The role of convexity on saddle-point dynamics: Lyapunov function and robustness

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***Abstract*—This paper studies the projected saddle-point dynamics associated to a convex-concave function, which we term as saddle function. The dynamics consists of gradient descent of the saddle function in variables corresponding to convexity and (projected) gradient ascent in variables corresponding to concavity. Under the assumption that the saddle function is twice continuously differentiable, we provide a novel characterization of the omega-limit set of the trajectories of this dynamics in terms of the diagonal blocks of the Hessian. Using this characterization, we establish global asymptotic convergence of the dynamics under local strong convexity-concavity of the saddle function. When strong convexity-concavity holds globally, we establish three results. First, we identify a Lyapunov function for the projected saddle-point dynamics when the saddle function corresponds to the Lagrangian of a general constrained optimization problem. Second, when the saddle function is the Lagrangian of an optimization problem with equality constraints, we show input-to-state stability of the saddle-point dynamics by providing an ISS Lyapunov function. Third, we design an opportunistic state-triggered implementation of the dynamics. Various examples illustrate our results.**

I. INTRODUCTION

Saddle-point dynamics and its variations have been used extensively in the design and analysis of distributed feedback controllers and optimization algorithms in several domains, including power networks, network flow problems, and zero-sum games. The analysis of the global convergence of this class of dynamics typically relies on some global strong/strict convexity-concavity property of the saddle function defining the dynamics. The main aim of this paper is to refine this analysis by unveiling two ways in which convexity-concavity of the saddle function plays a role. First, we show that local strong convexity-concavity is enough to conclude global asymptotic convergence, thus generalizing previous results that rely on global strong/strict convexity-concavity instead. Second, we show that global strong convexity-concavity in turn implies a stronger form of convergence, that is, input-to-state stability (ISS) of the dynamics. We also explore some of the important implications of this property in the practical implementation of the saddle-point dynamics.

A preliminary version of this work will appear at the 2016 Allerton Conference on Communication, Control, and Computing, Monticello, Illinois as [1].

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Literature review: The analysis of the convergence properties of (projected) saddle-point dynamics to the set of saddle points goes back to [2], [3], motivated by the study of nonlinear programming and optimization. These works employed direct methods, examining the approximate evolution of the distance of the trajectories to the saddle point and concluding attractivity by showing it to be decreasing. Subsequently, motivated by the extensive use of the saddle-point dynamics in congestion control problems, the literature on communication networks developed a Lyapunov-based and passivity-based asymptotic stability analysis, see e.g. [4] and references therein. Motivated by network optimization, more recent works [5], [6] have employed indirect, LaSalle-type arguments to analyze asymptotic convergence. For this class of problems, the aggregate nature of the objective function and the local computability of the constraints make the saddle-point dynamics corresponding to the Lagrangian naturally distributed. Many other works exploit this dynamics to solve network optimization problems for various applications, e.g., distributed convex optimization [6], [7], distributed linear programming [8], bargaining problems [9], and power networks [10], [11], [12], [13], [14]. Another area of application is game theory, where saddle-point dynamics is applied to find the Nash equilibria of two-person zero-sum games [15], [16]. In the context of distributed optimization, the recent work [17] employs a (strict) Lyapunov function approach to ensure asymptotic convergence of saddle-point-like dynamics. The work [18] examines the asymptotic behavior of the saddle-point dynamics when the set of saddle points is not asymptotically stable and, instead, trajectories exhibit oscillatory behavior. Our previous work has established global asymptotic convergence of the saddle-point dynamics [19] and the projected saddle-point dynamics [20] under global strict convexity-concavity assumptions. The works mentioned above require similar or stronger global assumptions on the convexity-concavity properties of the saddle function to ensure convergence. Our results here directly generalize the convergence properties reported above. Specifically, we show that traditional assumptions on the problem setup can be relaxed if convergence of the dynamics is the desired property: global convergence of the projected saddle-point dynamics can be guaranteed under local strong convexity-concavity assumptions. Furthermore, if traditional assumptions do hold, then a stronger notion of convergence, that also implies robustness, is guaranteed: if strong convexity-concavity holds globally, the dynamics admits a Lyapunov function and in the absence of projection, the dynamics is ISS, admitting an ISS Lyapunov function.

Statement of contributions: Our starting point is the definition of the projected saddle-point dynamics for a differentiable convex-concave function, referred to as saddle function. The dynamics has three components: gradient descent, projected gradient ascent, and gradient ascent of the saddle function, where each gradient is with respect to a subset of the arguments of the function. This unified formulation encompasses all forms of the saddle-point dynamics mentioned in the literature review above. Our contributions shed light on the effect that the convexity-concavity of the saddle function has on the convergence attributes of the projected saddle-point dynamics. Our first contribution is a novel characterization of the omega-limit set of the trajectories of the projected saddle-point dynamics in terms of the diagonal Hessian blocks of the saddle function. To this end, we use the distance to a saddle point as a LaSalle function, express the Lie derivative of this function in terms of the Hessian blocks, and show it is nonpositive using second-order properties of the saddle function. Building on this characterization, our second contribution establishes global asymptotic convergence of the projected saddle-point dynamics to a saddle point assuming only local strong convexity-concavity of the saddle function. Our third contribution identifies a novel Lyapunov function for the projected saddle-point dynamics for the case when strong convexity-concavity holds globally and the saddle function can be written as the Lagrangian of a constrained optimization problem. This discontinuous Lyapunov function can be interpreted as multiple continuously differentiable Lyapunov functions, one for each set in a particular partition of the domain determined by the projection operator of the dynamics. Interestingly, the identified Lyapunov function is the sum of two previously known and independently considered LaSalle functions. When the saddle function takes the form of the Lagrangian of an equality constrained optimization, then no projection is present. In such scenarios, if the saddle function satisfies global strong convexity-concavity, our fourth contribution establishes input-to-state stability (ISS) of the dynamics with respect to the saddle point by providing an ISS Lyapunov function. Our last contribution uses this function to design an opportunistic state-triggered implementation of the saddle-point dynamics. We show that the trajectories of this discrete-time system converge asymptotically to the saddle points and that executions are Zeno-free, i.e., that the difference between any two consecutive triggering times is lower bounded by a common positive quantity. Various examples illustrate our results.

II. Preliminaries

This section introduces our notation and preliminary notions on convex-concave functions, discontinuous dynamical systems, and input-to-state stability.

A. Notation

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{N}$ denote the set of real, nonnegative real, and natural numbers, respectively. We let $\|\cdot\|$ denote the 2-norm on $\mathbb{R}^n$ and the respective induced norm on $\mathbb{R}^{n\times m}$. Given $x, y \in \mathbb{R}^n$, x_i denotes the i-th component of x, and $x \leq y$ denotes $x_i \leq y_i$ for $i \in \{1, \dots, n\}$. For vectors $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, the vector $(u; w) \in \mathbb{R}^{n+m}$ denotes their concatenation. For $a \in \mathbb{R}$ and $b \in \mathbb{R}_{\geq 0}$, we let

$$[a]_b^+ = \begin{cases} a, & \text{if } b > 0, \\ \max\{0, a\}, & \text{if } b = 0. \end{cases}$$

For vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n_{\geq 0}$, $[a]_b^+$ denotes the vector whose i-th component is $[a_i]_{b_i}^+$, for $i \in \{1, \dots, n\}$. Given a set $\mathcal{S} \subset \mathbb{R}^n$, we denote by $\mathrm{cl}(\mathcal{S})$, $\mathrm{int}(\mathcal{S})$, and $|\mathcal{S}|$ its closure, interior, and cardinality, respectively. The distance of a point $x \in \mathbb{R}^n$ to the set $\mathcal{S} \subset \mathbb{R}^n$ in 2-norm is $\|x\|_{\mathcal{S}} = \inf_{y\in\mathcal{S}} \|x - y\|$. The projection of x onto a closed set $\mathcal{S}$ is defined as the set $\mathrm{proj}_{\mathcal{S}}(x) = \{y \in \mathcal{S} \mid \|x - y\| = \|x\|_{\mathcal{S}}\}$. When $\mathcal{S}$ is also convex, $\mathrm{proj}_{\mathcal{S}}(x)$ is a singleton for any $x \in \mathbb{R}^n$. For a matrix $A \in \mathbb{R}^{n\times n}$, we use $A \succeq 0$, $A \succ 0$, $A \preceq 0$, and $A \prec 0$ to denote that A is positive semidefinite, positive definite, negative semidefinite, and negative definite, respectively. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of A. For a real-valued function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $(x, y) \mapsto F(x, y)$, we denote by $\nabla_x F$ and $\nabla_y F$ the column vector of partial derivatives of F with respect to the first and second arguments, respectively. Higher-order derivatives follow the convention $\nabla_{xy} F = \frac{\partial^2 F}{\partial x \partial y}$, $\nabla_{xx} F = \frac{\partial^2 F}{\partial x^2}$, and so on. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. The set of unbounded class $\mathcal{K}$ functions are called $\mathcal{K}_\infty$ functions. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{KL}$ if for any $t \in \mathbb{R}_{\geq 0}$, $x \mapsto \beta(x, t)$ is class $\mathcal{K}$ and for any $x \in \mathbb{R}_{\geq 0}$, $t \mapsto \beta(x, t)$ is continuous, decreasing with $\beta(t, x) \to 0$ as $t \to \infty$.

B. Saddle points and convex-concave functions

Here, we review notions of convexity, concavity, and saddle points from [21]. A function $f : \mathcal{X} \to \mathbb{R}$ is *convex* if

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda) f(x'),$$

for all $x, x' \in \mathcal{X}$ (where $\mathcal{X}$ is a convex domain) and all $\lambda \in [0, 1]$. A convex differentiable f satisfies the following *first-order convexity condition*

$$f(x') \geq f(x) + (x' - x)^\top \nabla f(x),$$

for all $x, x' \in \mathcal{X}$. A twice differentiable function f is *locally strongly convex* at $x \in \mathcal{X}$ if f is convex and $\nabla^2 f(x) \succeq mI$ for some $m > 0$. Moreover, a twice differentiable f is *strongly convex* if $\nabla^2 f(x) \succeq mI$ for all $x \in \mathcal{X}$ for some $m > 0$. A function $f : \mathcal{X} \to \mathbb{R}$ is *concave*, *locally strongly concave*, or *strongly concave* if $-f$ is convex, locally strongly convex, or strongly convex, respectively. A function $F : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is *convex-concave* (on $\mathcal{X}\times\mathcal{Y}$) if, given any point $(\tilde{x}, \tilde{y}) \in \mathcal{X}\times\mathcal{Y}$, $x \mapsto F(x, \tilde{y})$ is convex and $y \mapsto F(\tilde{x}, y)$ is concave. When the space $\mathcal{X}\times\mathcal{Y}$ is clear from the context, we refer to this property as F being convex-concave in (x, y). A point $(x_*, y_*) \in \mathcal{X} \times \mathcal{Y}$ is a *saddle point* of F on the set $\mathcal{X} \times \mathcal{Y}$ if $F(x_*, y) \leq F(x_*, y_*) \leq F(x, y_*)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The set of saddle points of a convex-concave function F is convex. The function F is *locally strongly convex-concave* at a saddle point

(x, y) if it is convex-concave and either $\nabla_{xx}F(x,y) \succeq mI$ or $\nabla_{yy}F(x,y) \preceq -mI$ for some $m > 0$. Finally, F is *globally strongly convex-concave* if it is convex-concave and either $x \mapsto F(x,y)$ is strongly convex for all $y \in \mathcal{Y}$ or $y \mapsto F(x,y)$ is strongly concave for all $x \in \mathcal{X}$.

