Decentralized Nash equilibrium learning by strategic generators for economic dispatch

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Abstract— **This paper studies an electricity market consisting of an independent system operator (ISO) and a group of generators. The goal is to solve the economic dispatch (ED) problem, i.e., make the generators collectively meet a given amount of power demand while minimizing the aggregate generation cost. The ISO by itself cannot solve the ED problem as the generators are strategic and do not share their cost functions. Instead, each generator submits to the ISO the price per unit of electricity at which it is willing to provide power to the ISO, which constitutes its bid. Based on the bids, the ISO decides how much production to allocate to each generator. The resulting Bertrand competition model defines the game among the generators where the actions are the bids and the payoffs are the profits. We provide a provably correct, decentralized strategy, termed NE SEEKING ALGORITHM, that takes the generators' bids to a neighborhood of the efficient Nash equilibrium and show that the optimal production of the generators converges to the optimizer of the ED problem. During the play, each generator only knows the amount of power the ISO requests it to produce and is not aware of the number of players, their actions, or their payoffs. Our algorithm can be understood as "learning via repeated play", where generators are "myopically selfish", changing their bid at each iteration with the sole aim of maximizing their payoff.**

I. INTRODUCTION

The envisioned architecture of the future power grid calls for different types of agents interacting with each other across multiple layers to enable the robust integration of distributed energy resources. Depending on the specific scenario, these interactions may fall anywhere within the cooperative-competitive spectrum. A particularly relevant problem is economic dispatch (ED), where a group of generators faces the task of providing a required load while minimizing the total generation cost. In this paper, we are interested in the strategic version of this problem and, particularly, in the policies that individual generators, in conjunction with independent system operator (ISO), can implement to solve it while acting in a selfish and rational fashion.

Literature review: Competition in electricity markets is a classical topic of study [1], [2]. Two extensively studied models are the supply function bidding and the Cournot (capacity) bidding, see [3], [4], respectively, and references therein. In our setup, we consider price-based bidding, a simplification of the supply function bidding. Most of these studies have revolved around pricing mechanisms, the resulting game, the existence and efficiency of the Nash equilibria of the game. On the other hand, only a few works [5], [6], [7], [8] deal with the issue of computing the Nash equilibrium via iterative algorithms designed for the players. The factors differentiating these setups are: pricing mechanisms, bidding functions, nature of demand (elastic or inelastic); and consideration of power flow constraints. However, these works either assume the generators know the costs, bids or actions of other generators, or the demand of each generator is a continuous function of the bids. We relax these assumptions here which, in turn, also rules out the possibility of using various other Nash equilibrium learning algorithms, such as best-response [9], fictitious play [10], and extremum seeking [11]. Our electricity market game falls into the broader class of multi-leader-follower games [12], [13], [14]. Equilibria of such games can be thought of as optimizers of mathematical programs with equilibrium constraints (MPEC) [15], [16] or equilibrium problem with equilibrium constraints (EPEC) [17], [18]. An overview of centralized solvers for MPEC problems is given in [19]. In [20], the authors present a distributed direct search method to find the equilibria of the MPEC problem but the follower's (the ISO in our case) optimization is required to have a unique equilibrium for each action of the leaders (the generators), which is in general not the case for electricity markets. Finally, our work has close connections with the growing interest in the design of provably correct distributed algorithms for the cooperative solution of the ED problem, see [21], [22], [23], [24], [25] and references therein.

Statement of contributions: Our starting point is the formulation of the inelastic electricity market game. Generators are strategic and do not share their cost functions, so the ISO cannot solve the ED problem by itself. Instead, each generator submits to the ISO a bid (the price per unit of electricity at which it is willing to provide power). Based on the collected bids, the ISO decides how much production to allocate to each generator. The resulting Bertrand competition model defines the game among the generators, where the actions are the bids and the payoffs are the profits. We define concepts of Nash and efficient Nash equilibrium, emphasizing the importance of the latter for the optimal dispatch of generators. Our first contribution establishes the existence of an efficient Nash equilibrium for the inelastic electricity market game and a sufficient condition for uniqueness. Our second and main contribution is the design and correctness analysis of the NE SEEKING ALGORITHM. We show that this decentralized iterative strategy is guaranteed to take the bids of the generators to a neighborhood of the unique efficient Nash equilibrium of the game. The NE SEEKING ALGORITHM can be interpreted as "learning via repeated

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play", where at each iteration, generators act rationally and selfishly, trying to maximize their own profit. Along the execution, the only information available to the generators is their bid and the amount of generation that the ISO request from them. In particular, generators are not aware of the number of other generators, their costs, bids, or payoffs. Simulations illustrate our results. For reasons of space, most proofs are omitted and will appear elsewhere.

