_j, \tag{22a}$$

for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1]$ while the constraints (19b) reduce to

$$\sum_{i=j_1}^{j_f-1} \frac{\phi_i}{R_i} \leq \theta_{j_f} - \theta_{j_1}, \tag{22b}$$

for each $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j_f - 1]$. The data capacity is

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) = \max_{\substack{\phi_j \in \mathbb{Z}_{\geq 0}, \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (18), (22) hold}}} n \sum_{j=j_0}^{j_f-1} \phi_j. \tag{23}$$

Note that for $J = 0$ all the constraints, (18) and (22) are linear, though ϕ_j are still restricted to be integers. This brings us to the next result.

Proposition 4.5: (A sub-optimal solution and quantification of sub-optimality in the case of $J = 0$). Suppose the channel is such that $J = 0$ for all $j \in \mathcal{J} = \{j_0, \ldots, j_f\}$. Let $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \triangleq n\sum_{j=j_0}^{j_f-1} \phi_j^N$ where $\phi^N \triangleq \lfloor \phi^r \rfloor \triangleq (\lfloor \phi_{j_0}^r \rfloor, \ldots, \lfloor \phi_{j_f-1}^r \rfloor)$,

$$\phi^r = \operatorname*{argmax}_{\substack{\phi_j \in \mathbb{R}_{\geq 0}, \ \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (18), (22) hold}}} \sum_{j=j_0}^{j_f-1} \phi_j.$$

Then ϕ^N is a sub-optimal solution to (23), i.e., $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \leq \mathcal{D}(\theta_{j_0}, \theta_{j_f})$ and

$$\begin{aligned}\mathcal{D}(\theta_{j_0}, \theta_{j_f}) &- \mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \\ &\leq n|\{j \in \mathbb{Z}_{\geq 0} \cap [j_0, j_{f-1}] : \bar{\pi}_j > 0\}|.\end{aligned}$$

2) No guaranteed channel availability: If $J > 0$, we forgo optimality in favor of an easily computable lower bound of the data capacity. With a slight abuse of notation, we let

$$\phi_j^N = \lfloor R_j(\theta_{j+1} - \theta_j) \rfloor, \quad j \in \mathbb{Z}_{\geq 0},$$

which is the number of bits that can be communicated (transmitted and received) during the time slot $I_j = [\theta_j, \theta_{j+1})$. Hence, $\{\phi_j^N\}_{j \in \mathbb{Z}_{\geq 0}}$ is a feasible solution and, again with an abuse of notation, we denote

$$\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f-1} \phi_j^N,$$

which is a sub-optimal lower bound of the data capacity.

D. Computing data capacity in real time

As mentioned earlier, we want the encoder to compute a lower bound for the data capacity up to the end of the next blackout period. However, the computation of $\mathcal{D}_s(\tau_1, \tau_2)$ in the case of $J = 0$ involves solving a linear program and hence may not be suitable for real-time computation. Thus, given $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ (or $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f})$), we propose a simpler procedure to compute a lower bound on $\mathcal{D}(t, \theta_{j_f})$ (or $\mathcal{D}_s(t, \theta_{j_f})$) for any $t \in [\theta_{j_0}, \theta_{j_0+1})$. We present the procedure in the following result.

Proposition 4.6: (Real-time computation of data capacity). Let ϕ^* (or ϕ^N) be any optimizing solution to $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ (or $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f})$). Let

$$\hat{\mathcal{D}}(t, \theta_{j_f}) \triangleq \left[n \left\lfloor \phi^*_{j_0} - R_{j_0}(t - \theta_{j_0}) \right\rfloor\right]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi^*_j$$

$$\hat{\mathcal{D}}_s(t, \theta_{j_f}) \triangleq \left[n \left\lfloor \phi^N_{j_0} - R_{j_0}(t - \theta_{j_0}) \right\rfloor\right]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi^N_j,$$

for any $t \in [\theta_{j_0}, \theta_{j_0+1})$. Then, $0 \leq \mathcal{D}(t, \theta_{j_f}) - \hat{\mathcal{D}}(t, \theta_{j_f}) \leq n$ and $0 \leq \mathcal{D}_s(t, \theta_{j_f}) - \hat{\mathcal{D}}_s(t, \theta_{j_f}) \leq n$.

Proposition 4.6 provides a method to reuse a previously computed solution to find a tight sub-optimal solution to the data capacity problem in real-time.

V. Event-triggered stabilization

In this section, we address the problem of event-triggered control under a time-varying channel. Section V-A address the case with no channel blackouts. Section V-B builds on this design and analysis to deal with channel blackouts.