C. Discontinuous dynamical systems

Here we present notions of discontinuous dynamical systems [22], [23]. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lebesgue measurable and locally bounded. Consider the differential equation

$$\dot{x} = f(x). \tag{1}$$

A map $\gamma : [0,T) \to \mathbb{R}^n$ is a *(Caratheodory) solution* of (1) on the interval $[0,T)$ if it is absolutely continuous on $[0,T)$ and satisfies $\dot{\gamma}(t) = f(\gamma(t))$ almost everywhere in $[0,T)$. We use the terms solution and trajectory interchangeably. A set $\mathcal{S} \subset \mathbb{R}^n$ is *invariant* under (1) if every solution starting in $\mathcal{S}$ remains in $\mathcal{S}$. For a solution γ of (1) defined on the time interval $[0,\infty)$, the *omega-limit* set $\Omega(\gamma)$ is defined by

$$\Omega(\gamma) = \{y \in \mathbb{R}^n \mid \exists \{t_k\}_{k=1}^{\infty} \subset [0,\infty) \text{ with } \lim_{k\to\infty} t_k = \infty \text{ and } \lim_{k\to\infty} \gamma(t_k) = y\}.$$

If the solution γ is bounded, then $\Omega(\gamma) \neq \emptyset$ by the Bolzano-Weierstrass theorem [24, p. 33]. Given a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, the *Lie derivative of V along* (1) at $x \in \mathbb{R}^n$ is $\mathcal{L}_f V(x) = \nabla V(x)^\top f(x)$. The next result is a simplified version of [22, Proposition 3].

Proposition 2.1: (Invariance principle for discontinuous Caratheodory systems): Let $\mathcal{S} \in \mathbb{R}^n$ be compact and invariant. Assume that, for each point $x_0 \in \mathcal{S}$, there exists a unique solution of (1) starting at x_0 and that its omega-limit set is invariant too. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable map such that $\mathcal{L}_f V(x) \le 0$ for all $x \in \mathcal{S}$. Then, any solution of (1) starting at $\mathcal{S}$ converges to the largest invariant set in $\mathrm{cl}(\{x \in \mathcal{S} \mid \mathcal{L}_f V(x) = 0\})$.

D. Input-to-state stability

Here, we review the notion of input-to-state stability (ISS) following [25]. Consider a system

$$\dot{x} = f(x,u), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ is the input that is measurable and locally essentially bounded, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Assume that starting from any point in $\mathbb{R}^n$, the trajectory of (2) is defined on $\mathbb{R}_{\ge 0}$ for any given control. Let $\mathrm{Eq}(f) \subset \mathbb{R}^n$ be the set of equilibrium points of the unforced system. Then, the system (2) is *input-to-state stable* (ISS) with respect to $\mathrm{Eq}(f)$ if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that each trajectory $t \mapsto x(t)$ of (2) satisfies

$$\|x(t)\|_{\mathrm{Eq}(f)} \le \beta(\|(x(0)\|_{\mathrm{Eq}(f)}, t) + \gamma(\|u\|_\infty)$$

for all $t \ge 0$, where $\|u\|_\infty = \mathrm{ess\ sup}_{t\ge 0}\|u(t)\|$ is the essential supremum (see [24, p. 185] for the definition) of u. This notion captures the graceful degradation of the asymptotic convergence properties of the unforced system as the size of the disturbance input grows. One convenient way of showing ISS is by finding an ISS-Lyapunov function. An *ISS-Lyapunov function* with respect to the set $\mathrm{Eq}(f)$ for system (2) is a differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|_{\mathrm{Eq}(f)}) \le V(x) \le \alpha_2(\|x\|_{\mathrm{Eq}(f)}); \tag{3}$$

(ii) there exists a continuous, positive definite function $\alpha_3 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\nabla V(x)^\top f(x,v) \le -\alpha_3(\|x\|_{\mathrm{Eq}(f)}) \tag{4}$$

for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}^m$ for which $\|x\|_{\mathrm{Eq}(f)} \ge \gamma(\|v\|)$.

Proposition 2.2: (ISS-Lyapunov function implies ISS): If (2) admits an ISS-Lyapunov function, then it is ISS.

III. Problem statement

In this section, we provide a formal statement of the problem of interest. Consider a twice continuously differentiable function $F : \mathbb{R}^n \times \mathbb{R}^p_{\ge 0} \times \mathbb{R}^m \to \mathbb{R}$, $(x,y,z) \mapsto F(x,y,z)$, which we refer to as *saddle function*. With the notation of Section II-B, we set $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^p_{\ge 0} \times \mathbb{R}^m$, and assume that F is convex-concave on $(\mathbb{R}^n) \times (\mathbb{R}^p_{\ge 0} \times \mathbb{R}^m)$. Let $\mathrm{Saddle}(F)$ denote its (non-empty) set of saddle points. We define the *projected saddle-point dynamics* for F as

$$\dot{x} = -\nabla_x F(x,y,z), \tag{5a}$$
$$\dot{y} = [\nabla_y F(x,y,z)]^+_y, \tag{5b}$$
$$\dot{z} = \nabla_z F(x,y,z). \tag{5c}$$

When convenient, we use the map $X_{\text{p-sp}} : \mathbb{R}^n \times \mathbb{R}^p_{\ge 0} \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$ to refer to the dynamics (5). Note that the domain $\mathbb{R}^n \times \mathbb{R}^p_{\ge 0} \times \mathbb{R}^m$ is invariant under $X_{\text{p-sp}}$ (this follows from the definition of the projection operator) and its set of equilibrium points precisely corresponds to $\mathrm{Saddle}(F)$ (this follows from the defining property of saddle points and the first-order condition for convexity-concavity of F). Thus, a saddle point (x_*, y_*, z_*) satisfies

$$\nabla_x F(x_*,y_*,z_*) = 0, \quad \nabla_z F(x_*,y_*,z_*) = 0, \tag{6a}$$
$$\nabla_y F(x_*,y_*,z_*) \le 0, \quad y_*^\top \nabla_y F(x_*,y_*,z_*) = 0. \tag{6b}$$

Our interest in the dynamics (5) is motivated by two bodies of work in the literature: one that analyzes primal-dual dynamics, corresponding to (5a) together with (5b), for solving inequality constrained network optimization problems, see e.g., [3], [5], [14], [11]; and the other one analyzing saddle-point dynamics, corresponding to (5a) together with (5c), for solving equality constrained problems and finding Nash equilibrium of zero-sum games, see e.g., [19] and references therein. By considering (5a)-(5c) together, we aim to unify these lines of work.

Our main objectives are to identify conditions that guarantee that the set of saddle points is globally asymptotically stable under the dynamics (5) and formally characterize the robustness properties using the concept of input-to-state stability. We also seek to use the latter to explore the design of opportunistic state-triggered implementations of the dynamics for scenarios where the hardware imposes limits on the sampling rate.

IV. Local properties of the saddle function imply global convergence

Our first result of this section provides a novel characterization of the omega-limit set of the trajectories of the projected saddle-point dynamics (5).

Proposition 4.1: (Characterization of the omega-limit set of solutions of $X_{\text{p-sp}}$): Given a twice continuously differentiable, convex-concave function F, the set $\text{Saddle}(F)$ is stable under the projected saddle-point dynamics $X_{\text{p-sp}}$ and the omega-limit set of every solution is contained in the largest invariant set $\mathcal{M}$ in $\mathcal{E}(F)$, where

$$\begin{aligned}\mathcal{E}(F) = \{&(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \mid \\ &(x-x_*; y-y_*; z-z_*) \in \ker(\overline{H}(x,y,z,x_*,y_*,z_*)), \\ &\text{for all } (x_*,y_*,z_*) \in \text{Saddle}(F)\}, \end{aligned} \tag{7}$$

and

$$\begin{aligned}\overline{H}(x,y,z,x_*,y_*,z_*) &= \int_0^1 H(x(s),y(s),z(s))ds, \\ (x(s),y(s),z(s)) &= (x_*,y_*,z_*) + s(x-x_*, y-y_*, z-z_*), \\ H(x,y,z) &= \begin{bmatrix} -\nabla_{xx}F & 0 & 0 \\ 0 & \nabla_{yy}F & \nabla_{yz}F \\ 0 & \nabla_{zy} & \nabla_{zz}F \end{bmatrix}_{(x,y,z)}. \end{aligned} \tag{8}$$

Proof: The proof follows from the application of the LaSalle Invariance Principle for discontinuous Caratheodory systems (cf. Proposition 2.1). Let $(x_*,y_*,z_*) \in \text{Saddle}(F)$ and $V_1 : \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ be defined as

$$V_1(x,y,z) = \frac{1}{2}\big(\|x-x_*\|^2 + \|y-y_*\|^2 + \|z-z_*\|^2\big). \tag{9}$$

The Lie derivative of V_1 along (5) is

$$\begin{aligned}&\mathcal{L}_{X_{\text{p-sp}}} V_1(x,y,z) \\ &\quad = -(x-x_*)^\top \nabla_x F(x,y,z) + (y-y_*)^\top [\nabla_y F(x,y,z)]^+_y \\ &\qquad + (z-z_*)^\top \nabla_z F(x,y,z) \\ &\quad = -(x-x_*)^\top \nabla_x F(x,y,z) + (y-y_*)^\top \nabla_y F(x,y,z) \\ &\qquad + (z-z_*)^\top \nabla_z F(x,y,z) \\ &\qquad + (y-y_*)^\top ([\nabla_y F(x,y,z)]^+_y - \nabla_y F(x,y,z)) \\ &\quad \leq -(x-x_*)^\top \nabla_x F(x,y,z) + (y-y_*)^\top \nabla_y F(x,y,z) \\ &\qquad + (z-z_*)^\top \nabla_z F(x,y,z), \end{aligned} \tag{10}$$

where the last inequality follows from the fact that $T_i = (y-y_*)_i([\nabla_y F(x,y,z)]^+_y - \nabla_y F(x,y,z))_i \leq 0$ for each $i \in \{1,\dots,p\}$. Indeed if $y_i > 0$, then $T_i = 0$ and if $y_i = 0$, then $(y-y_*)_i \leq 0$ and $([\nabla_y F(x,y,z)]^+_y - \nabla_y F(x,y,z))_i \geq 0$ which implies that $T_i \leq 0$. Next, denoting $\lambda = (y;z)$ and $\lambda_* = (y_*,z_*)$, we simplify the above inequality as

$$\begin{aligned}&\mathcal{L}_{X_{\text{p-sp}}} V_1(x,y,z) \\ &\quad \leq -(x-x_*)^\top \nabla_x F(x,\lambda) + (\lambda-\lambda_*)^\top \nabla_\lambda F(x,\lambda) \\ &\quad \overset{(a)}{=} -(x-x_*)^\top \int_0^1 \Big(\nabla_{xx} F(x(s),\lambda(s))(x-x_*) \\ &\qquad + \nabla_{\lambda x} F(x(s),\lambda(s))(\lambda-\lambda_*)\Big) ds \\ &\qquad + (\lambda-\lambda_*)^\top \int_0^1 \Big(\nabla_{x\lambda} F(x(s),\lambda(s))(x-x_*) \\ &\qquad + \nabla_{\lambda\lambda} F(x(s),\lambda(s))(\lambda-\lambda_*)\Big) ds \\ &\quad \overset{(b)}{=} [x-x_*; \lambda-\lambda_*]^\top \overline{H}(x,\lambda,x_*,\lambda_*) \begin{bmatrix} x-x_* \\ \lambda-\lambda_* \end{bmatrix} \overset{(c)}{\leq} 0, \end{aligned}$$

where (a) follows from the fundamental theorem of calculus using the notation $x(s) = x_* + s(x-x_*)$ and $\lambda(s) = \lambda_* + s(\lambda-\lambda_*)$ and recalling from (6) that $\nabla_x F(x_*,\lambda_*) = 0$ and $(\lambda-\lambda_*)^\top \nabla_\lambda F(x_*,\lambda_*) \leq 0$; (b) follows from the definition of $\overline{H}$ using $(\nabla_{\lambda x} F(x,\lambda))^\top = \nabla_{x\lambda} F(x,\lambda)$; and (c) follows from the fact that $\overline{H}$ is negative semi-definite. Now using this fact that $\mathcal{L}_{X_{\text{p-sp}}} V_1$ is nonpositive at any point, one can deduce, see e.g. [20, Lemma 4.2-4.4], that starting from any point $(x(0),y(0),z(0))$ a unique trajectory of $X_{\text{p-sp}}$ exists, is contained in the compact set $V_1^{-1}(V_1(x(0),y(0),z(0))) \cap (\mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m)$ at all times, and its omega-limit set is invariant. These facts imply that the hypotheses of Proposition 2.1 hold and so, we deduce that the solutions of the dynamics $X_{\text{p-sp}}$ converge to the largest invariant set where the Lie derivative is zero, that is, the set

$$\begin{aligned}&\mathcal{E}(F,x_*,y_*,z_*) = \{(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \mid \\ &(x;y;z) - (x_*;y_*;z_*) \in \ker(\overline{H}(x,y,z,x_*,y_*,z_*))\}. \end{aligned} \tag{11}$$