Organization: Section II introduces the problem statement. Section III establishes the existence and uniqueness of efficient Nash equilibrium of the inelastic electricity market game. Section III presents the NE SEEKING ALGORITHM and its convergence properties. Section V provides the simulation example and Section VI gathers our conclusions and ideas for future work.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 1}$ denote the set of real, nonnegative real, and positive integer numbers, respectively. The 1- and 2-norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|_1$ and $\|\cdot\|$, respectively. Let $B_\delta(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \delta\}$ denote the open ball centered at $x \in \mathbb{R}^n$ with radius $\delta > 0$. The projection of a point $x \in \mathbb{R}^n$ onto a closed and convex set $\mathcal{D} \subset \mathbb{R}^n$, denoted $\text{proj}_{\mathcal{D}}(x)$ satisfies $\|x - \text{proj}_{\mathcal{D}}(x)\| = \min_{y \in \mathcal{D}} \|x - y\|$. For $x \in \mathbb{R}$, the ceiling operator $\lceil x \rceil$ gives the smallest integer greater than x. Given $x, y \in \mathbb{R}^n$, x_i denotes the i-th component of x, and $x \leq y$ denotes $x_i \leq y_i$ for $i \in \{1, \ldots, n\}$. We use $\mathbf{0}_N = (0, \ldots, 0) \in \mathbb{R}^N$ and $\mathbf{1}_N = (1, \ldots, 1) \in \mathbb{R}^N$. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist $\delta_x, L_x > 0$ such that $\|f(y_1) - f(y_2)\| \leq L_x \|y_1 - y_2\|$ for any $y_1, y_2 \in B_{\delta_x}(x)$. If f is locally Lipschitz at every $x \in \mathcal{S} \subset \mathbb{R}^n$, then we simply say that f is locally Lipschitz on $\mathcal{S}$. The map f is Lipschitz with constant $L > 0$ on $\mathcal{S} \subset \mathbb{R}^n$ if $\|f(x) - f(y)\| \leq L\|x - y\|$ for any $x, y \in \mathcal{S}$. Note that if f is locally Lipschitz on $\mathcal{S} \subset \mathbb{R}^n$, then it is Lipschitz on every compact set $\mathcal{S}_c \subset \mathcal{S}$. A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex on $\mathcal{S} \subset \mathbb{R}^n$ with $\mu > 0$ if it is convex and its Hessian satisfies $\nabla^2 f(x) > \mu$ for all $x \in \mathcal{S}$.

II. PROBLEM STATEMENT

Consider a group of $N \in \mathbb{Z}_{\geq 1}$ generators that aim to collectively meet the inelastic demand of the consumers through a competitive bidding process in the electricity market. The cost of power generation for each generator $n \in \{1, \ldots, N\}$ is given by a twice continuously differentiable and μ-strongly convex function $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $f_n(0) = 0$ and $\nabla f_n(0) \geq 0$ (in particular, the inverse $(\nabla f_n)^{-1}$ of the gradient is a well-defined single-valued function on the interval $[\nabla f_n(0), \infty)$). The cost of generating $x_n \in \mathbb{R}_{\geq 0}$ amount of power by the n-th generator is $f_n(x_n)$. For these functions, we assume that ∇f_n and $(\nabla f_n)^{-1}$ are locally Lipschitz on $[0, \infty)$ and $(\nabla f_n(0), \infty)$, respectively. Given a power allocation $x = (x_1, \ldots, x_N) \in \mathbb{R}^N_{\geq 0}$ for the group of generators, the aggregate cost is $\sum_{n=1}^N f_n(x_n)$. We assume the total power demand from the consumers to be inelastic, denoted by $y > 0$, a quantity known to the Independent System Operator (ISO). An ISO is the central regulatory authority of the electricity market. The ISO can interact with the generators, whereas each generator can only communicate with the ISO and is not even aware of the number of other generators participating in the market and their respective cost functions. The goal for the ISO is to seek the allocation $x \in \mathbb{R}^N_{\geq 0}$ that minimizes the total cost incurred by the generators in catering the demand y,

$$\underset{x}{\text{minimize}} \quad \textstyle\sum_{n=1}^N f_n(x_n), \tag{1a}$$

$$\text{subject to} \quad \textstyle\sum_{n=1}^N x_n = y, \tag{1b}$$

$$x \geq \mathbf{0}_N. \tag{1c}$$

This problem is termed as the *economic dispatch (ED) problem* or welfare optimization problem, see e.g. [3]. We assume (1) is feasible. Note that its optimizer, denoted x^*, is unique as the cost functions are μ-strongly convex [26].