A. Control in the absence of channel blackouts

In the case of no channel blackouts, the encoder may choose to transmit at any time and, in addition, we assume the channel rate R is sufficiently high at all times so that there is no need to resort to the computation of data capacity. For this reason, we are able to consider arbitrary (not necessarily piecewise constant) functions $t \mapsto R(t)$. For $p \in \mathbb{Z}_{\geq 0}$, let

$$T_M(p) = \sigma \min\{\Gamma_1(1, 1), T, T^*(p)\}, \quad (24)$$

where $\sigma \in (0, 1)$ is a design parameter, T is the parameter chosen in (11) and T^* is as defined in Lemma 3.3. As we show in the sequel, if $T_M(p)$ is an upper bound on the communication delay when $b = np$ bits are transmitted, then it is sufficient to design an event-triggering rule that guarantees the control objective is met.

In the presence of communication delays, we need to make sure that (i) the control objective is not violated between a transmission and the resulting control update and (ii) at the control update times, the encoding error is sufficiently small to ensure future performance. To this end, we define

$$\mathcal{L}_1(t) \triangleq \bar{h}_{\text{pf}}\left(T_M(\bar{p}(t)), h_{\text{pf}}(t), \epsilon(t)\right), \quad (25a)$$

$$\mathcal{L}_2(t) \triangleq \bar{h}_{\text{ch}}\left(T_M(\bar{p}(t)), h_{\text{pf}}(t), \epsilon(t), \bar{p}(t)\right), \quad (25b)$$

to take care of each of these requirements. If up to $\bar{b} = n\bar{p}$ bits are transmitted at time t, then $\mathcal{L}_1(t)$ provides an upper bound on the performance-trigger function h_{pf} at the reception time which would be less than $t + T_M(\bar{p}(t))$, while $\mathcal{L}_2(t)$ provides an upper bound on the channel-trigger function h_{ch} if the control is updated as soon as the packet is received.

Theorem 5.1: (Event-triggered control in the absence of blackouts). Suppose $t \mapsto \bar{p}(t)$ is piecewise constant, as in (17b), with a uniform lower bound 1 (i.e., no blackouts) and a uniform upper bound $p^{\max}$. Assume that

$$R(t) \geq \frac{p}{T_M(p)}, \ \forall p \in \{1, \ldots, \bar{p}(t)\}, \ \forall t. \quad (26)$$

Consider the system (1) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$ determined recursively by

$$t_{k+1} = \min\{t \geq \tilde{r}_k : \mathcal{L}_1(t) \geq 1 \ \vee \ \mathcal{L}_1(t^+) \geq 1 \ \vee \ \mathcal{L}_2(t) \geq 1 \ \vee \ \mathcal{L}_2(t^+) \geq 1\}. \quad (27)$$

Let $\{r_k\}_{k \in \mathbb{Z}_{\geq 0}}$ and $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0 = t_0$ and $\tilde{r}_k = r_k \leq t_k + \Delta_k$ for $k \in \mathbb{Z}_{>0}$. Assume the encoding scheme is such that (6) is satisfied for all $t \geq t_0$. Also assume that $\mathcal{L}_1(t_0) \leq 1$, $\mathcal{L}_2(t_0) \leq 1$ and that (8) holds. Let $\underline{p_k}$ be

$$\underline{p_k} \triangleq \min\{p \in \mathbb{Z}_{>0} : \bar{h}_{\text{ch}}\left(\frac{p}{R(t_k)}, h_{\text{pf}}(t_k), \epsilon(t_k), p\right) \leq 1\}.$$

Then, the following hold:

(i) $\underline{p_1} \leq \bar{p}(t_1)$. Further for each $k \in \mathbb{Z}_{>0}$, if $p_k \in \mathbb{Z}_{>0} \cap [\underline{p_k}, \bar{p}(t_k)]$, then $\underline{p_{k+1}} \leq \bar{p}(t_{k+1})$.

(ii) the inter-transmission times $\{t_{k+1} - t_k\}_{k \in \mathbb{Z}_{>0}}$ and inter-update times $\{\tilde{r}_{k+1} - \tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$ have a uniform positive lower bound,

(iii) the origin is exponentially stable for the closed-loop system, with $V(x(t)) \leq V_d(t_0)e^{-\beta(t-t_0)}$ for $t \geq t_0$.