Finally, since (x_*,y_*,z_*) was chosen arbitrary, we get that the solutions converge to the largest invariant set $\mathcal{M}$ contained in $\mathcal{E}(F) = \bigcap_{(x_*,y_*,z_*) \in \text{Saddle}(F)} \mathcal{E}(F,x_*,y_*,z_*)$, concluding the proof. ∎

Note that the proof of Proposition 4.1 shows that the Lie derivative of the function V_1 is negative, but not strictly negative, outside the set $\text{Saddle}(F)$. The next result shows that local strong convexity-concavity around a saddle point together with global convexity-concavity of the saddle function are enough to guarantee global convergence.

Theorem 4.2: (Global asymptotic stability of the set of saddle points under $X_{\text{p-sp}}$): Given a twice continuously differentiable, convex-concave function F which is locally strongly convex-concave at a saddle point, the set $\text{Saddle}(F)$ is globally asymptotically stable under the projected saddle-point dynamics $X_{\text{p-sp}}$ and the convergence of trajectories is to a point.

Proof: Our proof proceeds by characterizing the set $\mathcal{E}(F)$ defined in (7). Let (x_*,y_*,z_*) be a saddle point at which F is locally strongly convex-concave. Without loss of generality, assume that $\nabla_{xx} F(x_*,y_*,z_*) \succ 0$ (the case of negative definiteness of the other Hessian block can be reasoned analogously). Let $(x,y,z) \in \mathcal{E}(F,x_*,y_*,z_*)$ (recall the definition of this set in (11)). Since $\nabla_{xx} F(x_*,y_*,z_*) \succ 0$ and F is twice continuously differentiable, we have that $\nabla_{xx} F$ is positive definite in a neighborhood of (x_*,y_*,z_*) and so

$$\int_0^1 \nabla_{xx} F(x(s),y(s),z(s)) ds \succ 0,$$

where $x(s) = x_* + s(x-x_*)$, $y(s) = y_* + s(y-y_*)$, and $z(s) = z_* + s(z-z_*)$. Therefore, by definition of $\mathcal{E}(F,x_*,y_*,z_*)$, it follows that $x = x_*$ and so, $\mathcal{E}(F,x_*,y_*,z_*) \subseteq \{x_*\} \times (\mathbb{R}^p_{\geq 0} \times \mathbb{R}^m)$. From Proposition 4.1 the trajectories of $X_{\text{p-sp}}$ converge to the largest invariant set $\mathcal{M}$ contained in $\mathcal{E}(F,x_*,y_*,z_*)$.

To characterize this set, let $(x_*, y, z) \in \mathcal{M}$ and $t \mapsto (x_*, y(t), z(t))$ be a trajectory of $X_{\text{p-sp}}$ that is contained in $\mathcal{M}$ and hence in $\mathcal{E}(F, x_*, y_*, z_*)$. From (10), we get

$$\begin{aligned}
&\mathcal{L}_{X_{\text{p-sp}}} V_1(x, y, z) \\
&\leq -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top \nabla_y F(x, y, z) \\
&\quad + (z - z_*)^\top \nabla_z F(x, y, z) \\
&\leq F(x, y, z) - F(x, y_*, z_*) + F(x_*, y, z) - F(x, y, z) \\
&\leq F(x_*, y_*, z_*) - F(x, y_*, z_*) + F(x_*, y, z) \\
&\quad - F(x_*, y_*, z_*) \leq 0,
\end{aligned} \tag{12}$$

where in the second inequality we have used the first-order convexity and concavity property of the maps $x \mapsto F(x, y, z)$ and $(y, z) \mapsto F(x, y, z)$. Now since $\mathcal{E}(F, x_*, y_*, z_*) = \{(x_*, y, z) \mid \mathcal{L}_{X_{\text{p-sp}}} V_1(x_*, y, z) = 0\}$, using the above inequality, we get $F(x_*, y(t), z(t)) = F(x_*, y_*, z_*)$ for all $t \geq 0$. Thus, for all $t \geq 0$, $\mathcal{L}_{X_{\text{p-sp}}} F(x_*, y(t), z(t)) = 0$ which yields

$$\begin{aligned}
\nabla_y F(x_*, y(t), z(t))^\top [\nabla_y F(x_*, y(t), z(t))]^+_{y(t)} \\
+ \|\nabla_z F(x_*, y(t), z(t))\|^2 = 0
\end{aligned}$$

Note that both terms in the above expression are nonnegative and so, we get $[\nabla_y F(x_*, y(t), z(t))]^+_{y(t)} = 0$ and $\nabla_z F(x_*, y(t), z(t)) = 0$ for all $t \geq 0$. In particular, this holds at $t = 0$ and so, $(x, y, z) \in \text{Saddle}(F)$, and we conclude $\mathcal{M} \subset \text{Saddle}(F)$. Hence $\text{Saddle}(F)$ is globally asymptotically stable. Combining this with the fact that individual saddle points are stable, one deduces the pointwise convergence of trajectories along the same lines as in [26, Corollary 5.2]. ∎

A closer look at the proof of the above result reveals that the same conclusion also holds under milder conditions on the saddle function. In particular, F need only be twice continuously differentiable in a neighborhood of the saddle point and the local strong convexity-concavity can be relaxed to a condition on the line integral of Hessian blocks of F. We state next this stronger result.

Theorem 4.3: (Global asymptotic stability of the set of saddle points under $X_{\text{p-sp}}$): Let F be convex-concave and continuously differentiable with locally Lipschitz gradient. Suppose there is a saddle point (x_*, y_*, z_*) and a neighborhood of this point $\mathcal{U}_* \subset \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m$ such that F is twice continuously differentiable on $\mathcal{U}_*$ and either of the following holds

(i) for all $(x, y, z) \in \mathcal{U}_*$,

$$\int_0^1 \nabla_{xx} F(x(s), y(s), z(s)) ds \succ 0,$$

(ii) for all $(x, y, z) \in \mathcal{U}_*$,

$$\int_0^1 \begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x(s), y(s), z(s))} ds \prec 0,$$

where $(x(s), y(s), z(s))$ are given in (8). Then, $\text{Saddle}(F)$ is globally asymptotically stable under the projected saddle-point dynamics $X_{\text{p-sp}}$ and the convergence of trajectories is to a point.

We omit the proof of this result for space reasons: the argument is analogous to the proof of Theorem 4.2, where one replaces the integral of Hessian blocks by the integral of generalized Hessian blocks (see [27, Chapter 2] for the definition of the latter), as the function is not twice continuously differentiable everywhere.

Example 4.4: (Illustration of global asymptotic convergence): Consider $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ given as

$$F(x, y, z) = f(x) + y(-x_1 - 1) + z(x_1 - x_2), \tag{13}$$

where

$$f(x) = \begin{cases} \|x\|^4, & \text{if } \|x\| \leq \frac{1}{2}, \\ \frac{1}{16} + \frac{1}{2}(\|x\| - \frac{1}{2}), & \text{if } \|x\| \geq \frac{1}{2}. \end{cases}$$

Note that F is convex-concave on $(\mathbb{R}^2) \times (\mathbb{R}_{\geq 0} \times \mathbb{R})$ and $\text{Saddle}(F) = \{0\}$. Also, F is continuously differentiable on the entire domain and its gradient is locally Lipschitz. Finally, F is twice continuously differentiable on the neighborhood $\mathcal{U}_* = B_{1/2}(0) \cap (\mathbb{R}^2 \times \mathbb{R}_{\geq 0} \times \mathbb{R})$ of the saddle point 0 and hypothesis (i) of Theorem 4.3 holds on $\mathcal{U}_*$. Therefore, we conclude from Theorem 4.3 that the trajectories of the projected saddle-point dynamics of F converge globally asymptotically to the saddle point 0. Figure 1 shows an execution. •

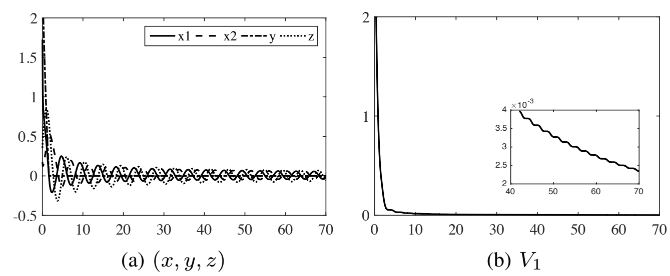


Fig. 1. Execution of the projected saddle-point dynamics (5) starting from $(1.7256, 0.1793, 2.4696, 0.3532)$ for Example 4.4. As guaranteed by Theorem 4.3, the trajectory converges to the unique saddle point 0 and the function V_1 defined in (9) decreases monotonically.

Remark 4.5: (Comparison with the literature): Theorems 4.2 and 4.3 complement the available results in the literature concerning the asymptotic convergence properties of saddle-point [3], [19], [17] and primal-dual dynamics [5], [20]. The former dynamics corresponds to (5) when the variable y is absent and the later to (5) when the variable z is absent. For both saddle-point and primal-dual dynamics, existing global asymptotic stability results require assumptions on the global properties of F, in addition to the global convexity-concavity of F, such as global strong convexity-concavity [3], global strict convexity-concavity, and its generalizations [19]. In contrast, the novelty of our results lies in establishing that certain local properties of the saddle function are enough to guarantee global asymptotic convergence. •

V. Lyapunov function for constrained optimization problems

Our discussion above has established the global asymptotic stability of the set of saddle points resorting to LaSalle-type arguments (because the function V_1 defined in (9) is not a strict Lyapunov function). In this section, we identify instead a strict Lyapunov function for the projected saddle-point dynamics when the saddle function F corresponds to the Lagrangian of a constrained optimization problem. The relevance of this result stems from two facts. On the one hand, the projected saddle-point dynamics has been employed profusely to solve network optimization problems. On the other hand, although the conclusions on the asymptotic convergence of this dynamics that can be obtained with the identified Lyapunov function are the same as in the previous section, having a Lyapunov function available is advantageous for a number of reasons, including the study of robustness against disturbances, the characterization of the algorithm convergence rate, or as a design tool for developing opportunistic state-triggered implementations. We come back to this point later.