The ISO could solve the above problem provided it knows all the cost functions of the generators. However, this information is not available to the ISO when generators are strategic and power allocation takes place by following a bidding process, resulting into a game-theoretic formulation of the dispatch. Instead of sharing their cost with the ISO, the generators bid the price per unit of power that they are willing to provide the power at. This kind of price-based bidding is also well known in the economics literature as price or Bertrand competition [27, Chapter 12]. Specifically, generator n bids the cost per unit power $b_n \in \mathbb{R}_{\geq 0}$ and, when convenient, we represent the bids of all other generators except n by $b_{-n} = (b_1, \ldots, b_{n-1}, b_{n+1}, \ldots, b_N)$. Given all the bids $b = (b_1, \ldots, b_N) \in \mathbb{R}^N_{\geq 0}$, the ISO solves the following *strategic economic dispatch (SED) problem*

$$\underset{x}{\text{minimize}} \quad \textstyle\sum_{n=1}^N b_n x_n, \tag{2a}$$

$$\text{subject to} \quad \textstyle\sum_{n=1}^N x_n = y, \tag{2b}$$

$$x \geq \mathbf{0}_N. \tag{2c}$$

Note that the difference between (2) and (1) is the objective function which is linear in the former and nonlinear, convex in the latter. For problem (2), at the optimizer, the generator with the minimum bid (assuming it is unique) gets to provide y units of power and the rest produce nothing. We refer to the generator with the minimum bid as the *winner* of the bid. If more than one generator has the minimum bid, then there exists infinitely many optimizers where y is provided collectively by the set of winners. Given the fact that the ISO solves (2) once all the bids are gathered, the objective of each generator n is to bid a quantity $b_n \geq 0$ that maximizes its payoff $u_n : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$,

$$u_n(b_n, x_n^{\text{opt}}(b_n, b_{-n})) = b_n x_n^{\text{opt}}(b_n, b_{-n}) - f_n(x_n^{\text{opt}}(b_n, b_{-n})), \tag{3}$$

where $x_n^{\text{opt}}(b_n, b_{-n})$ is the n-th component of an optimizer $x^{\text{opt}}(b_n, b_{-n})$ of the SED problem (2) corresponding to the bidding (b_n, b_{-n}). For convenience, we use $x^{\text{opt}}(b_n, b_{-n})$ and $x^{\text{opt}}(b)$ interchangeably to denote an optimizer of (2). Note that the payoff of the players is not only defined by the bids of other players but also by the optimizer of (2) that the ISO selects. For this reason, the definition of the pure Nash equilibrium for the game described below will be slightly different from the standard one, see e.g. [28].

Definition 2.1: (Inelastic electricity market game): The *inelastic electricity market* game is defined by the following

(i) Players: the set of generators $\{1, \dots, N\}$,
(ii) Action: for each player n, the bid $b_n \in \mathbb{R}_{\geq 0}$,
(iii) Payoff: for each player n, the payoff u_n in (3). •

The *(pure) Nash equilibrium* of the inelastic electricity market game is the bid profile of the group $b^* \in \mathbb{R}^N_{\geq 0}$ for which there exists an optimizer $x^{\text{opt}}(b^*)$ of the optimization (2) that satisfies the following: for each $n \in \{1, \dots, N\}$, and each bid $b_n \in \mathbb{R}_{\geq 0}$ with $b_n \neq b^*_n$, and each optimizer $x^{\text{opt}}(b_n, b^*_{-n})$ of (2) for the bid profile (b_n, b^*_{-n}), we have

$$u_n(b_n, x^{\text{opt}}_n(b_n, b^*_{-n})) \leq u_n(b^*_n, x^{\text{opt}}_n(b^*)). \tag{4}$$

An *efficient Nash equilibrium* b^* of the inelastic electricity market game is a Nash equilibrium for which the optimizer x^* of (1) is also an optimizer of (2) given bids b^* and

$$x^*_n = \text{argmax}_{x \geq 0} b^*_n x - f_n(x), \tag{5}$$

for all $n \in \{1, \dots, N\}$. Note that the right-hand side of (5) is unique as the functions are μ-strongly convex. At the efficient Nash equilibrium, the production that the generators are willing to provide, maximizing their profit, coincides with the optimal generation for the ED problem (1). This property justifies the study of efficient Nash equilibria.

