Reachability metrics for bilinear complex networks

Yingbo Zhao and Jorge Cortés

Abstract— **Controllability metrics based on the controllability Gramian have been widely used in linear control theory, and have recently seen renewed interests in the study of complex networks of dynamical systems. For example, the minimum eigenvalue and the trace of the Gramian are related to the worst-case and average minimum input energy, respectively, to steer the state from the origin to a target state. This paper explores similar questions that remain unanswered for bilinear control systems. In the context of complex networks, bilinear systems characterize scenarios where an actuator not only can affect the state of a node, but also can affect the strength of the interconnections among some neighboring nodes. Under the assumption that the infinity norm of the input is bounded by some function of the network dynamic matrices, we derive a lower bound on the minimum input energy to steer the state of a bilinear network from the origin to any reachable target state based on the generalized reachability Gramian of bilinear systems. We also provide a lower bound on the average minimum input energy over all target states on the unit hypersphere in the state space. Based on the reachability metrics proposed, we propose an actuator selection method that provides guaranteed minimum average input energy.**

I. INTRODUCTION

Complex networks such as electrical power grids, brain networks and transportation networks, are an essential part of modern society. A complex network typically consists of many dynamical subsystems (known as nodes) that interact with each other. An important question is how to manipulate the behavior of a large scale, complex network through controlling a few selected nodes. Answering this question facilitates the analysis and design of an engineering network. So far, existing results on the control of a complex network rely on the assumption that an external control input can directly affect the state of a node without affecting its interaction with other nodes. In this paper, we study the class of complex networks where the control input may not only affect directly the states of controlled nodes, but also change the interconnections among neighboring nodes in the network.

Literature review: Controllability of a complex network, that is, how to steer the state of the entire network through changing the states of some subsystems using external inputs is a fundamental problem and has been studied by many researchers [1], [2], [3].

A complex network is controllable if one can steer its state from any starting point to any terminal point in the state space. Using graph-theoretic tools, Liu *et al.* [2] relate the number of control nodes necessary to ensure controllability of a complex network to the network's degree distribution. Also by exploring the properties of a network's connection graph, Rahmani *et al.* [1] consider linear consensus-type networks, while Aguilar *et al.* [3] consider nonlinear consensus-type networks.

Controllability is a binary, qualitative property that does not characterize the difficulty in a control task. As a result, various quantitative controllability metrics have been proposed based on the reachability Gramian[1] [4], [5], [6]. Pasqualetti *et al.* [4] consider the selection of control nodes in a complex network to reduce the worst-case minimum energy to drive its state from the origin to a target state. Summers *et al.* [6] propose an optimal sensor and actuator placement strategy in complex dynamical networks to reduce the average minimum control energy over random target states.

Most of the existing results on controllability of complex networks rely on the assumption that the external inputs do not affect the interconnections among the subsystems in the network. This critical assumption may fail for certain classes of complex networks. For example, in the study of brain connectivity, it has been shown that external inputs not only have direct effects on brain states in a particular area (which corresponds to a node in the brain network), but can also change the intrinsic or latent connections that couple the states among different areas in the brain [7]. To deal with the scenario when the external inputs for control purposes may alter the interconnections among subsystems in a complex network, we use bilinear models to characterize the network instead of linear ones.

Bilinear systems are one of the simplest classes of nonlinear systems but can be used to represent a wide range of physical, chemical, economical and biological systems that cannot be effectively modeled using linear systems [8], [9], [10]. In the context of complex networks, bilinear systems characterize scenarios where an actuator not only can affect the state of a node, but also can affect the strength of the interconnections among some neighboring nodes. In this paper, we study reachability metrics for bilinear networks that characterize the minimum input energy required to steer the state of the entire network from the origin to another point in the state space.

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[1]For a linear time-invariant (LTI) system, the reachability Gramian is the same as the controllability Gramian. This is not true for bilinear systems. Since we only discuss reachability problems in this paper, we will use the term reachability Gramian.

