Event-triggered stabilization of scalar linear systems under packet drops

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Abstract— **This paper considers scalar linear systems with process noise and packet drops between the sensor and the controller. Motivated by considerations about the efficient use of resources, we design an event-triggered transmission policy to ensure exponential convergence of the second moment of the plant state to an ultimate bound in finite time. Our technical approach evaluates the satisfaction of the control objective in an online fashion and designs an event-triggering policy that is specifically tailored to it. In addition to formally establishing that the event-triggered policy meets the desired objective, we also quantify its efficiency by providing an upper bound on the fraction of expected number of transmissions over the system executions. Simulations illustrate the results.**

I. Introduction

The idea of networked control systems is one of the fundamental abstractions of cyber-physical systems. The main characteristic feature of a networked control system is that feedback signals are communicated over a communication channel or a network, and hence control must be performed under communication constraints such as quantization, unreliability, and latency. These limitations underscore the importance of carefully tying together the use of the available resources to the desired level of performance of the task. In this paper, we explore the design of event-triggered transmission policies for second moment stabilization of the state of the plant under packet drops.

Literature review

The increasing ubiquity of cyberphysical systems has brought to the forefront the need for integrated and systematic design methodologies that go beyond adhoc approaches [1], [2]. Of particular relevance to this work is the body of work dealing with feedback control under communication constraints, see [3]–[5] and references therein, particularly packet drops or erasure channels, see e.g., [6], [7]. In the past decade, event-triggered control and, in general, opportunistic state-triggered control methods [8]–[10] have gained a lot of popularity in designing transmission policies that seek to efficiently use the communication resources in networked control systems. This body of work identifies triggering criteria driven by the satisfaction of a desired level of performance to opportunistically execute certain actions (e.g., update the actuation signal, sample some data, or communicate some information). With few exceptions, the emphasis in terms of communications is on minimizing the number of transmissions while largely ignoring other limitations of the channel, such as unreliability, finite precision, or latency. Our previous work [11], [12] has considered such limitations for deterministic models of channel behavior.

Although the body of literature on opportunistic state-triggered control is by now fairly extensive, the application of these ideas in the stochastic setting is still relatively limited, even though one of the first works on event-triggered control [13] was in this setting. Further, event-triggering methods in the stochastic setting have almost exclusively been utilized in finite or infinite horizon optimal control problems with fixed threshold-based triggering. Among these, the works [14]–[16] also incorporate transmission costs in the cost function and analyze the optimal transmission costs. On the other hand, [17], [18] analyze the transmission rates. In addition, [16]–[19] also consider packet drops. In [20], the authors show optimality of certainty equivalence in event-triggered control for certain finite horizon problems. In contrast to starting with an event-triggered control policy, the work [21] formulates an optimal control problem over a horizon of length N with the constraint that at most M transmissions may occur and the optimal control policy turns out to be event-triggered.

Stochastic stability, in the sense of moment stability, with event-triggered control has received much less attention. The work [22] follows in the spirit of [8] to study self-triggered sampling for second-moment stability of state-feedback controlled SDE systems. The work [23] proposes a fixed threshold-based event-triggered anytime control policy under packet drops. Under the assumption that the controller has knowledge of the transmission times, including when a packet is dropped, the policy guarantees second moment stability with exponential convergence to a finite bound asymptotically. Both [22], [23] are applicable to multidimensional nonlinear systems.