Remark 2.2: (Cooperation versus competition in economic dispatch): In our previous work [29], [21] and in other distributed solution strategies for economic dispatch, e.g. [22], [23], [24], [25], the generators cooperatively find the optimizer of the ED problem. In a cooperative framework, generators are willing to share their information (state, gradient of the cost function, or Lagrange multiplier) with their neighboring generators. However, in the strategic framework considered here, generators aim only to maximize their own profits and do not share information with any other generator or the ISO. Both frameworks fit well into the envisioned hierarchical architecture of the future grid [30] that aims to integrate efficiently distributed energy resources (DERs) and flexible loads into the bulk power grid. At the top level, distributed energy resource providers (DERPs or aggregators) are strategic and compete in the electricity market regulated by the ISO. At a lower level, each DERP does not own generation but, instead, has agreements in place with a set of heterogeneous DERs and flexible loads that work cooperatively to meet the power demand entasked to the DERP. Recent works, see e.g. [3], [4] and references therein, consider different scenarios of strategic generators in economic dispatch but do not provide strategies for the generators to find the equilibrium of the resulting games. •

III. Existence and uniqueness of efficient Nash equilibrium

In this section, we first establish the existence of an efficient Nash equilibrium of the inelastic electricity market game described in Section II and later provide a condition under which this equilibrium is unique.

Proposition 3.1: (Existence of efficient Nash equilibrium for the inelastic electricity market game): Let $x^* = (x^*_1, x^*_2, \dots, x^*_N) \in \mathbb{R}^N_{\geq 0}$ be the solution of the ED problem (1). Let $x^*_n > 0$ for some $n \in \{1, \dots, N\}$. Then, $b^* = \nabla f_n(x^*_n)\mathbf{1}_N$ is an efficient Nash equilibrium of the inelastic electricity market game.

Proof: The Lagrangian of the optimization (1) is

$$L(x, \nu, \lambda) = \sum_{n=1}^N f_n(x_n) + \nu\Big(\sum_{n=1}^N x_n - y\Big) - \lambda^\top x,$$

where $\nu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N_{\geq 0}$ are Lagrange multipliers corresponding to constraints (1b) and (1c), respectively. Since constraints (1b) and (1c) are affine and the feasibility set is nonempty, the refined Slater condition is satisfied for (1) and hence, the duality gap between the primal and the dual optimization problems is zero [26]. Under this condition, a primal-dual optimizer (x^*, ν^*, λ^*) satisfies the following Karush-Kuhn-Tucker (KKT) conditions

$$\nabla f_n(x^*_n) + \nu^* - \lambda^*_n = 0, \text{ for all } n, \tag{6a}$$

$$\textstyle\sum_{n=1}^N x^*_n = y, \quad x^* \geq \mathbf{0}_N, \tag{6b}$$

$$\lambda^* \geq \mathbf{0}_N, \quad x^*_n \lambda^*_n = 0, \text{ for all } n. \tag{6c}$$

By hypothesis, $x^*_n > 0$ for some n and so, from (6c), $\lambda^*_n = 0$ and from (6a), $\nu^* = -\nabla f_n(x^*_n)$. Now consider the bid profile $b^* = \nabla f_n(x^*_n)\mathbf{1}_N = -\nu^* \mathbf{1}_N$. First we show that

$$x^*_n = \text{argmax}_{x \geq 0} b^*_n x - f_n(x), \tag{7}$$

for all $n \in \{1, \dots, N\}$, which establishes the efficiency (5) for the bid profile b^*. For each n, consider the optimization $\max_{x \geq 0} b^*_n x - f_n(x)$. Because zero duality holds for this optimization, a point $x_o \in \mathbb{R}_{\geq 0}$ is an optimizer if and only if it satisfies the KKT conditions

$$b^*_n - \nabla f_n(x_o) + \mu_o = 0,$$
$$\mu_o \geq 0, \quad x_o \geq 0, \quad \mu_o x_o = 0,$$

where μ_o is the dual optimizer. Since x^*_n satisfies the above conditions with $\mu_o = \lambda^*_n$, the expression (7) holds.