While the reachability/controllability of bilinear systems has been widely investigated [11], [12], [13], [14], [15], [16], few results are available for (Gramian-based) quantitative controllability or reachability metrics. Although a notion of reachability Gramian has been proposed for bilinear systems, its relation with the input energy functional is less understood. Under the assumption that at least one of the coefficient matrices of the bilinear terms is nonsingular, the target state x_f belongs to a neighborhood of the origin, and an integrability condition[2], Gray *et al.* [19] show that for a continuous-time stable bilinear system with reachability Gramian $\mathcal{W}_c$, the input energy required to drive the state from the origin to x_f is always greater than $x_f^T \mathcal{W}_c^{-1} x_f$. Instead of the integrability condition, Benner *et al.* [18] assume that the reachability Gramian $\mathcal{W}_c$ is diagonal and prove similar results for all $x_f = \epsilon e_j$ where ϵ is some positive real number and e_j is any canonical unit vector in $\mathbb{R}^n$.

Statement of contributions: Unlike linear control theory, a (global) quantitative reachability metric is currently lacking for bilinear systems. In this paper, we propose a reachability Gramian-based lower bound to the minimum input energy required to steer the system state from the origin to any reachable target state. In particular, under the assumption that the infinity norm of the input is bounded by some function of the system matrices, we show that, like in linear control theory, a Gramian-based lower bound on the minimum input energy functional holds for any reachable target state. We provide an example to illustrate the tightness of this lower bound. We also provide a lower bound on the average minimum input energy over all target states on the unit hypersphere in the state space in terms of the trace of the reachability Gramian. We then propose a controller selection method in a bilinear network to maximize this lower bound. Through an example, we show that: 1. the lower bound is a good estimate of the actual performance; 2. maximizing this lower bound actually leads to the optimization of the actual performance. For space reasons, most proofs are removed and will appear elsewhere.

Organization: Section II introduces discrete-time bilinear control systems and states the problem of interest. Section III details basic properties of the associated reachability Gramian and Section IV establishes its relationship with the input energy functional. Motivated by this result, Section V explores the problem of selecting actuators in order to maximize the trace of the Gramian. Section VI provides our conclusions and ideas for future work.

Notation: For a vector $x \in \mathbb{R}^n$, we use x_i to denote its i-th element and $\|x\|_\infty$ to denote its infinity norm. For a matrix $M \in \mathbb{R}^{n\times m}$, we use $M_i \in \mathbb{R}^n$ to denote its i-th column, i.e., $M = \begin{bmatrix} M_1 & M_2 & \dots & M_m \end{bmatrix}$. $vec(M) = \begin{bmatrix} M_1^T & M_2^T & \dots & M_m^T \end{bmatrix}^T$ is the vector generated by stacking the columns of matrix M. A matrix is Schur stable if the magnitudes of all its eigenvalues are all smaller than 1. For a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_{min}(M)$ and $\lambda_{max}(M)$ denote its minimum and maximum eigenvalues, respectively. For symmetric matrices, we use the notation $M > 0$ (resp. $M \geq 0$) to denote that M is positive definite (resp. M is positive semidefinite). The spectral norm (maximum singular value) of a matrix M is denoted by $\|M\|$. The n-vector with all zero elements is denoted by $\mathbf{0}_n$ and the $m \times n$ matrix with all zero elements is denoted by $\mathbf{0}_{m\times n}$. The identity matrix of dimension $n \times n$ is denoted by I_n. For any $j \leq k \in \mathbb{Z}_{\geq 0}$, we use $\{x\}_j^k$ to denote the series $\{x(j), x(j+1), \dots, x(k)\}$, and omit j if $j = 0$. We use $\text{diag}(A_1, \dots, A_n)$ to denote a block-diagonal matrix. The symbol $\otimes$ represents the Kronecker product.

[2]The integrability condition may not hold for a general continuous-time bilinear system (see [17], [18] for detailed discussions).

II. Problem Formulation

We consider the class of discrete-time bilinear control systems with state-space representation

$$x(k+1) = Ax(k) + \sum_{j=1}^{m} (F_j x(k) + B_j) u_j(k), \qquad (1)$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time index, $x(k) \in \mathbb{R}^n$ is the system state, $u_j(k) \in \mathbb{R}$ is the control input and $A, F_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^n$ are the system matrices. When convenient, we simply refer to the bilinear control system (1) by (A, F, B). Throughout the paper, we assume that A is Schur stable.