Statement of contributions

We start by formulating the problem of second moment stabilization of scalar linear systems under process noise and independent identically distributed packet drops between the sensor and the controller. The control objective is the exponential convergence of the second moment of the plant state to an ultimate bound in finite time. Our first contribution is the design of an event-triggered transmission policy that is driven by the stated control objective. The synthesis of our policy is based on a two-step design procedure. First,

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we consider a nominal quasi-time-triggered policy where no transmission occurs for a given number of timesteps, and then transmissions occur on every time step thereafter. Second, we define the event-trigger policy by evaluating the expectation of the system performance at the next reception time given the current information under the nominal policy, and prescribe a transmission if this expectation does not meet the objective. Although this approach results in a transmission policy more complex than a threshold-based triggering, it has the advantage that it is tuned to the control objective and hence results in fewer transmissions. Our second contribution is the rigorous characterization of the system evolution under the proposed event-triggered transmission policy, which allows us to formally establish that it indeed satisfies the control objective. Finally, our third contribution characterizes the efficiency of the proposed design by providing an upper bound on the fraction of the expected number of transmissions over an infinite time horizon. Various simulations illustrate our results. Some proofs are omitted for reasons of space and will appear elsewhere.

Organization

Section II describes the problem setup. Section III presents the two-step design of the event-triggered transmission policy and Section IV analyzes the dynamic evolution of the system under it. Section V presents simulations. We gather our conclusions and ideas for future work in Section VI.

Notation

We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ denote the set of real, non-negative real numbers, integers, positive integers and non-negative integers respectively. We use the notation $[a, b]_{\mathbb{Z}}$ and $(a, b)_{\mathbb{Z}}$ to denote $[a, b] \cap \mathbb{Z}$ and $(a, b) \cap \mathbb{Z}$, respectively. Given a set A, we denote its indicator function by $\mathbf{1}_A$, i.e., $\mathbf{1}_A(x) = 0$ if $x \notin A$ and $\mathbf{1}_A(x) = 1$ if $x \in A$. We use 'w.p.' as a shorthand for 'with probability'. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ be two sub-sigma fields of $\mathcal{F}$. Then, the *tower property* of conditional expectation is

$$\mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{G}_2\right] \mid \mathcal{G}_1\right] = \mathbb{E}\left[X \mid \mathcal{G}_1\right] = \mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{G}_1\right] \mid \mathcal{G}_2\right].$$

II. Problem Statement

Here we specify the overall problem setup: our model for the plant dynamics, our assumptions on the sensor, the actuator, and the unreliable communication channel between them, and the control objective.

Plant, sensor, and actuator: Consider a scalar discrete-time linear time-invariant system evolving according to

$$x_{k+1} = a x_k + u_k + v_k, \tag{1}$$

for $k \in \mathbb{N}_0$. Here $x \in \mathbb{R}$ denotes the state of the plant, $a \in \mathbb{R}$ defines the system internal dynamics, $u \in \mathbb{R}$ is the control input, and v is a zero-mean independent and identically distributed process noise with covariance $M > 0$. We also assume that the process noise is uncorrelated with the system state.

A sensor measures the plant state x_k at time k. The sensor is not co-located with the controller, and therefore communicates with it over an unreliable communication channel (which is described in detail below). We assume that the communication channel supports the transmission of a packet containing the complete state, although the transmitted packet may be dropped by the channel. We let $\hat{x}^+$ be the state of the controller, which results in a control action given by $u_k = L\hat{x}_k^+$. Thus, on successful communication, i.e. reception of a packet, $\hat{x}_k^+ = x_k$. During the time between two receptions, the controller itself estimates the plant state. We assume that the sensor has an identical state estimator with state estimate $\hat{x}_k$ at time step k, which is used by the sensor to determine the transmission instants. This is possible with identical initialization of the estimators at the sensor and the controller and acknowledgments from the controller to the sensor on successful reception times. We denote the *estimation error* as $e_k \triangleq x_k - \hat{x}_k$, which is known to the sensor at all times, but not to the controller.