The interpretation of the three claims of the result is as follows. Claim (i) essentially states that if the number of bits transmitted in the past is according to the given recommendation, then in the future, the sufficient number of bits $\underline{b_k} = n\underline{p_k}$ to guarantee continued performance will respect the time-varying channel constraints. Claim (ii) is sufficient to guarantee non-Zeno behavior and claim (iii) states that indeed the control objective is met.

B. Control in the presence of channel blackouts

Here, we address the scenario of channel blackouts building on our developments in Section V-A. The main difficulty comes from the fact that in the presence of blackouts, the channel might be completely unavailable. Thus, the event-triggering condition not only needs to be based on the functions $\mathcal{L}_1$ and $\mathcal{L}_2$ in (25), but also on the available data capacity up to the next blackout.

Throughout the section, we assume both R and $\bar{p}$ are piecewise constant functions, as in (17) and, without loss of generality, that time slots with $\bar{p} = 0$ are not consecutive. We let $B_k \triangleq (\theta_{j_k}, \theta_{j_k+1}]$ denote the k^{th} blackout slot, with $k \in \mathbb{Z}_{>0}$. Also, for any $t \geq t_0$, we let

$$\tau_l(t) \triangleq \min\{s \geq t : \bar{p}(s) = 0\},$$

$$\tau_u(t) \triangleq \min\{s \geq \tau_l(t) : \bar{p}(s) > 0\},$$

give, respectively, the beginning and the end times of the next channel blackout slot from the current time t. When there is no confusion, we simply use τ_l and τ_u, dropping the argument t. Hence, for $t \in [t_0, \theta_{j_1})$, we have $\tau_l(t) = \theta_{j_1}$ and $\tau_u(t) = \theta_{j_1+1}$. Similarly, for any $k \in \mathbb{Z}_{>0}$ and $t \in (\theta_{j_k}, \theta_{j_{k+1}}]$, we have $\tau_l(t) = \theta_{j_{k+1}}$ and $\tau_u(t) = \theta_{j_{k+1}+1}$. At time t, the length of the next channel blackout slot, $T_b(t) \triangleq \tau_u(t) - \tau_l(t)$, determines a sufficient upper bound on the encoding error $d_e(\tau_l)$, or equivalently $\epsilon(\tau_l)$, for non-violation of the control objective during the blackout or immediately subsequent to it. We quantify it next.

Lemma 5.2: (Upper bound on required ϵ before blackout). For $t \in [t_0, \infty)$, suppose

$$\epsilon(\tau_l(t)) \leq \epsilon_r(t) \triangleq \min \left\{ \frac{(e^{wT_b(t)} - 1)(w + \mu)}{W(e^{(w+\mu)T_b(t)} - 1)}, \frac{1}{e^{\bar{\mu} T_b(t)}} \right\}, \tag{28}$$

where $\bar{\mu} \triangleq \|A\|_\infty + \frac{\beta}{2}$. If $h_{\text{pf}}(\tau_l(t)) \leq 1$, then $h_{\text{pf}}(s) \leq 1$ for all $s \in [\tau_l(t), \tau_u(t)]$ and $h_{\text{ch}}(\tau_u(t)) \leq 1$ (in particular $\epsilon(\tau_u(t)) \leq 1$).

The ability to ensure that $\epsilon(\tau_l)$ is sufficiently small is determined by the data capacity $\mathcal{D}(t, \tau_l)$. To have a real-time implementation, we make use of the sub-optimal lower bound $\hat{\mathcal{D}}_s(t, \tau_l)$ instead. However, notice that maximizing the data throughput and satisfying the primary control goal of exponential convergence at a desired rate may not be compatible in general. Thus, to still be able to use the intuition and the building blocks from Section V-A, we need to impose a time-varying artificial bound on the allowed packet size in place of $\bar{p}(t)$ that prevents the system from affecting the data capacity until the next blackout. To this end, we store in the variable $\mathcal{P}_j$ the value of ϕ_j^N, where ϕ^N is as defined in Section IV-C for $\mathcal{D}_s(\theta_j, \tau_l(\theta_j))$. Define

$$\Phi^{\tau_l}(t) \triangleq [\lfloor \mathcal{P}_j - R_j(t - \theta_j) \rfloor]_+, \ t \in (\theta_j, \theta_{j+1}]. \tag{29}$$

We notice from Proposition 4.6 that $n\Phi^{\tau_l}(t)$ is the optimal number of bits to be transmitted during $(t, \theta_{j+1}]$ to obtain the sub-optimal data capacity $\hat{\mathcal{D}}_s(t, \tau_l(t))$. Note that some of $n\Phi^{\tau_l}(t)$ bits may be received after θ_{j+1}. Now, we let

$$\psi^{\tau_l}(t) \triangleq \min\{\bar{p}(t), \Phi^{\tau_l}(t)\} \tag{30}$$

be the artificial bound on the packet size for transmissions. Notice that $\Phi^{\tau_l}(t)$ may at times be zero, even when $\bar{p}(t) > 0$, which means letting $\psi^{\tau_l}(t)$ be the bound on packet size may itself introduce *artificial blackouts*. However, we can state how long artificial blackouts may be, as the next result shows.