The system (1) is controllable in a set $\mathbb{S}$ if, for any given pair of initial state and target state in $\mathbb{S}$, there exists a finite control sequence that drives the system from one to the other. The notion of reachability corresponds to controllability from the origin, i.e., the existence of a finite control sequence that takes the state from the origin to an arbitrary target state in $\mathbb{S}$. Controllability and reachability are qualitative measures of a system that do not precisely characterize how easy or difficult, in terms of control effort, it is for the system to go from one state to another.

Our objective in this paper is to provide quantitative measures of the degree of controllability for the bilinear control system (1). Formally, consider the minimum-energy optimal control problem for a given target state x_f and a time horizon $K \in \mathbb{Z}_{>0}$, defined by

$$\begin{aligned} \min_{\{u\}^{K-1}} \quad & \textstyle\sum_{k=0}^{K-1} u^T(k) u(k) \\ \text{s.t.} \quad & \forall k = 0, \dots, K-1, \\ & \text{Dynamics (1) holds}, \\ & x(0) = \mathbf{0}_n, \quad x(K) = x_f. \end{aligned} \qquad (2)$$

Our aim then can be understood as seeking to characterize the value of the optimal solution of (2) in terms of the data (A, F, B) defining the bilinear control system. Given that, in the case of linear systems, this characterization relies on the notion of reachability Gramian, our ensuing discussion presents a generalized notion of Gramian for bilinear control systems and explores its properties in detail.

III. Reachability Gramian for bilinear control systems

This section introduces the notion of reachability Gramian for stable discrete-time bilinear systems and characterizes some useful properties. Our discussion here is the basis for our analysis later where we establish the relationship of the reachability Gramian with the minimum-energy optimal control problem (2).

A. Reachability Gramian

The reachability Gramian for a stable discrete-time bilinear system (A, F, B) is, cf. [20],

$$\mathcal{W} = \sum_{i=1}^{\infty} \mathcal{W}_i, \tag{3}$$

where

$$\mathcal{W}_i = \sum_{k_1,\dots,k_i=0}^{\infty} \mathcal{P}_i(\{k\}_1^i)\mathcal{P}_i^T(\{k\}_1^i),$$
$$\mathcal{P}_1(k) = A^k B \in \mathbb{R}^{n\times m},$$
$$\mathcal{P}_i(\{k\}_1^i) = A^{k_i} F(I_m \otimes \mathcal{P}_{i-1}(\{k\}_1^{i-1})) \in \mathbb{R}^{n\times m^i}, \quad i \geq 2.$$

The reachability Gramian for continuous-time bilinear systems is defined analogously, see e.g., [21], [22]. This notion of reachability Gramian is widely used in model order reduction of bilinear systems [23], [24] and linear switched systems [25]. Notice that, for linear control systems (i.e., $F = \mathbf{0}_{n\times nm}$ in (1)), the reachability Gramian in (3)

$$\mathcal{W} = \mathcal{W}_1 = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k$$

is the reachability Gramian associated to the corresponding discrete-time linear time-invariant system.

Throughout this paper, we assume that (A, F, B) are such that the series in (3) converges and the resulting matrix is positive definite. A sufficient condition for the latter is that $(A, \mathbf{0}_{n\times mn}, B)$ is controllable, which in turn is equivalent to $\mathcal{W}_1 > 0$. Our next section discusses sufficient conditions for the convergence of the series in (3).

B. Properties of the Gramian

For linear control systems, it is well known that the reachability Gramian is a solution of the Lyapunov equation [26]. Similarly, the generalized reachability Gramian is a solution of a generalized Lyapunov equation [19], [22]. The following result has appeared in [20], [18]. For the sake of completeness, we state the result and provide a formal proof.

Theorem 1: (Generalized Lyapunov equation). The generalized reachability Gramian $\mathcal{W}$ satisfies the following generalized Lyapunov equation

$$A\mathcal{W}A^T - \mathcal{W} + \sum_{j=1}^{m} F_j \mathcal{W} F_j^T + BB^T = \mathbf{0}_{n\times n}. \tag{4}$$