Theorem 5.1: (Lyapunov function for $X_{\text{p-sp}}$): Let $F : \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ be defined as

$$F(x, y, z) = f(x) + y^\top g(x) + z^\top (Ax - b), \tag{14}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex, twice continuously differentiable, $g : \mathbb{R}^n \to \mathbb{R}^p$ is convex, twice continuously differentiable, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. For each $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m$, define the index set of active constraints

$$\mathcal{J}(x, y, z) = \{j \in \{1, \dots, p\} \mid y_j = 0 \text{ and } (\nabla_y F(x, y, z))_j < 0\}.$$

Then, the function $V_2 : \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$,

$$\begin{aligned} V_2(x, y, z) = \frac{1}{2}\Big(&\|\nabla_x F(x, y, z)\|^2 + \|\nabla_z F(x, y, z)\|^2 \\ &+ \sum_{j\in\{1,\dots,p\}\setminus\mathcal{J}(x,y,z)} ((\nabla_y F(x, y, z))_j)^2\Big) \\ &+ \frac{1}{2}\|(x, y, z)\|^2_{\text{Saddle}(F)} \end{aligned}$$

satisfies the following

(i) $V_2(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m$ and $V_2(x, y, z) = 0$ if and only if $(x, y, z) \in \text{Saddle}(F)$,

(ii) for any trajectory $t \mapsto (x(t), y(t), z(t))$ of $X_{\text{p-sp}}$, the map $t \mapsto V_2(x(t), y(t), z(t))$ is right-continuous, almost everywhere differentiable, satisfying (a) $\frac{d}{dt}V_2(x(t), y(t), z(t)) < 0$ for all $t \geq 0$ where the derivative exists and $(x(t), y(t), z(t)) \notin \text{Saddle}(F)$ (b) $V_2(x(t'), y(t'), z(t')) \leq \lim_{t\uparrow t'} V_2(x(t), y(t), z(t))$ for all $t' \geq 0$.

As a consequence, the set $\text{Saddle}(F)$ is globally asymptotically stable under $X_{\text{p-sp}}$ and convergence of trajectories is to a point.

Proof: We start by partitioning the domain so that the function V_2 is continuously differentiable in the interior of each of the sets of the partition. Let $\mathcal{I} \subset \{1, \dots, p\}$ and

$$\mathcal{D}(\mathcal{I}) = \{(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m \mid \mathcal{J}(x, y, z) = \mathcal{I}\}.$$

Note that for $\mathcal{I}_1, \mathcal{I}_2 \subset \{1, \dots, p\}$, $\mathcal{I}_1 \neq \mathcal{I}_2$, we have $\mathcal{D}(\mathcal{I}_1) \cap \mathcal{D}(\mathcal{I}_2) = \emptyset$. Moreover,

$$\mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m = \bigcup_{\mathcal{I}\subset\{1,\dots,p\}} \mathcal{D}(\mathcal{I}).$$

Next, consider a trajectory $t \mapsto (x(t), y(t), z(t))$ of $X_{\text{p-sp}}$ starting at some point $(x(0), y(0), z(0)) \in \mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m$. Let $(a, b) \subset \mathbb{R}_{\geq 0}$ be a time interval for which there exists a set $\mathcal{I}' \subset \{1, \dots, p\}$ such that $(x(s), y(s), z(s)) \in \mathcal{D}(\mathcal{I}')$ for all $s \in (a, b)$. That is, the trajectory does not switch domains in the interval (a, b). We wish to show that $\frac{d}{dt}V_2(x(t), y(t), z(t))$ exists almost everywhere on (a, b) and its value is less than zero at all times $s \in (a, b)$ whenever $(x(s), y(s), z(s)) \notin \text{Saddle}(F)$. To this end, define the function

$$\begin{aligned} V_2^{\mathcal{I}'}(x, y, z) = \frac{1}{2}\Big(&\|\nabla_x F(x, y, z)\|^2 + \|\nabla_z F(x, y, z)\|^2 \\ &+ \sum_{j\notin\mathcal{I}'} ((\nabla_y F(x, y, z))_j)^2\Big) + \frac{1}{2}\|(x, y, z)\|^2_{\text{Saddle}(F)}. \end{aligned}$$

Since $V_2^{\mathcal{I}'}$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^p_{\geq 0} \times \mathbb{R}^m$ and $t \mapsto (x(t), y(t), z(t))$ is absolutely continuous, we deduce that $t \mapsto V_2^{\mathcal{I}'}(x(t), y(t), z(t))$ is absolutely continuous. Therefore, by Rademacher's Theorem [27], the map $t \mapsto V_2^{\mathcal{I}'}(x(t), y(t), z(t))$ is differentiable almost everywhere. By definition, $V_2(x(s), y(s), z(s)) = V_2^{\mathcal{I}'}(x(s), y(s), z(s))$ for all $s \in (a, b)$. Therefore

$$\frac{d}{dt}V_2(x(t), y(t), z(t))\Big|_{t=s} = \frac{d}{dt}V_2^{\mathcal{I}'}(x(t), y(t), z(t))\Big|_{t=s}$$

for almost all $s \in (a, b)$. Further, since $V_2^{\mathcal{I}'}$ is continuously differentiable, we have

$$\frac{d}{dt}V_2^{\mathcal{I}'}(x(t), y(t), z(t))\Big|_{t=s} = \mathcal{L}_{X_{\text{p-sp}}}V_2^{\mathcal{I}'}(x(s), y(s), z(s)).$$

Now consider any $(x, y, z) \in \mathcal{D}(\mathcal{I}') \setminus \text{Saddle}(F)$. Then,

$$\begin{aligned} &\mathcal{L}_{X_{\text{p-sp}}}V_2^{\mathcal{I}'}(x, y, z) \\ &= -\nabla_x F(x, y, z)^\top \nabla_{xx} F(x, y, z) \nabla_x F(x, y, z) \\ &\quad + \begin{bmatrix} [\nabla_y F(x, y, z)]^+_y \\ \nabla_z F(x, y, z) \end{bmatrix}^\top \begin{bmatrix} \nabla_{yy} F & \nabla_{yz} F \\ \nabla_{zy} F & \nabla_{zz} F \end{bmatrix}_{(x,y,z)} \begin{bmatrix} [\nabla_y F(x, y, z)]^+_y \\ \nabla_z F(x, y, z) \end{bmatrix} \\ &\quad + \mathcal{L}_{X_{\text{p-sp}}}\Big(\frac{1}{2}\|(x, y, z)\|^2_{\text{Saddle}(F)}\Big). \end{aligned} \tag{15}$$

The first two terms in the above expression are the Lie derivative of $(x, y, z) \mapsto V_2^{\mathcal{I}'}(x, y, z) - \frac{1}{2}\|(x, y, z)\|^2_{\text{Saddle}(F)}$. This computation can be shown using the properties of the operator $[\cdot]^+_y$. Now let $(x_*, y_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x, y, z)$. Then, by Danskin's Theorem [28, p. 99], we have

$$\nabla\|(x, y, z)\|^2_{\text{Saddle}(F)} = 2(x - x_*; y - y_*; z - z_*) \tag{16}$$

Using this expression, we get

$$\begin{aligned} &\mathcal{L}_{X_{\text{p-sp}}}\Big(\frac{1}{2}\|(x, y, z)\|^2_{\text{Saddle}(F)}\Big) \\ &= -(x - x_*)^\top \nabla_x F(x, y, z) + (y - y_*)^\top [\nabla_y F(x, y, z)]^+_y \\ &\quad + (z - z_*)^\top \nabla_z F(x, y, z) \\ &\leq F(x_*, y, z) - F(x_*, y_*, z_*) + F(x_*, y_*, z_*) \\ &\quad - F(x, y_*, z_*), \end{aligned}$$

where the last inequality follows from (12). Now using the above expression in (15) we get

$$
\begin{aligned}
&\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x,y,z)\\
&\le -\nabla_x F(x,y,z)\nabla_{xx}F(x,y,z)\nabla_x F(x,y,z)\\
&\quad + \begin{bmatrix}[\nabla_y F(x,y,z)]_y^+\\ \nabla_z F(x,y,z)\end{bmatrix}^\top \begin{bmatrix}\nabla_{yy}F & \nabla_{yz}F\\ \nabla_{zy}F & \nabla_{zz}F\end{bmatrix}_{(x,y,z)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \begin{bmatrix}[\nabla_y F(x,y,z)]_y^+\\ \nabla_z F(x,y,z)\end{bmatrix}\\
&\quad + F(x_*,y,z) - F(x_*,y_*,z_*) + F(x_*,y_*,z_*)\\
&\quad - F(x,y_*,z_*) \le 0.
\end{aligned}
$$

If $\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x,y,z) = 0$, then (a) $\nabla_x F(x,y,z) = 0$; (b) $x = x_*$; and (c) $F(x_*,y,z) = F(x_*,y_*,z_*)$. From (b) and (6), we conclude that $\nabla_z F(x,y,z) = 0$. From (c) and (14), we deduce that $(y - y_*)^\top g(x_*) = 0$. Note that for each $i \in \{1,\dots,p\}$, we have $(y_i - (y_*)_i)(g(x_*))_i \le 0$. This is because either $(g(x_*))_i = 0$ in which case it is trivial or $(g(x_*))_i < 0$ in which case $(y_*)_i = 0$ (as y_* maximizes the map $y \mapsto y^\top g(x_*)$) thereby making $y_i - (y_*)_i \ge 0$. Since, $(y_i - (y_*)_i)(g(x_*))_i \le 0$ for each i and $(y - y_*)^\top g(x_*) = 0$, we get that for each $i \in \{1,\dots,p\}$, either $(g(x_*))_i = 0$ or $y_i = (y_*)_i$. Thus, $[\nabla_y F(x,y,z)]_y^+ = 0$. These facts imply that $(x,y,z) \in \text{Saddle}(F)$. Therefore, if $(x,y,z) \in \mathcal{D}(\mathcal{I}') \setminus \text{Saddle}(F)$ then $\mathcal{L}_{X_{\text{p-sp}}} V_2^{\mathcal{I}'}(x,y,z) < 0$.

The final step is to show that there is only a countable number of time instances when the trajectory switches its domain. To this end, let $t' > 0$ be a time instant for which there exists a $\delta > 0$ such that $(x(t'),y(t'),z(t')) \in \mathcal{D}(\mathcal{I}')$ and $(x(t),y(t),z(t)) \in \mathcal{D}(\mathcal{I})$ for all $t \in (t'-\delta,t')$ for some $\mathcal{I},\mathcal{I}' \subset \{1,\dots,p\}$. One can deduce by definition that if $\mathcal{I}' \setminus \mathcal{I} \neq \emptyset$, then $\lim_{t\uparrow t'} V_2(x(t),y(t),z(t)) > V_2(x(t'),y(t'),z(t'))$. Thus, at each point of discontinuity of V_2, its value decreases. This implies that there is only a countable number of discontinuities, as (1) V_2 is lower bounded by zero, (2) rationals are countable, and (3) at each point of discontinuity, there is rational between the left limit and the value of V_2 at that point. Further, because there is a finite number of subsets of $\{1,\dots,p\}$, there is a finite number of domain switchings between any two consecutive time instances where V_2 is discontinuous. This is because any domain switch that makes the index set corresponding to the domain of trajectory bigger causes a discontinuity in V_2. With this, we conclude that there are only a countable number of time instances when the trajectory switches its domain, completing the proof. ■

Remark 5.2: (Multiple Lyapunov functions): The Lyapunov function V_2 is discontinuous on the domain $\mathbb{R}^n \times \mathbb{R}^p_{\ge 0} \times \mathbb{R}^m$. However, it can be seen as multiple (continuously differentiable) Lyapunov functions [29], each valid on a domain, patched together in an appropriate way such that along the trajectories of $X_{\text{p-sp}}$, the evolution of V_2 is continuously differentiable with negative derivative at intervals where it is continuous and at times of discontinuity the value of V_2 only decreases. Note that in the absence of the projection in $X_{\text{p-sp}}$ (that is, no y-component of the dynamics), the function V_2 takes a much simpler form with no discontinuities and is continuously differentiable on the entire domain. •