Communication channel: The sensor can transmit the plant state to the controller with infinite precision and instantaneously at time steps of its choosing. However, packets might be lost. Thus, we define a (binary-valued) *transmission process* $\{t_k\}_{k \in \mathbb{N}_0}$ as

$$t_k \triangleq \begin{cases} 1, & \text{if a packet is transmitted at } k, \\ 0, & \text{if no packet is transmitted at } k. \end{cases} \tag{2}$$

This process is determined by a transmission policy $\mathcal{T}$, which is to be specified by the designer. Similarly, we define a (binary-valued) *reception process* $\{r_k\}_{k \in \mathbb{N}_0}$, with r_k being 1 or 0 depending on whether a packet is received or not at time step k. The transmission and reception processes may differ due to Bernoulli-distributed packet drops. Formally, if $p \in (0, 1]$ denotes the probability of successful transmission, the reception process is defined as

$$r_k \triangleq \begin{cases} 1, & \text{w.p. } p \text{ if } t_k = 1, \\ 0, & \text{if } t_k = 0 \text{ or w.p. } (1-p) \text{ if } t_k = 1. \end{cases} \tag{3}$$

We denote the sequence of reception times as $\{S_j\}_{j \in \mathbb{N}_0}$, i.e.,

$$S_0 = 0, \quad S_{j+1} \triangleq \min\{k > S_j : r_k = 1\}, \tag{4}$$

where we have assumed, without loss of generality, that $S_0 = 0$ and hence also $r_0 = 1$. We also denote the *latest reception time before* k and *latest reception time up to* k by S_{j_k} and $S_{j_k}^+$, respectively. Formally,

$$S_{j_k} \triangleq \max\{i < k : r_i = 1\}, \tag{5a}$$

$$S_{j_k}^+ \triangleq \max\{i \leq k : r_i = 1\}. \tag{5b}$$

Note that both coincide if $r_k = 0$. The need for the separate notions would become clearer as the discussion progresses: the notion of S_{j_k} is useful in the design of the triggering rule, while the notion of $S^+_{j_k}$ is useful for analysis of the system evolution.

System evolution: Given the communication model between the sensor and the controller described above, we can describe the system evolution and the controller's estimate, respectively, as

$$x_{k+1} = ax_k + L\hat{x}^+_k + v_k = \bar{a}x_k - Le^+_k + v_k, \qquad (6a)$$
$$\hat{x}_{k+1} = \bar{a}\hat{x}^+_k, \qquad (6b)$$

where $\bar{a} = a + L$ and $e^+_k \triangleq x_k - \hat{x}^+_k$, with

$$\hat{x}^+_k \triangleq \begin{cases} \hat{x}_k & \text{if } r_k = 0, \\ x_k, & \text{if } r_k = 1. \end{cases} \qquad (6c)$$

The need for the notation $\hat{x}^+$ and e^+ arises because we are interested in designing a state-triggered transmission policy. Specifically, the decision to transmit at time k is determined by the sensor based on the data x_k and $\hat{x}_k$ (or equivalently e_k), while the plant state at $k+1$ depends on whether a packet was received or not at k. Capturing this necessitates the additional notation $\hat{x}^+$ and e^+_k. We denote by $I_k \triangleq (k, x_k, e_k, S_{j_k}, x_{S_{j_k}})$ the information available to the sensor at time k, based on which it decides whether to transmit or not. We also let $I^+_k \triangleq (k, x_k, e^+_k, S^+_{j_k}, x_{S^+_{j_k}})$, which differs from I_k only if k is a reception time, i.e., $r_k = 1$ (equivalently, only if $k = S_j$ for some j). For the closed-loop system to be completely defined, the last element we need to specify is the transmission policy $\mathcal{T}$ that determines the transmission process (2).

Control objective: Our main objective is to ensure the stability of the plant dynamics with a guaranteed level of performance. Given that the plant evolution is stochastic due to both the presence of random disturbances and the unreliable communication channel, we consider stochastic stability. Formally, we seek to synthesize a transmission policy $\mathcal{T}$ that ensures

$$\mathbb{E}_{\mathcal{T}}\left[x^2_k \mid I^+_0\right] \leq \max\{c^{2k}x^2_0, B\}, \quad \forall k \in \mathbb{N}, \qquad (7)$$

which corresponds to the expected value of the second moment of the plant state, conditional on the initial information, converging at an exponential rate $c \in (0, 1)$ to its ultimate bound $B \geq 0$.