Our next step is to show the Nash equilibrium condition (4) for the bid profile b^*. Since all bids are same at this bid profile, x^* is one of the optimizers of the SED problem (2) and so we set $x^{\text{opt}}(b^*) = x^*$. Further, for each n, the payoff at the bid profile b^* and the optimizer $x^{\text{opt}}(b^*) = x^*$ is nonnegative. Specifically, if $x^*_n = 0$, then $u_n(b^*_n, x^{\text{opt}}_n(b^*)) = 0$. If $x^*_n > 0$ then from the KKT condition (6a) we get $b^*_n = \nabla f_n(x^*_n)$. Since ∇f_n is increasing, we get $\nabla f_n(x) \leq b^*_n$ for all $x \in [0, x^*_n]$. Hence,

$$\begin{aligned} u_n(b^*_n, x^{\text{opt}}_n(b^*)) &= b^*_n x^*_n - f_n(x^*_n) \\ &= \int_0^{x^*_n} \nabla(b^*_n x - f_n(x))dx \\ &= \int_0^{x^*_n} (b^*_n - \nabla f_n(x))dx \geq 0. \end{aligned}$$

Now pick any generator $n \in \{1, \dots, N\}$. For bid $b_n \neq b^*_n$ we have two cases, first, $b_n > b^*_n$ and second, $b_n \leq b^*_n$. For the first case, any optimizer $x^{\text{opt}}(b_n, b^*_{-n})$ of (2) satisfies $x_n^{\mathrm{opt}}(b_n, b_{-n}^*) = 0$ as n does not win the bid. Hence, $u(b_n, x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) = 0 \le u(b_n^*, x_n^*)$. For the second case,

$$\begin{aligned} u(b_n, x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) &= b_n x_n^{\mathrm{opt}}(b_n, b_{-n}^*) - f_n(x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) \\ &\le b_n^* x_n^{\mathrm{opt}}(b_n, b_{-n}^*) - f_n(x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) \\ &\le b_n^* x_n^* - f_n(x_n^*) = u(b_n^*, x_n^*), \end{aligned}$$

where in the first inequality we have used $b_n \le b_n^*$ and in the second we have used (7). This shows (4) for the bid profile b^*, concluding the proof. ■

From the KKT conditions (6) in the proof of Proposition 3.1, one can deduce that for any $n_1, n_2 \in \{1, \ldots, N\}$ if $x_{n_1}^* > 0$ and $x_{n_2}^* > 0$, one has $\nabla f_{n_1}(x_{n_1}^*) = \nabla f_{n_2}(x_{n_2}^*)$. For convenience, hereafter, we denote

$$b_{\mathrm{NE}} = \nabla f_n(x_n^*), \text{ for any } x_n^* > 0, \tag{8}$$

and the corresponding efficient Nash equilibrium is $b_{\mathrm{NE}}\mathbf{1}_N$. Note that $b_{\mathrm{NE}} > 0$ because following the assumption on the costs, $\nabla f_n(x) > 0$ for all $x > 0$ and all $n \in \{1, \ldots, N\}$.

Next we provide a sufficient condition that ensures uniqueness of $b_{\mathrm{NE}}\mathbf{1}_N$ as the efficient Nash equilibrium of the inelastic electricity market game.

Lemma 3.2: (Uniqueness of the efficient Nash equilibrium of the inelastic electricity market game): If the optimizer x^* of (1) satisfies $x^* > \mathbf{0}_N$, then $b_{\mathrm{NE}}\mathbf{1}_N$ is the unique efficient Nash equilibrium of the inelastic electricity market game.

In the reminder of the paper, we assume that the sufficient condition in Lemma 3.2 holds, unless otherwise stated.

IV. The NE Seeking Algorithm and its convergence properties

In this section, we introduce a decentralized Nash equilibrium learning algorithm, termed NE SEEKING ALGORITHM, and show that any of its executions takes the generators to the unique efficient Nash equilibrium (and consequently, to the optimizer of the ED problem (1)).