Remark 5.3: (Connection with the literature: II): The two functions whose sum defines V_2 are, individually by themselves, sufficient to establish asymptotic convergence of $X_{\text{p-sp}}$ using LaSalle Invariance arguments, see e.g., [5], [20]. However, the fact that their combination results in a strict Lyapunov function for the projected saddle-point dynamics is a novelty of our analysis here. In [17], a different Lyapunov function is proposed and an exponential rate of convergence is established for a saddle-point-like dynamics which is similar to $X_{\text{p-sp}}$ but without projection components. •

VI. ISS AND SELF-TRIGGERED IMPLEMENTATION OF THE SADDLE-POINT DYNAMICS

Here, we build on the novel Lyapunov function identified in Section V to explore other properties of the projected saddle-point dynamics beyond global asymptotic convergence. Throughout this section, we consider saddle functions F that corresponds to the Lagrangian of an equality-constrained optimization problem, i.e.,

$$F(x,z) = f(x) + z^\top(Ax - b), \tag{17}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}$. The reason behind this focus is that, in this case, the dynamics (5) is smooth and the Lyapunov function identified in Theorem 5.1 is continuously differentiable. These simplifications allow us to analyze input-to-state stability of the dynamics using the theory of ISS-Lyapunov functions (cf. Section II-D). On the other hand, we do not know of such a theory for projected systems, which precludes us from carrying out ISS analysis for dynamics (5) for a general saddle function. The projected saddle-point dynamics (5) for the class of saddle functions given in (17) takes the form

$$\dot{x} = -\nabla_x F(x,z) = -\nabla f(x) - A^\top z, \tag{18a}$$
$$\dot{z} = \nabla_z F(x,z) = Ax - b, \tag{18b}$$

corresponding to equations (5a) and (5c). We term these dynamics simply *saddle-point dynamics* and denote it as $X_{\text{sp}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$.

A. Input-to-state stability

Here, we establish that the saddle-point dynamics (18) is ISS with respect to the set $\text{Saddle}(F)$ when disturbance inputs affect it additively. Disturbance inputs can arise when implementing the saddle-point dynamics as a controller of a physical system because of a variety of malfunctions, including errors in the gradient computation, noise in state measurements, and errors in the controller implementation. In such scenarios, the following result shows that the dynamics (18) exhibits a graceful degradation of its convergence properties, one that scales with the size of the disturbance.

Theorem 6.1: (ISS of saddle-point dynamics): Let the saddle function F be of the form (17), with f strongly convex, twice continuously differentiable, and satisfying $mI \preceq \nabla^2 f(x) \preceq MI$ for all $x \in \mathbb{R}^n$ and some constants $0 < m \le$

$M < \infty$. Then, the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\nabla_x F(x,z) \\ \nabla_z F(x,z) \end{bmatrix} + \begin{bmatrix} u_x \\ u_z \end{bmatrix}, \tag{19}$$

where $(u_x, u_z) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \mathbb{R}^m$ is a measurable and locally essentially bounded map, is ISS with respect to Saddle(F).

Proof: For notational convenience, we refer to (19) by $X_{\text{sp}}^{\text{p}} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$. Our proof consists of establishing that the function $V_3 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$,

$$V_3(x,z) = \frac{\beta_1}{2} \|X_{\text{sp}}(x,z)\|^2 + \frac{\beta_2}{2} \|(x,z)\|^2_{\text{Saddle}(F)} \tag{20}$$

with $\beta_1 > 0$, $\beta_2 = \frac{4\beta_1 M^4}{m^2}$, is an ISS-Lyapunov function with respect to Saddle(F) for X_{sp}^{p}. The statement then directly follows from Proposition 2.2.

We first show (3) for V_3, that is, there exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \|(x,z)\|^2_{\text{Saddle}(F)} \leq V_3(x,z) \leq \alpha_2 \|(x,z)\|^2_{\text{Saddle}(F)}$ for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$. The lower bound follows by choosing $\alpha_1 = \beta_2/2$. For the upper bound, define the function $U : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ by

$$U(x_1, x_2) = \int_0^1 \nabla^2 f(x_1 + s(x_2 - x_1)) ds. \tag{21}$$

By assumption, it holds that $mI \preceq U(x_1, x_2) \preceq MI$ for all $x_1, x_2 \in \mathbb{R}^n$. Also, from the fundamental theorem of calculus, we have $\nabla f(x_2) - \nabla f(x_1) = U(x_1, x_2)(x_2 - x_1)$ for all $x_1, x_2 \in \mathbb{R}^n$. Now pick any $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $(x_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x,z)$, that is, the projection of (x,z) on the set Saddle(F). This projection is unique as Saddle(F) is convex. Then, one can write

$$\begin{aligned} \nabla_x F(x,z) &= \nabla_x F(x_*, z_*) + \int_0^1 \nabla_{xx} F(x(s), z(s))(x - x_*) ds \\ &\quad + \int_0^1 \nabla_{zx} F(x(s), z(s))(z - z_*) ds, \\ &= U(x_*, x)(x - x_*) + A^\top (z - z_*), \end{aligned} \tag{22}$$

where $x(s) = x_* + s(x - x_*)$ and $z(s) = z_* + s(z - z_*)$. Also, note that

$$\begin{aligned} \nabla_z F(x,z) &= \nabla_z F(x_*, z_*) + \int_0^1 \nabla_{xz} F(x(s), z(s))(x - x_*) ds \\ &= A(x - x_*). \end{aligned} \tag{23}$$

The expressions (22) and (23) use $\nabla_x F(x_*, z_*) = 0$, $\nabla_z F(x_*, z_*) = 0$, and $\nabla_{zx} F(x,z) = \nabla_{xz} F(x,z)^\top = A^\top$ for all (x,z). From (22) and (23), we get

$$\begin{aligned} \|X_{\text{sp}}(x,z)\|^2 &\leq \tilde{\alpha}_2 (\|x - x_*\|^2 + \|z - z_*\|^2) \\ &= \tilde{\alpha}_2 \|(x,z)\|^2_{\text{Saddle}(F)}, \end{aligned}$$

where $\tilde{\alpha}_2 = \frac{3}{2}(M^2 + \|A\|^2)$. In the above computation, we have used the inequality $(a+b)^2 \leq 3(a^2 + b^2)$ for any $a, b \in \mathbb{R}$. The above inequality gives the upper bound $V_3(x,z) \leq \alpha_2 \|(x,z)\|^2_{\text{Saddle}(F)}$, where $\alpha_2 = \frac{3\beta_1}{2}(M^2 + \|A\|^2) + \frac{\beta_2}{2}$.

The next step is to show that the Lie derivative of V_3 along the dynamics X_{sp}^{p} satisfies the ISS property (4). Again, pick any $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$ and let $(x_*, z_*) = \text{proj}_{\text{Saddle}(F)}(x,z)$. Then, by Danskin's Theorem [28, p. 99], we get

$$\nabla \|(x,z)\|^2_{\text{Saddle}(F)} = 2(x - x_*; z - z_*).$$

Using the above expression, one can compute the Lie derivative of V_3 along the dynamics X_{sp}^{p} as

$$\begin{aligned} \mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) &= -\beta_1 \nabla_x F(x,z) \nabla_{xx} F(x,z) \nabla_x F(x,z) \\ &\quad - \beta_2 (x - x_*)^\top \nabla_x F(x,z) + \beta_2 (z - z_*)^\top \nabla_z F(x,z) \\ &\quad + \beta_1 \nabla_x F(x,z)^\top \nabla_{xx} F(x,z) u_x \\ &\quad + \beta_1 \nabla_x F(x,z)^\top \nabla_{xz} F(x,z) u_z \\ &\quad + \beta_1 \nabla_z F(x,z)^\top \nabla_{zx} F(x,z) u_x \\ &\quad + \beta_2 (x - x_*)^\top u_x + \beta_2 (z - z_*)^\top u_z. \end{aligned}$$

Due to the particular form of F, we have

$$\begin{aligned} \nabla_x F(x,z) &= \nabla f(x) + A^\top z, & \nabla_z F(x,z) &= Ax - b, \\ \nabla_{xx} F(x,z) &= \nabla^2 f(x), & \nabla_{xz} F(x,z) &= A^\top, \\ \nabla_{zx} F(x,z) &= A, & \nabla_{zz} F(x,z) &= 0. \end{aligned}$$

Also, $\nabla_x F(x_*, z_*) = \nabla_x f(x_*) + A^\top z_* = 0$ and $\nabla_z F(x_*, z_*) = Ax_* - b = 0$. Substituting these values in the expression of $\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3$, replacing $\nabla_x F(x,z) = \nabla_x F(x,z) - \nabla_x F(x_*, z_*) = \nabla f(x) - \nabla f(x_*) + A^\top (z - z_*) = U(x_*, x)(x - x_*) + A^\top (z - z_*)$, and simplifying,

$$\begin{aligned} &\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) = \\ &- \beta_1 (U(x_*, x)(x - x_*))^\top \nabla^2 f(x) (U(x_*, x)(x - x_*)) \\ &- \beta_1 (z - z_*)^\top A \nabla^2 f(x) A^\top (z - z_*) \\ &- \beta_1 (U(x_*, x)(x - x_*))^\top \nabla^2 f(x) A^\top (z - z_*) \\ &- \beta_1 (z - z_*)^\top A \nabla^2 f(x) (U(x_*, x)(x - x_*)) \\ &- (x - x_*)^\top U(x_*, x)(x - x_*) \\ &+ \beta_1 (U(x_*, x)(x - x_*) + A^\top (z - z_*))^\top \nabla^2 f(x) u_x \\ &+ \beta_1 (U(x_*, x)(x - x_*) + A^\top (z - z_*))^\top A^\top u_z \\ &+ \beta_2 (x - x_*)^\top u_x + \beta_1 (A(x - x_*))^\top A u_x + \beta_2 (z - z_*)^\top u_z. \end{aligned}$$

Upper bounding now the terms using $\|\nabla^2 f(x)\|, \|U(x_*, x)\| \leq M$ for all $x \in \mathbb{R}^n$ yields

$$\begin{aligned} &\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) \\ &\quad \leq -[x - x_*; \, A^\top (z - z_*)]^\top \overline{U}(x_*, x)[x - x_*; \, A^\top (z - z_*)] \\ &\qquad + C_x(x,z)\|u_x\| + C_z(x,z)\|u_z\|, \end{aligned} \tag{24}$$

where

$$\begin{aligned} C_x(x,z) &= \Big(\beta_1 M^2 \|x - x_*\| + \beta_1 M \|A\| \|z - z_*\| \\ &\qquad + \beta_2 \|x - x_*\| + \beta_1 \|A\|^2 \|x - x_*\|\Big), \\ C_z(x,z) &= \Big(\beta_1 M \|A\| \|x - x_*\| + \beta_1 \|A\|^2 \|z - z_*\| \\ &\qquad + \beta_2 \|z - z_*\|\Big), \end{aligned}$$

and $\overline{U}(x_*, x)$ is

$$\begin{bmatrix} \beta_1 U \nabla^2 f(x) U + \beta_2 U & \beta_1 U \nabla^2 f(x) \\ \beta_1 \nabla^2 f(x) U & \beta_1 \nabla^2 f(x) \end{bmatrix}.$$

where $U = U(x_*, x)$. Note that $C_x(x,z) \le \tilde{C}_x \|x - x_*; z - z_*\| = \tilde{C}_x \|(x,z)\|_{\text{Saddle}(F)}$ and $C_z(x,z) \le \tilde{C}_z \|x - x_*; z - z_*\| = \tilde{C}_z \|(x,z)\|_{\text{Saddle}(F)}$, where

$$\tilde{C}_x = \beta_1 M^2 + \beta_1 M \|A\| + \beta_2 + \beta_1 \|A\|^2,$$
$$\tilde{C}_z = \beta_1 M \|A\| + \beta_1 \|A\|^2 + \beta_2.$$