A possible transmission policy would be to simply transmit at every time instant. However, such a policy would presumably lead to an inefficient use of the resources (in this case, the communication channel) since it does not take into account the plant state to make decisions about transmissions. Instead, our goal here is to synthesize an event-triggered transmission policy $\mathcal{T}$ (i.e., an online policy in which the decision to transmit or not at a given time step is determined by a state-based criterion that uses the information available) that guarantees (7).

Standing assumptions: We assume that the drift constant a is such that $|a| > 1$, so that control is necessary. We also assume that $\bar{a}^2 < c^2 < 1$, so that the performance function is always non-positive under zero noise and no packet drops. Finally, we assume that $a^2(1-p) < 1$, so that stabilization under packet drops is possible.

III. Event-triggered transmission policy

This section provides an alternative control objective and shows that its satisfaction implies the original one. Inspired by this reformulated objective, we then design the event-triggered transmission policy.

A. Working control objective

Given the initial condition, the control objective stated in (7) prescribes a property on the whole trajectory of the system in a priori fashion. Here, instead, we describe an alternative control objective which prescribes a property on the trajectory in an online fashion, as the system evolution is progressing. The reason for this is twofold: on the one hand, this objective is easier to handle with regards to the design of the transmission policy; on the other hand, we show that this objective implies the original one. To define this alternative control objective, consider the *performance function*,

$$h_k = x^2_k - \max\{c^{2(k-S_{j_k})}x^2_{S_{j_k}}, B\}, \qquad (8)$$

which has the interpretation of capturing the desired performance at time k with respect to the state at the latest reception time before k. Given this interpretation, consider the alternative control objective that consists of ensuring that

$$\mathbb{E}_{\mathcal{T}}\left[h_k \mid I^+_{S_{j_k}}\right] \leq 0, \quad \forall k \in \mathbb{N}. \qquad (9)$$

The next result shows that the satisfaction of (9) ensures that the original control objective (7) is also met.

Lemma 3.1: (A stronger control objective). If a transmission policy $\mathcal{T}$ ensures the working objective (9), then it also guarantees the control objective (7).

Proof: The proof relies on the use of induction. Note that the two objectives coincide for $k \in [0, S_1]_{\mathbb{Z}}$. Now, assume that (9) implies that (7) is guaranteed for all $k \in [0, S_j]_{\mathbb{Z}}$ for some $j \in \mathbb{N}$. Then, letting $y_k \triangleq x^2_k - \max\{c^{2k}x^2_0, B\}$, notice that for all $k \in [S_j + 1, S_{j+1}]_{\mathbb{Z}}$, we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{T}}\left[y_k \mid I^+_0\right] = \mathbb{E}_{\mathcal{T}}\left[\mathbb{E}_{\mathcal{T}}\left[y_k \mid I^+_0\right] \mid I^+_{S_j}\right] \\
&\leq \mathbb{E}_{\mathcal{T}}\left[\mathbb{E}_{\mathcal{T}}\left[x^2_k - \max\{c^{2(k-S_j)}x^2_{S_j}, B\} \mid I^+_0\right] \mid I^+_{S_j}\right] \\
&= \mathbb{E}_{\mathcal{T}}\left[\mathbb{E}_{\mathcal{T}}\left[x^2_k - \max\{c^{2(k-S_j)}x^2_{S_j}, B\} \mid I^+_{S_j}\right] \mid I^+_0\right] \\
&= \mathbb{E}_{\mathcal{T}}\left[\mathbb{E}_{\mathcal{T}}\left[h_k \mid I^+_{S_j}\right] \mid I^+_0\right] \leq 0,
\end{aligned}$$

where we have first used the tower property, then the assumption made for the induction step - specifically that $\mathbb{E}_{\mathcal{T}}\left[y_{S_j} \mid I^+_0\right] \leq 0$, the tower property again, the definition of h_k and finally the hypothesis of the Lemma that (9) is satisfied. The chain of inequalities thus proves the result. ■

Given Lemma 3.1, in the remainder of the paper, we focus on the stronger but easier to handle control objective (9) as a means of ensuring that our original control objective (7) is met.