A. NE Seeking Algorithm

We start with an informal description of the NE SEEKING ALGORITHM. The algorithm is iterative and can be interpreted as "learning via repeated play" of the inelastic electricity market game by the generators. Both ISO and generators have bounded rationality, with each generator trying to maximize its own profit and the ISO trying to maximize the welfare of the entities.

> *[Informal description]:* At each iteration k, generators decide on a bid and send it to the ISO. Once the ISO has obtained the bids, it computes an optimizer of the SED problem (2) and sends the production level of each generator at the optimizer to the respective generator. At the $(k+1)$-th iteration, generators adjust their bid based on their previous bid, the quantity that maximizes their payoff for the previous bid, and the allocation of generation assigned by the ISO. The iterative process starts with the generators arbitrarily selecting initial bids.

The NE SEEKING ALGORITHM is formally presented in Algorithm 1. In the NE SEEKING ALGORITHM, the role of the ISO is to compute an optimizer of the SED problem after the bids are submitted. The bid adjustment at each iteration is done by the generators in a "myopically selfish" and rational fashion, since their sole aim is to maximize their payoff and not to make the other generators converge to a strategy that might make its payoff higher in the future. Roughly speaking, the algorithm prescribes that

if n **loses the bid:** two things can happen. (i) n was willing to produce a positive quantity $q_n(k) > 0$ at bid $b_n(k)$ but the demand from ISO is $r_n(k) = 0$. Thus, the rational choice for n would be to decrease the bid in the next iteration to increase its chances of winning and getting positive payoff. (ii) n was willing to produce nothing $q_n(k) = 0$ at bid $b_n(k)$ and got $r_n(k)$. At this point, reducing the bid will not increase the payoff as it will not be willing to produce more at a lower bid. On the other hand increasing the bid will not make it win. Therefore, the bid stays put.

if n **wins the bid and gets** $r_n(k) = y$**:** then it would want to move the bid in the direction that makes its payoff higher in the next iteration, assuming that n wins the next round of play. If $q_n(k) < y$, then the amount demanded by the ISO is more than what the generator is willing to produce, so n increases its cost, i.e., the bid. If $q_n(k) > y$, then the demand is less than what the generator is willing to supply so n decreases its bid.

Remark 4.1: (Information structure and alternative learning approaches): The generators have no knowledge of the number of other players, their actions or their payoffs. The only information available to them at each iteration is their own bid and the amount that the ISO requests from them.

Algorithm 1: NE SEEKING ALGORITHM

```
Executed by: generators n ∈ {1,...,N} and ISO
Data       : cost f_n and stepsizes {β_k}_{k∈Z≥1} for each
             generator n, and load y for ISO
Initialize : Each generator n selects arbitrarily
             b_n(1) ≥ 0, sets k = 1, and jumps to step 4;
             ISO sets k = 1 and waits for step 6
1  while k > 0 do
       /* For each generator n:                          */
2      Receive r_n(k−1) from ISO
3      Set b_n(k) =
       proj_{R≥0}(b_n(k−1) + β_k(r_n(k−1) − q_n(k−1)))
4      Set q_n(k) = argmax_{q≥0} b_n(k)q − f_n(q)
5      Send b_n(k) to the ISO; set k = k+1
       /* For ISO:                                       */
6      Receive b_n(k) from each n ∈ {1,...,N}
7      Set N(k) = argmin_{n∈{1,...,N}} b_n(k)
8      Compute for all n ∈ {1,...,N},
           r_n(k) = { y,  if n = max{i | i ∈ N(k)},
                    { 0,  otherwise.
9      Send r_n(k) to each n ∈ {1,...,N}; set k = k+1
10 end
```

This information structure rules out the applicability of a number of Nash equilibrium learning methods, including best-response dynamics [9], fictitious play [10], or other gradient-based adjustments [7], all requiring some kind of information about other players. Methods that relax this requirement, such as the extremum seeking techniques used in [11], rely on the payoff functions being continuous in the actions of the players, which is not the case for the inelastic electricity market game. •

B. Convergence analysis

In this section, we show that for constant stepsize, the bids of the generators along any execution of NE SEEKING ALGORITHM converge to a neighborhood of the unique efficient Nash equilibrium $b_{\rm NE}\mathbf{1}_N$. The size of the neighborhood is a decreasing function of the stepsize and can be made arbitrarily small.