Proof: From the definition (3), one can see that $\mathcal{W}_1 = \sum_{k_1=0}^{\infty} A^{k_1} BB^T (A^{k_1})^T$ satisfies

$$A\mathcal{W}_1 A^T - \mathcal{W}_1 + BB^T = \mathbf{0}_{n\times n}. \tag{5}$$

For $i \geq 2$, we obtain

$$\begin{aligned}\mathcal{W}_i &= \sum_{k_1,\dots,k_i=0}^{\infty} \mathcal{P}_i(\{k\}_1^i)\mathcal{P}_i^T(\{k\}_1^i) \\ &= \sum_{k_1,\dots,k_i=0}^{\infty} A^{k_i} F(I \otimes \mathcal{P}_{i-1}\mathcal{P}_{i-1}^T) F^T (A^{k_i})^T \\ &= \sum_{k_i=0}^{\infty} A^{k_i} \Big(\sum_{j=1}^{m} F_j \sum_{k_1,\dots,k_{i-1}=0}^{\infty} \mathcal{P}_{i-1}\mathcal{P}_{i-1}^T F_j^T \Big)(A^{k_i})^T \\ &= \sum_{k_i=0}^{\infty} A^{k_i} \Big(\sum_{j=1}^{m} F_j \mathcal{W}_{i-1} F_j^T \Big)(A^{k_i})^T. \end{aligned} \tag{6}$$

Therefore,

$$A\mathcal{W}_i A^T - \mathcal{W}_i + \sum_{j=1}^{m} F_j \mathcal{W}_{i-1} F_j^T = \mathbf{0}_{n\times n}. \tag{7}$$

By summing (5) and (7) with i ranging from 2 to ∞, we obtain (4) and the proof is complete. ■

It is thus possible to obtain the reachability Gramian $\mathcal{W}$ through solving the generalized Lyapunov equation (4), which one can do by computing

$$\mathrm{vec}(\mathcal{W}) = (I_{n^2} - A \otimes A - \sum_{j=1}^{m} F_j \otimes F_j)^{-1} \mathrm{vec}(BB^T), \tag{8}$$

assuming that the inverse exists.

IV. Minimum input energy for reachability

In this section, we obtain a lower bound on the minimum input energy required to steer the state of a bilinear control system from the origin to any reachable state, under the assumption that the infinity norm of the input is upper bounded. This bound is expressed in terms of the reachability Gramian.

From the formulation (2) of the optimal control problem in Section II, the necessary optimality conditions for the solution $\{u^*\}^{K-1}$ lead to the following nonlinear two-point boundary-value problem for $k = 0, \dots, K-1$,

$$\begin{aligned} x(k+1) &= Ax(k) + \frac{1}{2}\sum_{j=1}^{m} (F_j x(k) + B_j) \\ &\quad \cdot (F_j x(k) + B_j)^T \eta(k), \\ \eta(k-1) &= A^T \eta(k) + \frac{1}{2}\sum_{j=1}^{m} (\eta^T(k) F_j x(k) \\ &\quad + \eta^T(k) B_j) F_j^T \eta(k) \in \mathbb{R}^n, \\ u_j^*(k) &= \frac{1}{2}(F_j x(k) + B_j)^T \eta(k). \end{aligned} \tag{9}$$

For a stable, controllable, linear time-invariant system $(A, \mathbf{0}_{n\times mn}, B)$, one can obtain analytically the optimal control sequence from (9),

$$u^*(k) = B^T (A^T)^{K-k-1}\mathcal{W}_{1,K}^{-1}x_f,$$

and associated minimum control energy,

$$\sum_{k=0}^{K-1}(u^*(k))^T u^*(k) = x_f^T\mathcal{W}_{1,K}^{-1}x_f > x_f^T\mathcal{W}_1^{-1}x_f, \quad (10)$$

where $\mathcal{W}_{1,K} \triangleq \sum_{k=0}^{K-1} A^k BB^T(A^T)^k$ denotes the K-step controllability Gramian of the linear time-invariant system. Unfortunately, the nonlinear two-point boundary-value problem (9) does not admit an analytical solution in general (motivating the use of numerical approaches such as successive approximations [27], [28] and iterative methods [29]). Given our objective in this paper, we do not try to find the optimal control sequence but instead focus on the expression for the minimum control energy and, specifically, on its connection with the reachability Gramian.

The next result shows how, when the infinity norm of the input is upper bounded by a specific function of the system matrices A, F, and B defining the bilinear control system, then the lower bound in (10) also holds.