From Lemma A.1, we have $\overline{U}(x_*, x) \succeq \lambda_m I$, where $\lambda_m > 0$. Employing these facts in (24), we obtain

$$\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) \le -\lambda_m (\|x - x_*\|^2 + \|A^\top (z - z_*)\|^2) + (\tilde{C}_x + \tilde{C}_z) \|(x,z)\|_{\text{Saddle}(F)} \|u\|$$

From Lemma A.2, we get

$$\begin{aligned}\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) &\le -\lambda_m (\|x - x_*\|^2 + \lambda_{\text{s}}(AA^\top) \|z - z_*\|^2 \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x,z)\|_{\text{Saddle}(F)} \|u\| \\ &\le -\tilde{\lambda}_m \|(x,z)\|^2_{\text{Saddle}(F)} \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x,z)\|_{\text{Saddle}(F)} \|u\|,\end{aligned}$$

where $\tilde{\lambda}_m = \lambda_m \min\{1, \lambda_{\text{s}}(AA^\top)\}$. Now pick any $\theta \in (0,1)$. Then,

$$\begin{aligned}\mathcal{L}_{X_{\text{sp}}^{\text{p}}} V_3(x,z) &\le -(1-\theta)\tilde{\lambda}_m \|(x,z)\|^2_{\text{Saddle}(F)} \\ &\quad - \theta \tilde{\lambda}_m \|(x,z)\|^2_{\text{Saddle}(F)} \\ &\quad + (\tilde{C}_x + \tilde{C}_z) \|(x,z)\|_{\text{Saddle}(F)} \|u\| \\ &\le -(1-\theta)\tilde{\lambda}_m \|(x,z)\|^2_{\text{Saddle}(F)},\end{aligned}$$

whenever $\|(x,z)\|_{\text{Saddle}(F)} \ge \frac{\tilde{C}_x + \tilde{C}_z}{\theta \tilde{\lambda}_m} \|u\|$, which proves the ISS property. ■

Remark 6.2: (Relaxing global bounds on Hessian of f): The assumption on the Hessian of f in Theorem 6.1 is restrictive, but there are functions other than quadratic that satisfy it, see e.g. [30, Section 6]. We conjecture that the global upper bound on the Hessian can be relaxed by resorting to the notion of semiglobal ISS, and we will explore this in the future. •

The above result has the following consequence.

Corollary 6.3: (Lyapunov function for saddle-point dynamics): Let the saddle function F be of the form (17), with f strongly convex, twice continuously differentiable, and satisfying $mI \preceq \nabla^2 f(x) \preceq MI$ for all $x \in \mathbb{R}^n$ and some constants $0 < m \le M < \infty$. Then, the function V_3 (20) is a Lyapunov function with respect to the set Saddle(F) for the saddle-point dynamics (18).

Remark 6.4: (ISS with respect to Saddle(F) does not imply bounded trajectories): Note that Theorem 6.1 bounds only the distance of the trajectories of (19) to Saddle(F). Thus, if Saddle(F) is unbounded, the trajectories of (19) can be unbounded under arbitrarily small constant disturbances. However, if matrix A has full row-rank, then Saddle(F) is a singleton and the ISS property implies that the trajectory of (19) remains bounded under bounded disturbances. •

As pointed out in the above remark, if Saddle(F) is not unique, then the trajectories of the dynamics might not be bounded. We next look at a particular type of disturbance input which guarantees bounded trajectories even when Saddle(F) is unbounded. Pick any $(x_*, z_*) \in$ Saddle(F) and define the function $\tilde{V}_3 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\ge 0}$ as

$$\tilde{V}_3(x,z) = \frac{\beta_1}{2} \|X_{\text{sp}}(x,z)\|^2 + \frac{\beta_2}{2} (\|x - x_*\|^2 + \|z - z_*\|^2)$$

with $\beta_1 > 0$, $\beta_2 = \frac{4\beta_1 M^4}{m^2}$. One can show, following similar steps as those of proof of Theorem 6.1, that the function $\tilde{V}_3$ is an ISS Lyapunov function with respect to the point (x_*, z_*) for the dynamics X_{sp}^{p} when the disturbance input to z-dynamics has the special structure $u_z = A\tilde{u}_z$, $\tilde{u}_z \in \mathbb{R}^n$. This type of disturbance is motivated by scenarios with measurement errors in the values of x and z used in (18) and without any computation error of the gradient term in the z-dynamics. The following statement makes precise the ISS property for this particular disturbance.

Corollary 6.5: (ISS of saddle-point dynamics): Let the saddle function F be of the form (17), with f strongly convex, twice continuously differentiable, and satisfying $mI \preceq \nabla^2 f(x) \preceq MI$ for all $x \in \mathbb{R}^n$ and some constants $0 < m \le M < \infty$. Then, the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\nabla_x F(x,z) \\ \nabla_z F(x,z) \end{bmatrix} + \begin{bmatrix} u_x \\ A\tilde{u}_z \end{bmatrix}, \tag{25}$$

where $(u_x, \tilde{u}_z) : \mathbb{R}_{\ge 0} \to \mathbb{R}^{2n}$ is measurable and locally essentially bounded input, is ISS with respect to every point of Saddle(F).

The proof is analogous to that of Theorem 6.1 with the key difference that the terms $C_x(x,z)$ and $C_z(x,z)$ appearing in (24) need to be upper bounded in terms of $\|x - x_*\|$ and $\|A^\top (z - z_*)\|$. This can be done due to the special structure of u_z. With these bounds, one arrives at the condition (4) for Lyapunov $\tilde{V}_3$ and dynamics (25). One can deduce from Corollary 6.5 that the trajectory of (25) remains bounded for bounded input even when Saddle(F) is unbounded.

Example 6.6: (ISS property of saddle-point dynamics): Consider $F : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$ of the form (17) with

$$f(x) = \|x\|^2, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}. \tag{26}$$

Then, Saddle$(F) = \{(x,z) \in \mathbb{R}^2 \times \mathbb{R}^3 \mid x = (1,1), z = (-1,-1,-1) + \lambda(1,-1,-1), \lambda \in \mathbb{R}\}$ is a continuum of points. Note that $\nabla^2 f(x) = 2I$, thus, satisfying the assumption of bounds on the Hessian of f. By Theorem 6.1, the saddle-point dynamics for this saddle function F is input-to-state stable with respect to the set Saddle(F). This fact is illustrated in Figure 2, which also depicts how the specific structure of the disturbance input in (25) affects the boundedness of the trajectories. •

B. Self-triggered implementation

In this section we develop an opportunistic state-triggered implementation of the (continuous-time) saddle-point dynamics. Our aim is to provide a discrete-time execution of the algorithm, either on a physical system or as an optimization strategy, that do not require the continuous evaluation of the vector field and instead adjust the stepsize based on the current state of the system. Formally, given a sequence of triggering

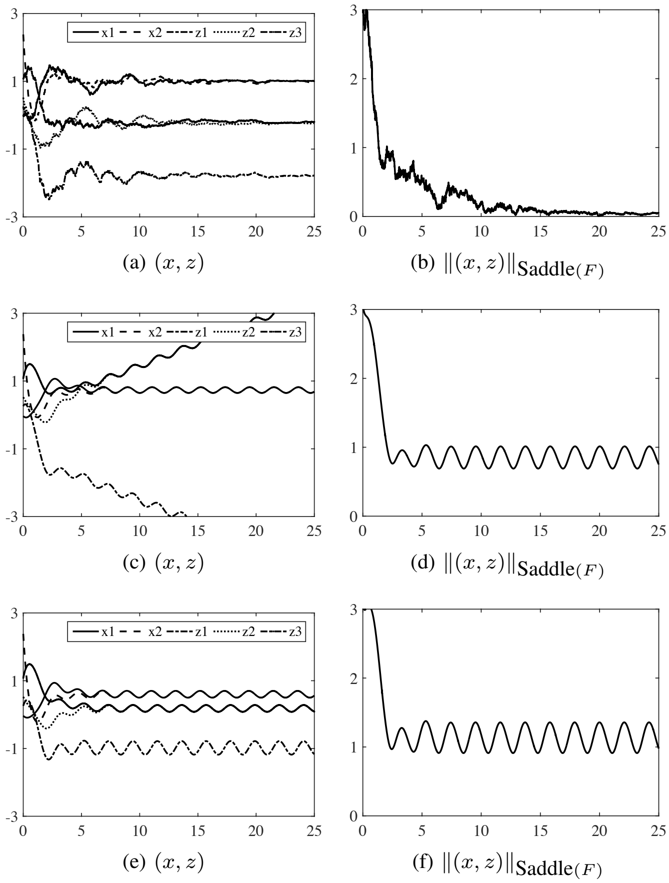


Fig. 2. Plots (a)-(b) show the ISS property, cf Theorem 6.1, of the dynamics (19) for the saddle function F defined by (26). The initial condition is $x(0) = (-0.0377, 2.3819)$ and $z(0) = (0.2580, 0.5229, 1.0799)$ and the input u is exponentially decaying in magnitude. As shown in (a)-(b), the trajectory converges asymptotically to a saddle point as the input is vanishing. Plots (c)-(d) have the same initial condition but the disturbance input consists of a constant plus a sinusoid. The trajectory is unbounded under bounded input while the distance to the set of saddle points remains bounded, cf. Remark 6.4. Plots (e)-(f) have the same initial condition but the disturbance input to the z-dynamics is of the form (25). In this case, the trajectory remains bounded as the dynamics is ISS with respect to each saddle point, cf. Corollary 6.5.

time instants $\{t_k\}_{k=0}^{\infty}$, with $t_0 = 0$, we consider the following implementation of the saddle-point dynamics

$$\dot{x}(t) = -\nabla_x F(x(t_k), z(t_k)), \tag{27a}$$
$$\dot{z}(t) = \nabla_z F(x(t_k), z(t_k)). \tag{27b}$$

for $t \in [t_k, t_{k+1})$ and $k \in \mathbb{Z}_{\geq 0}$. The objective is then to design a criterium to opportunistically select the sequence of triggering instants, guaranteeing at the same time the feasibility of the execution and global asymptotic convergence, see e.g., [31]. Towards this goal, we look at the evolution of the Lyapunov function V_3 in (20) along (27),

$$\begin{aligned}
&\nabla V_3(x(t), z(t))^\top X_{\mathrm{sp}}(x(t_k), z(t_k)) \\
&= \mathcal{L}_{X_{\mathrm{sp}}} V_3(x(t_k), z(t_k)) \\
&\quad + \Big(\nabla V_3(x(t), z(t)) - \nabla V_3(x(t_k), z(t_k))\Big)^\top \\
&\qquad\qquad\qquad\qquad X_{\mathrm{sp}}(x(t_k), z(t_k)).
\end{aligned} \tag{28}$$

We know from Corollary 6.3 that the first summand is negative outside $\mathrm{Saddle}(F)$. Clearly, for $t = t_k$, the second summand vanishes, and by continuity, for t sufficiently close to t_k, this summand remains smaller in magnitude than the first, ensuring the decrease of V_3. The next result is helpful in making this argument precise.