For each generator $n \in \{1,\dots,N\}$, the *optimum quantity function* $q_n^{\rm opt} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as

$$q_n^{\rm opt}(b_n) = \operatorname{argmax}_{q\geq 0} b_n q - f_n(q). \tag{9}$$

This represents the optimal generation of n given the bid $b_n \geq 0$. Also, at each iteration of the NE SEEKING ALGORITHM, $q_n(k) = q_n^{\rm opt}(b_n(k))$ for each n. The following result outlines the properties of $q_n^{\rm opt}$.

Lemma 4.2: (Properties of optimum quantity function): For each $n \in \{1,\dots,N\}$, the optimum quantity function $q_n^{\rm opt}$ given in (9) is single-valued on $\mathbb{R}_{\geq 0}$, strictly increasing on $[\nabla f_n(0), \infty)$, and locally Lipschitz on $(\nabla f_n(0), \infty)$.

Next, we present our main result regarding the convergence of the NE SEEKING ALGORITHM.

Theorem 4.3: (Convergence of NE SEEKING ALGORITHM *to the efficient Nash equilibrium):* Let $x^* > \mathbf{0}_N$ where x^* is the optimizer of (1). Then, for an execution of NE SEEKING ALGORITHM starting at any initial bid $b(1) = (b_1(1), b_2(1), \dots, b_N(1)) \in \mathbb{R}^N_{\geq 0}$ and having constant stepsize $\beta_k = \beta > 0$ for all $k \in \mathbb{Z}_{\geq 1}$, there exists a finite iteration $k_\beta \in \mathbb{Z}_{\geq 1}$ such that

$$b_n(k) \in [b_{\rm NE} - C_1(\beta), b_{\rm NE} + C_2(\beta)] \cap \mathbb{R}_{\geq 0}, \tag{10a}$$

$$q_n(k) \in [q_n^{\rm opt}(b_{\rm NE} - C_1(\beta)), q_n^{\rm opt}(b_{\rm NE} + C_2(\beta)], \tag{10b}$$

for all $k > k_\beta$ and all $n \in \{1,\dots,N\}$. Where functions

$$C_1(\beta) = \beta\Big((N-1)(2y+1) + N^2 L_g L_{g^{-1}}(2y+1) + N \max_n (q_n^{\rm opt}(b_{\rm NE} + 2\beta y + \beta))\Big), \tag{11a}$$

$$C_2(\beta) = \beta(2y+1), \tag{11b}$$

constant $L_g > 0$ is an upper bound on the Lipschitz constants of set of functions $\{\nabla f_n\}_{n=1}^N$ on the set of intervals $\{[0, q_n^{\rm opt}(b_{\rm NE})]\}_{n=1}^N$, respectively, and the constant $L_{g^{-1}} > 0$ is an upper bound on the Lipschitz constants of set of functions $\{q_n^{\rm opt}\}_{n=1}^N$ on the set $[b_{\rm NE}, b_{\rm NE} + 2\beta y + \beta]$.

Remark 4.4: (Accuracy versus convergence speed in reaching the Nash equilibrium): Theorem 4.3 states that the size of the neighborhood of the efficient Nash equilibrium that the bids reach is a decreasing function of the stepsize β. Simulations show that a smaller stepsize leads to an increasing convergence time, which presents a trade-off to the designer between accuracy and convergence speed. An alternative strategy to reach a small-sized neighborhood of the Nash equilibrium is to decrease the stepsize incrementally, i.e., to execute the algorithm with a specific β until the bids reach $[b_{\rm NE} - C_1(\beta), b_{\rm NE} + C_2(\beta)]$, then reduce the stepsize and follow the same procedure. Our current work seeks to explicitly characterize these trade-offs and the most efficient way of reaching the Nash equilibrium. •

Remark 4.5: (ED problem with $x^* \not> \mathbf{0}_N$): If the sufficient condition in Lemma 3.2 does not hold, i.e., the optimizer $x^* \not> \mathbf{0}_N$, then not all bids converge necessarily to the efficient Nash equilibrium $b_{\rm NE}\mathbf{1}_N$. However, using the same arguments as in the proof of Theorem 4.3, one can show that the bids of those generators for which $x_n^* > 0$ still converge to a neighborhood of $b_{\rm NE}$. For the remaining generators with $x_n^* = 0$, even though the bid might not converge to $b_{\rm NE}$, the optimal generation $q_n(k)$ becomes zero in finite time, which is exactly what the generator needs to provide at the optimizer of the ED problem. •