Lemma 5.3: (Upper bound on the length of artificial blackouts). Let $\tilde{B}_j \triangleq \{t \in I_j = (\theta_j, \theta_{j+1}] : \psi^{\tau_l}(t) = 0\}$. Then, for each $j \in \mathbb{Z}_{\geq 0}$, $\tilde{B}_j$ is an interval and if $\bar{\pi}_j > 0$, then the length of $\tilde{B}_j$ is less than $2/R_j = 2/R(\theta_{j+1})$.

With this in place, we define functions analogous to $\mathcal{L}_1$ and $\mathcal{L}_2$ to, respectively, monitor the compliance with the control objective and ensure the encoding error is sufficiently small at the control update times to ensure future performance. In addition, we define one more function to capture the effect of the data capacity,

$$\tilde{\mathcal{L}}_1(t) \triangleq \bar{h}_{\text{pf}}\left(\mathcal{T}(t), h_{\text{pf}}(t), \epsilon(t)\right), \tag{31a}$$

$$\tilde{\mathcal{L}}_2(t) \triangleq \bar{h}_{\text{ch}}\left(\mathcal{T}(t), h_{\text{pf}}(t), \epsilon(t), \psi^{\tau_l}(t)\right), \tag{31b}$$

$$\mathcal{L}_3(t, \epsilon) \triangleq n \log_2 \left(\frac{e^{\bar{\mu}(\tau_l(t) - t)} \epsilon}{\epsilon_r(t)} \right) - \sigma_1 \hat{\mathcal{D}}_s(t, \tau_l(t)), \tag{31c}$$

where $\sigma_1 \in (0, 1)$ is a design parameter and

$$\mathcal{T}(t) \triangleq \begin{cases} T_M(\psi^{\tau_l}(t)), & \text{if } \psi^{\tau_l}(t) \geq 1 \\ \frac{2}{R(t)}, & \text{if } \psi^{\tau_l}(t) = 0. \end{cases}$$

Theorem 5.4: (Event-triggered control in the presence of blackouts). Suppose $t \mapsto R(t)$ and $t \mapsto \bar{p}(t)$ are piecewise constant functions as in (17). Let $\{(\theta_{j_k}, \theta_{j_k+1}]\}_{k \in \mathbb{Z}_{>0}}$ be a sequence of channel blackout slots. Assume that $\bar{p}(t_0) > 0$ and that the piecewise constant function $\bar{p}$ is uniformly upper bounded by $p^{\max} \in \mathbb{Z}_{>0}$. Also, assume

$$R(t) \geq \frac{(p+2)}{T_M(p)}, \ \forall p \in \{1, \ldots, p^{\max}\}, \ \forall t. \tag{32}$$

Consider the system (1) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$ determined recursively by

$$\begin{aligned} t_{k+1} = \min \Big\{ t \geq \ \tilde{r}_k : \ \psi^{\tau_l}(t) \geq 1 \ \wedge \\ \Big(\max\{\tilde{\mathcal{L}}_1(t), \tilde{\mathcal{L}}_1(t^+), \tilde{\mathcal{L}}_2(t), \tilde{\mathcal{L}}_2(t^+)\} \geq 1 \ \vee \\ \max\{\tilde{\mathcal{L}}_3(t), \tilde{\mathcal{L}}_3(t^+)\} \geq 0 \Big) \Big\}, \end{aligned} \tag{33}$$

where $\tilde{\mathcal{L}}_3(t) \triangleq \mathcal{L}_3(t, \epsilon(t))$. Let $\{r_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0 = t_0$ and $r_k \leq t_k + \Delta_k$ for $k \in \mathbb{Z}_{>0}$. Let the update times $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0$ and for $k \in \mathbb{Z}_{>0}$

$$\tilde{r}_k = \min\{t \geq r_k : \psi^{\tau_l}(t) \geq 1 \ \vee \ \bar{p}(t) = 0\}. \tag{34}$$