Proposition 6.7: (Gradient of V_3 is locally Lipschitz): Let the saddle function F be of the form (17), with f twice differentiable, map $x \mapsto \nabla^2 f(x)$ Lipschitz with some constant $L > 0$, and $mI \preceq \nabla^2 f(x) \preceq MI$ for all $x \in \mathbb{R}^n$ and some constants $0 < m \leq M < \infty$. Then, for V_3 given in (20), the following holds

$$\|\nabla V_3(x_2, z_2) - \nabla V_3(x_1, z_1)\| \leq \xi(x_1, z_1)\|x_2 - x_1; z_2 - z_1\|,$$

for all $(x_1, z_1), (x_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^m$, where

$$\begin{aligned}
\xi(x_1, z_1) &= \sqrt{3}\Big(\beta_1^2(\xi_1(x_1, z_1)^2 + \|A\|^4 + \|A\|^2\xi_2^2) + \beta_2^2\Big)^{\frac{1}{2}}, \\
\xi_1(x_1, z_1) &= M\xi_2 + L\|\nabla_x F(x_1, z_1)\|, \\
\xi_2 &= \max\{M, \|A\|\}.
\end{aligned} \tag{29}$$

Proof: For the map $(x, z) \mapsto \nabla_x F(x, z)$, note that

$$\begin{aligned}
&\|\nabla_x F(x_2, z_2) - \nabla_x F(x_1, z_1)\| \\
&\quad = \Big\| \int_0^1 \nabla_{xx} F(x(s), z(s))(x_2 - x_1) ds \\
&\qquad\qquad + \int_0^1 \nabla_{zx} F(x(s), z(s))(z_2 - z_1)\Big\| \\
&\quad \leq M\|x_2 - x_1\| + \|A\|\|z_2 - z_1\| \\
&\quad \leq \xi_2\|x_2 - x_1; z_2 - z_1\|,
\end{aligned} \tag{30}$$

where $x(s) = x_1 + s(x_2 - x_1)$, $z(s) = z_1 + s(z_2 - z_1)$ and $\xi_2 = \max\{M, \|A\|\}$. In the above inequalities we have used the fact that $\|\nabla_{xx} F(x, z)\| = \|\nabla^2 f(x)\| \leq M$ for any (x, z). Further, the following Lipschitz condition holds by assumption

$$\|\nabla_{xx} F(x_2, z_2) - \nabla_{xx} F(x_1, z_1)\| \leq L\|x_2 - x_1\| \tag{31}$$

Using (30) and (31), we get

$$\begin{aligned}
&\|\nabla_{xx} F(x_2, z_2)\nabla_x F(x_2, z_2) - \nabla_{xx} F(x_1, z_1)\nabla_x F(x_1, z_1)\| \\
&\quad \leq \|\nabla_{xx} F(x_2, z_2)(\nabla_x F(x_2, z_2) - \nabla_x F(x_1, z_1))\| \\
&\qquad + \|(\nabla_{xx} F(x_2, z_2) - \nabla_{xx} F(x_1, z_1))\nabla_x F(x_1, z_1)\| \\
&\quad \leq \xi_1(x_1, z_1)\|x_2 - x_1; z_2 - z_1\|,
\end{aligned} \tag{32}$$

where $\xi_1(x_1, z_1) = M\xi_2 + L\|\nabla_x F(x_1, z_1)\|$. Also,

$$\begin{aligned}
\|\nabla_z F(x_2, z_2) - \nabla_z F(x_1, z_1)\| &= \|A(x_2 - x_1)\| \\
&\leq \|A\|\|x_2 - x_1; z_2 - z_1\|
\end{aligned} \tag{33}$$

Now note that

$$\begin{aligned}
\nabla_x V_3(x, z) &= \beta_1\Big(\nabla_{xx} F(x, z)\nabla_x F(x, z) + A^\top \nabla_z F(x, z)\Big) \\
&\qquad + \beta_2(x - x_*), \\
\nabla_z V_3(x, z) &= \beta_1 A \nabla_x F(x, z) + \beta_2(z - z_*).
\end{aligned}$$

Finally, using (30), (32), and (33), we get

$$\begin{aligned}
&\|\nabla V_3(x_2, z_2) - \nabla V_3(x_1, z_1)\|^2 = \|\nabla_x V_3(x_2, z_2) \\
&\qquad - \nabla_x V_3(x_1, z_1)\|^2 + \|\nabla_z V_3(x_2, z_2) - \nabla_z V_3(x_1, z_1)\|^2 \\
&\overset{(a)}{\leq} 3\beta_1^2\|\nabla_{xx} F(x_2, z_2)\nabla_x F(x_2, z_2)
\end{aligned}$$

$$
\begin{aligned}
&- \nabla_{xx}F(x_1,z_1)\nabla_x F(x_1,z_1)\|^2 \\
&+ 3\beta_1^2\|A^\top(\nabla_z F(x_2,z_2) - \nabla_z F(x_1,z_1))\|^2 + 3\beta_2^2\|x_2 - x_1\|^2 \\
&+ 3\beta_1^2\|A(\nabla_x F(x_2,z_2) - \nabla_x F(x_1,z_1))\|^2 + 3\beta_2^2\|z_2 - z_1\|^2 \\
&\le \xi(x_1,z_1)^2\|x_2 - x_1; z_2 - z_1\|^2,
\end{aligned}
$$

where in (a), we have used the inequality $(a+b)^2 \le 3(a^2+b^2)$ for any $a, b \in \mathbb{R}$. This concludes the proof. ■

Using Proposition 6.7 in (28), we obtain

$$
\begin{aligned}
&\nabla V_3(x(t),z(t))^\top X_{\text{sp}}(x(t_k),z(t_k)) \\
&\le \mathcal{L}_{X_{\text{sp}}}V_3(x(t_k),z(t_k)) + \xi(x(t_k),z(t_k)) \\
&\quad \|(x(t)-x(t_k));(z(t)-z(t_k))\|\|X_{\text{sp}}(x(t_k),z(t_k))\| \\
&= \mathcal{L}_{X_{\text{sp}}}V_3(x(t_k),z(t_k)) \\
&\quad + (t-t_k)\xi(x(t_k),z(t_k))\|X_{\text{sp}}(x(t_k),z(t_k))\|^2,
\end{aligned}
$$

where the equality follows from writing $(x(t),z(t))$ in terms of $(x(t_k),z(t_k))$ by integrating (27). Therefore, in order to ensure the monotonic decrease of V_3, we require the above expression to be nonpositive. That is,

$$
t_{k+1} \le t_k - \frac{\mathcal{L}_{X_{\text{sp}}}V_3(x(t_k),z(t_k))}{\xi(x(t_k),z(t_k))\|X_{\text{sp}}(x(t_k),z(t_k))\|^2}. \tag{34}
$$

Note that to set t_{k+1} equal to the right-hand side of the above expression, one needs to compute the Lie derivative at $(x(t_k),z(t_k))$. We then distinguish between two possibilities. If the self-triggered saddle-point dynamics acts as a closed-loop physical system and its equilibrium points are known, then computing the Lie derivative is feasible and one can use (34) to determine the triggering times. If, however, the dynamics is employed to seek the primal-dual optimizers of an optimization problem, then computing the Lie derivative is infeasible as it requires knowledge of the optimizer. To overcome this limitation, we propose the following alternative triggering criterium which satisfies (34) as shown later in our convergence analysis,

$$
t_{k+1} = t_k + \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)\xi(x(t_k),z(t_k))}, \tag{35}
$$

where $\tilde{\lambda}_m = \lambda_m \min\{1, \lambda_{\text{s}}(AA^\top)\}$, λ_m is given in Lemma A.1, and $\lambda_{\text{s}}(AA^\top)$ is the smallest nonzero eigenvalue of $AA^\top$. In either (34) or (35), the right-hand side depends only on the state $(x(t_k),z(t_k))$. These triggering times for the dynamics (27) define a first-order Euler discretization of the saddle-point dynamics with step-size selection based on the current state of the system. It is for this reason that we refer to (27) together with either the triggering criterium (34) or (35) as the *self-triggered saddle-point dynamics*. In integral form, this dynamics results in a discrete-time implementation of (18) given as

$$
\begin{bmatrix} x(t_{k+1}) \\ z(t_{k+1}) \end{bmatrix} = \begin{bmatrix} x(t_k) \\ z(t_k) \end{bmatrix} + (t_{k+1}-t_k)X_{\text{sp}}(x(t_k),z(t_k)).
$$

We understand the solution of (27) in the Caratheodory sense (note that this dynamics has a discontinuous right-hand side). The existence of such solutions, possibly defined only on a finite time interval, is guaranteed from the fact that along any trajectory of the dynamics there are only countable number of discontinuities encountered in the vector field. The next result however shows that solutions of (27) exist over the entire domain $[0,\infty)$ as the difference between consecutive triggering times of the solution is lower bounded by a positive constant. Also, it establishes the asymptotic convergence of solutions to the set of saddle points.

Theorem 6.8: (Convergence of the self-triggered saddle-point dynamics): Let the saddle function F be of the form (17), with A having full row rank, f strongly convex, twice differentiable, and satisfying $mI \preceq \nabla^2 f(x) \preceq MI$ for all $x \in \mathbb{R}^n$ and some constants $0 < m \le M < \infty$. Let the map $x \mapsto \nabla^2 f(x)$ be Lipschitz with some constant $L > 0$. Then, Saddle(F) is singleton. Let Saddle$(F) = \{(x_*, z_*)\}$. Then, for any initial condition $(x(0),z(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, we have

$$
\lim_{k\to\infty}(x(t_k),z(t_k)) = (x_*,z_*)
$$

for the solution of the self-triggered saddle-point dynamics, defined by (27) and (35), starting at $(x(0),z(0))$. Further, there exists $\mu_{(x(0),z(0))} > 0$ such that the triggering times of this solution satisfy

$$
t_{k+1} - t_k \ge \mu_{(x(0),z(0))}, \qquad \text{for all } k \in \mathbb{N}.
$$

Proof: Note that there is a unique equilibrium point to the saddle-point dynamics (18) for F satisfying the stated hypotheses. Therefore, the set of saddle point is singleton for this F. Now, given $(x(0),z(0)) \in \mathbb{R}^n \times \mathbb{R}^m$, let $V_3^0 = V_3(x(0),z(0))$ and define

$$
G = \max\{\|\nabla_x F(x,z)\| \mid (x,z) \in V_3^{-1}(\le V_3^0)\},
$$

where, we use the notation for the sublevel set of V_3 as

$$
V_3^{-1}(\le \alpha) = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \mid V_3(x,z) \le \alpha\}
$$

for any $\alpha \ge 0$. Since V_3 is radially unbounded, the set $V_3^{-1}(\le V_3^0)$ is compact and so, G is well-defined and finite. If the trajectory of the self-triggered saddle-point dynamics is contained in $V_3^{-1}(\le V_3^0)$, then we can bound the difference between triggering times in the following way. From Proposition 6.7 for all $(x,z) \in V_3^{-1}(\le V_3^0)$, we have $\xi_1(x,z) = M\xi_2 + L\|\nabla_x F(x,z)\| \le M\xi_2 + LG =: T_1$. Hence, for all $(x,z) \in V_3^{-1}(\le V_3^0)$, we get

$$
\begin{aligned}
\xi(x,z) &= \Big(\beta_1^2(\xi_1(x,z)^2 + \|A\|^4 + \|A\|^2\xi_2^2) + \beta_2^2\Big)^{\frac{1}{2}} \\
&\le \Big(\beta_1^2(T_1^2 + \|A\|^4 + \|A\|^2 + \xi_2^2) + \beta_2^2\Big)^{\frac{1}{2}} \\
&=: T_2.
\end{aligned}
$$

Using the above bound in (35), we get for all $k \in \mathbb{N}$

$$
\begin{aligned}
t_{k+1} - t_k &= \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)\xi(x(t_k),z(t_k))} \\
&\ge \frac{\tilde{\lambda}_m}{3(M^2 + \|A\|^2)T_2} > 0.
\end{aligned}
$$