V. SIMULATIONS

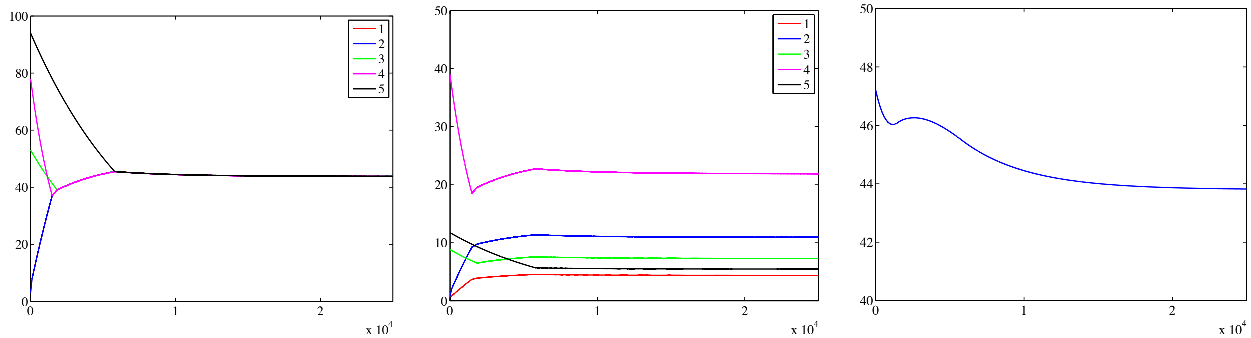
Here, we illustrate the application of the NE SEEKING ALGORITHM to find an efficient Nash equilibrium for an inelastic electricity market game with 5 generators. The cost functions for the generators are

$$f_n(x_n) = a_n x_n^2, \quad \text{where } a = (5,2,3,1,4). \tag{12}$$

The load is $y = 50$ units. The optimizer of the economic dispatch problem (1) defined with these cost functions and load is $x^* = (4.4, 10.9, 7.3, 21.9, 5.5)$. For the execution of the NE SEEKING ALGORITHM, the initial generator bids are arbitrarily selected to be $b(1) = (8, 3, 53, 78, 94)$ and the stepsizes β_k are chosen constant at value 0.001. Figure 1 shows the evolution of the bids and the optimal quantities that the generators would want to produce. As predicted by Theorem 4.3, the bids and the optimal quantities converge, respectively, to a neighborhood of the efficient Nash equilibrium $b^* = b_{\rm NE}\mathbf{1}_5$, $b_{\rm NE} = 43.79$, and the optimizer x^*.

VI. CONCLUSIONS

We have formulated the inelastic electricity market game and shown the existence of an efficient Nash equilibrium for it. We have designed the NE SEEKING ALGORITHM that is decentralized in implementation and that provably converges to a neighborhood of the efficient Nash equilibrium of the inelastic electricity market game. The algorithm can be interpreted as a repeated play of the game with minimal information available to the generators with selfish and rational decisions at each iteration. In the future, we aim to strengthen our results with time-varying stepsizes and the characterization of the time-to-convergence, extend the formulation of economic dispatch to include elastic demand, generator bounds, power flow constraints, and storage facilities, and study other bidding strategies, such as Cournot bidding, supply function bidding, and price-capacity bidding.



(a) Bids $b_n(k)$ (b) Optimal quantities $q_n(k)$ (c) Average bid $b_{\mathrm{avg}}(k)$

Fig. 1. Illustration of the execution of NE SEEKING ALGORITHM for the 5 generator example. The cost functions are given in (12) and the load is $y = 50$. The efficient Nash equilibrium of the inelastic electricity market game is $43.79\,\mathbf{1}_5$. As shown in (a), the bids converge to a neighborhood of the efficient Nash equilibrium in finite time (each color corresponding to a generator). (b) shows that the optimal quantity that the generators are willing to produce converges to a neighborhood of the optimizer $x^* = (4.4, 10.9, 7.3, 21.9, 5.5)$ of the ED problem (1). (c) shows the evolution of the average bid.

ACKNOWLEDGMENTS

The authors thank Dr. Wenyuan Tang and Dr. Subhonmesh Bose for useful discussions regarding electricity markets. This work was partially supported by NSF award ECCS-1307176.

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