Assume the encoding scheme is such that (6) is satisfied for all $t \geq t_0$. Further assume that (8) holds and that $\tilde{\mathcal{L}}_1(t_0) \leq 1$, $\tilde{\mathcal{L}}_2(t_0) \leq 1$, $\mathcal{L}_3(t_0, \epsilon(t_0)) \leq 0$ and, for each $k \in \mathbb{Z}_{>0}$, assume $\mathcal{L}_3(\theta_{j_k+1}, 1) \leq 0$. Let $\underline{p_k}$ be given by

$$\underline{p_k} \triangleq \min\{p \in \mathbb{Z}_{>0} : \bar{h}_{\text{ch}}\left(T_M(p), h_{\text{pf}}(t_k), \epsilon(t_k), p\right) \leq 1\}.$$

Then, the following hold:

(i) $\underline{p_1} \leq \psi^{\tau_l}(t_1)$. Further for each $k \in \mathbb{Z}_{>0}$, if $p_k \in \mathbb{Z}_{>0} \cap [\underline{p_k}, \psi^{\tau_l}(t_k)]$, then $\underline{p_{k+1}} \leq \psi^{\tau_l}(t_{k+1})$.

(ii) the inter-transmission times $\{t_{k+1} - t_k\}_{k \in \mathbb{Z}_{>0}}$ and inter-update times $\{\tilde{r}_{k+1} - \tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$ have a uniform positive lower bound,

(iii) the origin is exponentially stable for the closed-loop system, with $V(x(t)) \leq V_d(t_0) e^{-\beta(t - t_0)}$ for $t \geq t_0$.

Claim (i) in the result may be interpreted as the satisfaction of the constraints imposed by the channel. The use of ψ^{τ_l} in (33) and (34) also ensures that the data capacity is not lowered at any time in the future due to past transmissions. Figure 1 shows an implementation of the design in Theorem 5.4 on the system (1) with

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad K = \begin{bmatrix} -6 & -6 \end{bmatrix}.$$

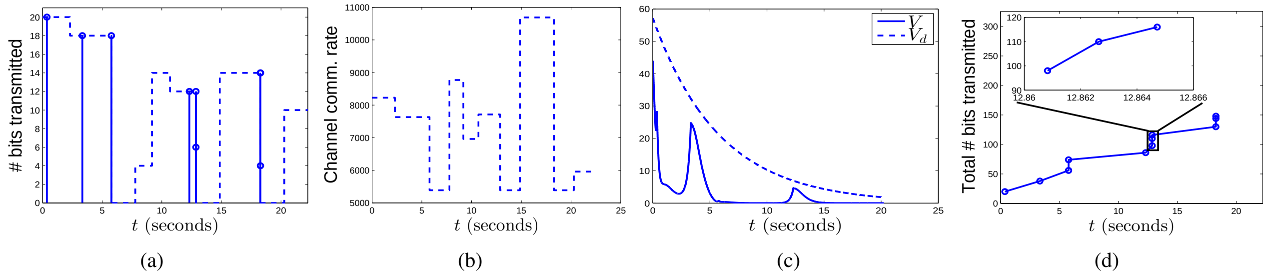

Fig. 1. Execution of the design in Theorem 5.4. On each transmission, the maximum possible number of bits are transmitted. The average inter-transmission interval is 1.84 and the minimum is 0.0017. (a) shows the transmission times, the number of bits transmitted on each transmission and the time-varying function $n\bar{p}$ (dashed line). The three intervals, $(5.77, 7.77]$, $(12.86, 14.86]$ and $(18.27, 20.27]$, with $\bar{p} = 0$ are the blackouts. (b) shows the time-varying function R. (c) shows the evolution of V and V_d and (d) shows the total number of bits transmitted.

VI. Conclusions

We have addressed the problem of event-triggered control of linear time-invariant systems under time-varying rate-limited communication channels. The class of channels we consider includes intermittent occurrence of channel blackouts. We have designed an event-triggered control scheme that, using prior knowledge of the channel, guarantees the exponential stabilization of the system at a desired convergence rate, even in the presence of intermittent channel blackouts. Key enablers of our design are the definition and analysis of the data capacity, which measures the maximum number of bits that can be communicated over a given time interval through one or more transmissions. We have also provided an efficient real-time algorithm to lower bound the data capacity for a time-slotted model of channel evolution. Future work will explore the reduction of the conservatism of the proposed design, scenarios with bounded disturbances, a stochastic model of channel evolution, and the trade-offs between the available information pattern at the encoder and the ability to perform event-triggered control.

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