This implies that as long as the trajectory is contained in $V_3^{-1}(\le V_3^0)$, the inter-trigger times are lower bounded by a positive quantity. Our next step is to show that the trajectory is contained in $V_3^{-1}(\le V_3^0)$. Note that if (34) is satisfied for the triggering condition (35), then the sequence $\{V_3(x(t_k), z(t_k))\}_{k\in\mathbb{N}}$ is strictly decreasing. Since V_3 is non-negative, this implies that $\lim_{k\to\infty} V_3(x(t_k), z(t_k)) = 0$ and so, by continuity, $\lim_{k\to\infty}(x(t_k), z(t_k)) = (x_*, z_*)$. Thus, it remains to show that (35) implies (34). To this end, first note the following inequalities shown in the proof of Theorem 6.1

$$\frac{\|X_{\text{sp}}(x,z)\|^2}{3(M^2+\|A\|^2)} \le \|(x-x_*);(z-z_*)\|^2, \tag{36a}$$

$$\left|\mathcal{L}_{X_{\text{sp}}} V_3(x,z)\right| \ge \tilde{\lambda}_m \|(x-x_*);(z-z_*)\|^2. \tag{36b}$$

Using these bounds, we get from (35)

$$\begin{aligned}
&t_{k+1} - t_k \\
&= \frac{\tilde{\lambda}_m}{3(M^2+\|A\|^2)\xi(x(t_k), z(t_k))} \\
&\overset{(a)}{=} \frac{\tilde{\lambda}_m \|X_{\text{sp}}(x(t_k), z(t_k))\|^2}{3(M^2+\|A\|^2)\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\
&\overset{(b)}{\le} \frac{\tilde{\lambda}_m \|(x(t_k)-x_*);(z(t_k)-z_*)\|^2}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\
&\overset{(c)}{\le} \frac{\left|\mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k))\right|}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2} \\
&= -\frac{\mathcal{L}_{X_{\text{sp}}} V_3(x(t_k), z(t_k))}{\xi(x(t_k), z(t_k))\|X_{\text{sp}}(x(t_k), z(t_k))\|^2},
\end{aligned}$$

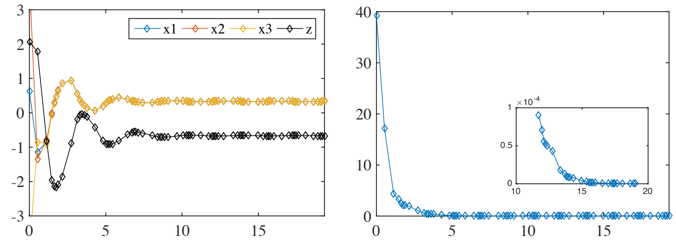
where (a) is valid as $\|X_{\text{sp}}(x(t_k), z(t_k))\| \neq 0$, (b) follows from (36a), and (c) follows from (36b). Thus, (35) implies (34) which completes the proof. ■

Note from the above proof that the convergence implication of Theorem 6.8 is also valid when the triggering criterium is given by (34) with the inequality replaced by the equality.

Example 6.9: (Self-triggered saddle-point dynamics): Consider the function $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$,

$$F(x,z) = \|x\|^2 + z(x_1 + x_2 + x_3 - 1). \tag{37}$$

Then, with the notation of (17), we have $f(x) = \|x\|^2$, $A = [1,1,1]$, and $b = 1$. The set of saddle points is a singleton, $\text{Saddle}(F) = \{((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), -\frac{2}{3})\}$. Note that $\nabla^2 f(x) = 2I$ and A has full row-rank, thus, the hypotheses of Theorem 6.8 are met. Hence, for this F, the self-triggered saddle-point dynamics (27) with triggering times (35) converges asymptotically to the saddle point of F. Moreover, the difference between two consecutive triggering times is lower bounded by a finite quantity. Figure 3 illustrates a simulation of dynamics (27) with triggering criteria (34) (replacing inequality with equality), showing that this triggering criteria also ensures convergence as commented above. •



(a) (x,z) (b) V_3

Fig. 3. Illustration of the self-triggered saddle-point dynamics defined by (27) with the triggering criterium (34). The saddle function F is defined in (37). With respect to the notation of Theorem 6.8, we have $m = M = 2$ and $\|A\| = \sqrt{3}$. We select $\beta_1 = 0.1$, then $\beta_2 = 1.6$, and from (29), $\xi_1 = 2$. These constants define functions V_3 (cf. (20)), ξ, and ξ_2 (cf. (29)) and also, the triggering times (35). In plot(a), the initial condition is $x(0) = (0.6210, 3.9201, -4.0817)$, $z(0) = 2.0675$. The trajectory converges to the unique saddle-point and the inter-trigger times are lower bounded by a positive quantity.

VII. Conclusions

This paper has studied the global convergence and robustness properties of the projected saddle-point dynamics. We have provided a characterization of the omega-limit set in terms of the Hessian blocks of the saddle function. Building on this result, we have established global asymptotic convergence assuming only local strong convexity-concavity of the saddle function. For the case when this strong convexity-concavity property is global, we have identified a Lyapunov function for the dynamics. In addition, when the saddle function takes the form of a Lagrangian of an equality constrained optimization problem, we have established the input-to-state stability of the saddle-point dynamics by identifying an ISS Lyapunov function, which we have used to design a self-triggered discrete-time implementation. In the future, we aim to generalize the ISS results to more general classes of saddle functions. In particular, we wish to define a "semi-global" ISS property that we conjecture will hold for the saddle-point dynamics when we relax the global upper bound on the Hessian block of the saddle function. Further, to extend the ISS results to the projected saddle-point dynamics, we plan to develop the theory of ISS for general projected dynamical systems. Finally, we intend to apply these theoretical guarantees to determine robustness margins and design opportunistic state-triggered implementations for frequency regulation controllers in power networks.

Appendix

Here we collect a couple of auxiliary results used in the proof of Theorem 6.1.

Lemma A.1: (Auxiliary result for Theorem 6.1: I): Let $B_1, B_2 \in \mathbb{R}^{n\times n}$ be symmetric matrices satisfying $mI \preceq B_1, B_2 \preceq MI$ for some $0 < m \le M < \infty$. Let $\beta_1 > 0$, $\beta_2 = \frac{4\beta_1 M^4}{m^2}$, and $\lambda_m = \min\{\frac{1}{2}\beta_1 m, \beta_1 m^3\}$. Then,

$$W := \begin{bmatrix} \beta_1 B_1 B_2 B_1 + \beta_2 B_1 & \beta_1 B_1 B_2 \\ \beta_1 B_2 B_1 & \beta_1 B_2 \end{bmatrix} \succ \lambda_m I.$$

Proof: Reasoning with Schur complement [21, Section A.5.5], the expression $W - \lambda_m I \succ 0$ holds if and only if the following hold

$$\begin{aligned}
&\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I \succ 0, \\
&\beta_1 B_2 - \lambda_m I - \\
&\beta_1 B_2 B_1 (\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \beta_1 B_1 B_2 \succ 0.
\end{aligned} \tag{A.38}$$

The first of the above inequalities is true since $\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I \succeq \beta_1 m^3 I + \beta_2 m I - \lambda_m I \succ 0$ as $\lambda_m \le \beta_1 m^3$. For the second inequality note that

$$\beta_1 B_2 - \lambda_m I$$

$$
\begin{aligned}
&- \beta_1 B_2 B_1 (\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \beta_1 B_1 B_2 \\
&\succeq (\beta_1 m - \lambda_m) I \\
&\quad - \beta_1^2 M^4 \lambda_{\max}\Big((\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)^{-1} \Big) I \\
&\succeq \Big(\frac{1}{2}\beta_1 m - \frac{\beta_1^2 M^4}{\lambda_{\min}(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)} \Big) I,
\end{aligned}
$$

where in the last inequality we have used the fact that $\lambda_m \leq \beta_1 m/2$. Note that $\lambda_{\min}\Big(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I\Big) \geq \beta_1 m^3 + \beta_2 m - \lambda_m \geq \beta_2 m$. Using this lower bound, the following holds

$$
\begin{aligned}
&\frac{1}{2}\beta_1 m - \frac{\beta_1^2 M^4}{\lambda_{\min}(\beta_1 B_1 B_2 B_1 + \beta_2 B_1 - \lambda_m I)} \\
&\qquad\qquad \geq \frac{1}{2}\beta_1 m - \frac{\beta_1^2 M^4}{\beta_2 m} = \frac{1}{4}\beta_1 m.
\end{aligned}
$$

The above set of inequalities show that the second inequality in (A.38) holds, which concludes the proof. ■

Lemma A.2: (Auxiliary result for Theorem 6.1: II): Let F be of the form (17) with f strongly convex. Let $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(x_*, z_*) = \mathrm{proj}_{\mathrm{Saddle}(F)}(x, z)$. Then, $z - z_*$ is orthogonal to the kernel of $A^\top$, and

$$
\|A^\top (z - z_*)\|^2 \geq \lambda_{\mathrm{s}}(AA^\top) \|z - z_*\|^2,
$$

where $\lambda_{\mathrm{s}}(AA^\top)$ is the smallest nonzero eigenvalue of $AA^\top$.

Proof: Our first step is to show that there exists $x_* \in \mathbb{R}^n$ such that if $(x, z) \in \mathrm{Saddle}(F)$, then $x = x_*$. By contradiction, assume that $(x_1, z_1), (x_2, z_2) \in \mathrm{Saddle}(F)$ and $x_1 \neq x_2$. The saddle point property at (x_1, z_1) and (x_2, z_2) yields

$$
F(x_1, z_1) \leq F(x_2, z_1) \leq F(x_2, z_2) \leq F(x_1, z_2) \leq F(x_1, z_1).
$$

This implies that $F(x_1, z_1) = F(x_2, z_1)$, which is a contradiction as $x \mapsto F(x, z_1)$ is strongly convex and x_1 is a minimizer of this map. Therefore, $\mathrm{Saddle}(F) = \{x_*\} \times \mathcal{Z}$, $\mathcal{Z} \subset \mathbb{R}^m$. Further, recall that the set of saddle points of F are the set of equilibrium points of the saddle point dynamics (18). Hence, $(x_*, z) \in \mathrm{Saddle}(F)$ if and only if

$$
\nabla f(x_*) + A^\top z = 0.
$$

We conclude from this that

$$
\mathcal{Z} = -(A^\top)^\dagger \nabla f(x_*) + \ker(A^\top), \tag{A.39}
$$

where $(A^\top)^\dagger$ and $\ker(A^\top)$ are the Moore-Penrose pseudoinverse [21, Section A.5.4] and the kernel of $A^\top$, respectively. By definition of the projection operator, if $(x_*, z_*) = \mathrm{proj}_{\mathrm{Saddle}(F)}(x, z)$, then $z_* = \mathrm{proj}_{\mathcal{Z}}(z)$ and so, from (A.39), we deduce that $(z - z_*)^\top v = 0$ for all $v \in \ker(A^\top)$. Using this fact, we conclude the proof by writing

$$
\begin{aligned}
\|A^\top (z - z_*)\|^2 &= (z - z_*)^\top AA^\top (z - z_*) \\
&\geq \lambda_{\mathrm{s}}(AA^\top) \|z - z_*\|^2,
\end{aligned}
$$

where the inequality follows by writing the eigenvalue decomposition of $AA^\top$, expanding the quadratic expression in $(z - z_*)$, and lower-bounding the terms. ■

ACKNOWLEDGMENTS

We would like to thank Simon K. Niederländer for discussions on Lyapunov functions for the saddle-point dynamics.

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