Theorem 2: (The generalized reachability Gramian is a metric for reachability). Consider the bilinear control system (1). If $\forall k = 1, 2, \ldots, K$,

$$\|u(k)\|_\infty \le 2^{-1}\big(\sum_{i,j=1}^{m}\|F_j^T\Psi F_i\|\big)^{-1}\beta, \quad (11)$$

where

$$\begin{aligned}\beta \triangleq &-\sum_{j=1}^{m}||A^T\Psi F_j + F_j^T\Psi A|| \\ &+ \big((\sum_{j=1}^{m}||A^T\Psi F_j + F_j^T\Psi A||)^2 \\ &- 4\sum_{i,j=1}^{m}||F_j^T\Psi F_i|| \cdot \lambda_{\max}(A^T\Psi A - \mathcal{W}^{-1})\big)^{1/2}, \\ \Psi \triangleq &\ \mathcal{W}^{-1} - \mathcal{W}^{-1}B(B^T\mathcal{W}^{-1}B - I_m)^{-1}B^T\mathcal{W}^{-1},\end{aligned}$$

then $\forall K \in \mathbb{Z}_{\ge 1}$,

$$\sum_{k=0}^{K-1} u^T(k)u(k) \ge x^T(K)\mathcal{W}^{-1}x(K). \quad (12)$$

The sufficient condition (11) is a magnitude constraint at every actuator. Theorem 2 provides a reachability Gramian-based lower bound to the minimum input energy required to drive the state from the origin to any reachable state. It is in this sense that we use the generalized reachability Gramian $\mathcal{W}$ as a reachability metric for bilinear systems.

Remark 1: (Positivity of the input upper bound in (11)). From the definition of β, it is obvious that the upper bound in (11) on the infinity norm of the input is positive if and only if the matrix $\mathcal{G}(A, B, F) = A^T\Psi A - \mathcal{W}^{-1}$ is negative definite. We have computed the upper bounds for hundreds of randomly generated matrix tuples (A, B, F) and they all turn out to be positive. However, to prove analytically the negative definiteness of $\mathcal{G}$ is in general difficult since $\mathcal{G}$ depends on A, B, F in a complicated manner. For scalar bilinear systems, one can prove the positivity of the upper bound easily.

The next result shows that Theorem 2 admits a simpler form for scalar bilinear systems.

Corollary 1: (Scalar case for Theorem 2). Consider the class of scalar bilinear systems (a, f, b). If $\forall k = 1, 2, \ldots, K$,

$$|u(k) + af^{-1}| \le \sqrt{a^2f^{-2} + 1}, \quad (13)$$

then inequality (12) holds $\forall K \in \mathbb{Z}_{\ge 1}$.

Next, we show through a counter example that the inequality (12) does not hold in general if the magnitude of the input is unconstrained.

Example 1: (Inequality (12) does not hold when the input is unconstrained). Consider the 2-step reachability problem for the scalar bilinear system $(a, f, 1)$,

$$\begin{aligned}x(k+1) &= ax(k) + fx(k)u(k) + u(k), \\ x(0) &= 0, \quad x(2) = x_f, \end{aligned} \quad (14)$$

whose reachability Gramian is given by $\mathcal{W} = (1 - a^2 - f^2)^{-1}$. It is easy to obtain from (14) that

$$u(0) = (a + fu(1))^{-1}(x_f - u(1)).$$

By denoting $x_f = Mu(1)$, $M \in \mathbb{R}$, we have

$$\begin{aligned}&u^2(0) + u^2(1) - x_f^T\mathcal{W}^{-1}x_f \\ &= \big(((a + fu(1))^{-2} - \mathcal{W}^{-1})M^2 - 2(a + fu(1))^{-2}M \\ &\quad + 1 + (a + fu(1))^{-2}\big)u^2(1).\end{aligned}$$

Choose $u(1)$ large enough such that $(a+fu(1))^2 > \mathcal{W}$, then there obviously exists M such that

$$u^2(0) + u^2(1) - x_f^T\mathcal{W}^{-1}x_f < 0.$$

Therefore, there exists $u(0)$, $u(1)$, x_f such that under the dynamics (14)), $u^2(0) + u^2(1) < x_f^T\mathcal{W}^{-1}x_f$. •

The following example illustrates the tightness of the Gramian-based lower bound (12) for the input energy functional.

Example 2: (Tightness of the Gramian-based lower bound in Theorem 2). Consider the following single-input bilinear control system taken from [30],

$$(A, f, b) : x(k+1) = Ax(k) + fu(k)x(k) + bu(k), \quad (15)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0.024 & 0 & 0 \\ 1 & 0 & -0.26 & 0 & 0 \\ 0 & 1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & -0.06 \\ 0 & 0 & 0.15 & 1 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.5 \end{bmatrix},$$

$$f = \mathrm{diag}(0.1, 0.2, 0.3, 0.4, 0.5).$$

We use (8) to compute the reachability Gramian $\mathcal{W}$ and use (11) to obtain the upper bound on $\|u(k)\|_\infty$ as

$$\|u(k)\|_\infty \leq 0.0011. \qquad (16)$$

Figure 1 compares the input energy functional $\sum_{i=0}^{k-1} u^2(i)$

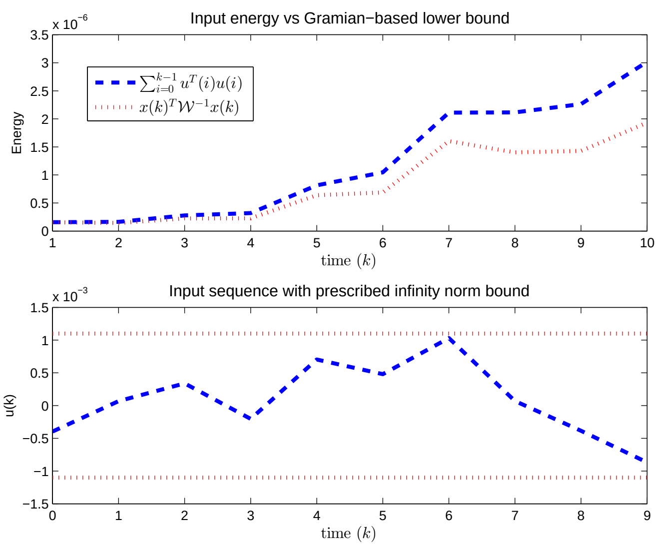

Fig. 1. Input energy vs Gramian-based lower bound.

with the Gramian-based lower bound $x^T(k)\mathcal{W}^{-1}x(k)$ for $k \leq K = 10$ and an arbitrarily chosen input sequence $\{u\}^{K-1}$ satisfying (16). Since the gap between the minimum input energy and the lower bound cannot be greater than that shown in the plot, Figure 1 shows that the Gramian-based lower bound is a good estimate of the minimum input energy required to drive the state from the origin to another state. •

V. Actuator Selection with Guaranteed Reachability Performance

This section explores the selection of actuators in light of our result in Section IV lower bounding the minimum energy required to steer the system from the origin to an arbitrary terminal state in terms of the reachability Gramian. Depending on the specific objective at hand, one can look at various metrics for actuator selection, such as the energy required in the worst case, in the average, etc. Particularly, by following the same approach as in [4], one can show that the worst-case control energy depends on the minimum eigenvalue of the reachability Gramian. Here, we focus on the average minimum control energy. Given (12) and the observation, cf. [6], that

$$\frac{\int_{\{x\in\mathbb{R}^n \mid ||x||=1\}} x^T\mathcal{W}^{-1}x dx}{\int_{\{x\in\mathbb{R}^n \mid ||x||=1\}} dx} = \frac{\operatorname{tr}(\mathcal{W}^{-1})}{n} \geq \frac{n}{\operatorname{tr}(\mathcal{W})}.$$

we focus on maximizing the trace of the reachability Gramian. Formally, the objective is to choose m actuators from a given group of M candidates ($m \leq M$) such that $\operatorname{tr}(\mathcal{W})$ is maximized. Denoting $V = \{1,\dots,M\}$ and $S = \{s_1,\dots,s_m\}$, we write the combinatorial optimization problem as

$$\max_{S\subseteq V} \quad \operatorname{tr}(\mathcal{W}(S)). \qquad (17)$$

We use $\mathcal{W}(S)$ instead of $\mathcal{W}$ to indicate its dependence on the choice of S. Similarly, we denote the input matrix B as $B_S = \begin{bmatrix} b_{s_1} & \dots & b_{s_m} \end{bmatrix}$, where $b_i \in \mathbb{R}^n$ is the input vector associated with actuator i for all $i \in V$. The optimization problem (17) may be difficult to solve for a large-scale network. In the following result, we provide a lower bound on $\operatorname{tr}(\mathcal{W})$.

Theorem 3: (A lower bound on $\mathcal{W}(S)$ that can be maximized globally). For any subset $S \subseteq V$, it holds that

$$\mathcal{W}(S) \geq \sum_{s\in S} \mathcal{W}(s). \qquad (18)$$

Remark 2: (Lower bound on $\operatorname{tr}(\mathcal{W}(S))$). Theorem 3 implies immediately that $\operatorname{tr}(\mathcal{W}(S)) \geq \sum_{s\in S} \operatorname{tr}(\mathcal{W}(s))$. To maximize this lower bound, one simply needs to compute $\operatorname{tr}(\mathcal{W}(s))$ individually for every $s \in V$. •

The following example shows that the lower bound in (18) provides a good reference for actuator selection.

Example 3: (Controller selection in a bilinear network system). Consider an augmented bilinear control system based on the model in Example 2,

$$x(k+1) = Ax(k) + \sum_{j\in S}(F_j x(k) + B_j)u_j(k),$$

where A, $B_0 = b$ and $F_0 = f$ are the same as those given in (15). The control node with index 0 represents the baseline controller that is always chosen to ensure that the reachability Gramian is nonsingular. This corresponds to the scenario when the network already has a set of actuators that provide controllability and one would like to add additional actuators to improve controllability.

The actuator candidates are (F_j, B_j) where B_j is the j-th canonical vector in $\mathbb{R}^5$ for $j = 1,2,3$. We let $F_1(1,2) = F_1(2,3) = 0.02$, $F_2(2,5) = 0.01$, $F_1(4,2) = 0.05$, and $F_3(1,1) = 0.05$, $F_3(4,5) = 0.02$, all the rest elements in F_j are zero for $j = 1,2,3$. Table I shows their individual and combinational contributions to the trace of the Gramian. We

TABLE I

Contribution of chosen sets of actuators to the trace of the Gramian.

S	$\operatorname{tr}(W(S))$	S	$\operatorname{tr}(W(S))$
$\{0\}$	14.4151	$\{0,2\}$	19.9132
$\{1\}$	5.0347	$\{0,3\}$	18.6893
$\{2\}$	4.0363	$\{0,1,2\}$	26.4962
$\{3\}$	3.0301	$\{0,1,3\}$	25.2767
$\{0,1\}$	20.9790	$\{0,2,3\}$	24.1914

make the following observations,

(i) $\sum_{s\in S} \operatorname{tr}(\mathcal{W}(s))$ is a good estimate of $\operatorname{tr}(\mathcal{W}(S))$. For example,

$$\frac{\operatorname{tr}(\mathcal{W}(\{0,1\})) - \sum_{s\in\{0,1\}} \operatorname{tr}(\mathcal{W}(s))}{\operatorname{tr}(\mathcal{W}(\{0,1\}))} = 0.0729,$$

which gives less than 8% estimation error.

(ii) Actuators with a large individual contribution provide a large combinational contribution. This fact suggests that ordering $\{\text{tr}(\mathcal{W}(s))\}_{s\in S}$ in decreasing order and selecting actuators sequentially is a sensible strategy.

It is worth mentioning that we have simulated this example for several sets of randomly generated (B_j, F_j) and observed similar results. •

VI. Conclusions

We have proposed a quantitative, Gramian-based reachability metric for discrete-time bilinear complex networks characterizing the minimum input energy required to steer the system state from the origin to an arbitrary point in the state space. Specifically, we showed that if the infinity norm of the input is upper bounded by some function of the system matrices, then the same relation between the reachability Gramian and input energy functional in linear control theory extends to bilinear systems. Further, we gave a lower bound on the average minimum input energy over all target states on the unit hypersphere in the state space. We also proposed an actuator selection method that maximizes this lower bound and provides guaranteed average minimum input energy.

Future work will include the consideration of other reachability metrics such as the worst-case minimum input energy and its relationship with the minimum eigenvalue of the reachability Gramian, the design of algorithms for selection of control inputs in complex networks, where both the nodes and the interconnection strength among neighboring nodes can be affected by actuators, and the study of observability metrics for bilinear control systems based on the generalized observability Gramian.

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