B. Two-step design strategy: nominal and event-triggered transmission policies

Our key idea to design the event-triggered transmission policy is the belief that, in the absence of reception of packets, the likelihood of violating the performance criterion must increase with time. We refer to this as the *monotonicity property*. Therefore, we design a transmission policy that overtly seeks to satisfy the performance criterion (9) only at the next (random) reception time. Later, our analysis will show that the monotonicity property above does indeed hold, in order to guarantee that the performance objective is not violated at any time step.

Given this discussion, we seek to design an event-triggered transmission policy $\mathcal{T}$ that would ensure

$$\mathbb{E}_{\mathcal{T}}\left[h_{S_{j+1}} \mid I_{S_j}^+\right] \leq 0, \quad \text{for each } j \in \mathbb{N}_0.$$

In general, computing $\mathbb{E}_{\mathcal{T}}\left[h_{S_{j+1}} \mid I_{S_j}^+\right]$ for an arbitrary event-triggered transmission policy $\mathcal{T}$ is a challenging task. This is because the evolution of the system state between consecutive reception times depends on the transmission instants, which are in turn determined online by the triggering function of the state and the specific realizations of the noise and the packet drops. Therefore, we take a two-step strategy to design the event-triggered transmission policy. First we consider a family of nominal quasi-time-triggered transmission policies $\mathcal{T}_k^D$, for which it is easy to compute $\mathbb{E}_{\mathcal{T}_k^D}\left[h_{S_{j_k+1}} \mid I_k\right]$. We then use this expectation under a nominal transmission policy to design our event-triggered transmission policy.

We start by defining a family of nominal transmission policies indexed by $k \in \mathbb{N}_0$ as

$$\mathcal{T}_k^D : t_i = \begin{cases} 0, & i \in \{k, \ldots, k+D-1\}, \\ 1, & i \geq k+D, \end{cases} \tag{10}$$

where $D \geq 1$. Under this nominal policy, no transmissions occur for the first D time steps from k to $k+D-1$, and transmissions occur on every time step thereafter (D is therefore the length of the interval from time k during which no transmissions occur). With the nominal policy, we associate the following *look-ahead* criterion,

$$\begin{aligned} G_k^D &\triangleq \mathbb{E}_{\mathcal{T}_k^D}\left[h_{S_{j_k+1}} \mid I_k\right] \qquad (11) \\ &= \sum_{s=D}^{\infty} \mathbb{E}\left[h_{S_{j_k+1}} \mid I_k, S_{j_k+1} = k+s\right](1-p)^{s-D}p, \end{aligned}$$

which is the conditional expectation of the performance function at the next reception time, given the information at k under the transmission policy $\mathcal{T}_k^D$. This interpretation gives rise to the central idea behind our proposed event-triggered transmission policy: if the criterion is positive (i.e., the performance objective is expected to be violated at the next reception time if no transmission occurs for D timesteps, and forever after), then we need to start transmitting earlier to try to revert the situation before it is too late. Formally, the event-triggered policy $\mathcal{T}_{\mathcal{E}}$ given the last successful reception time S_j is defined as

$$\mathcal{T}_{\mathcal{E}} : t_k = \begin{cases} 0, & \text{if } k \in \{S_j+1, \ldots, T_{S_j}-1\} \\ 1, & \text{if } k \in \{T_{S_j}, \ldots, S_{j+1}\}, \end{cases} \tag{12a}$$

where

$$T_{S_j} \triangleq \min\{k > S_j : G_k^D \geq 0\}. \tag{12b}$$

Thus, under the proposed policy, the sensor transmits on each time step starting at T_{S_j} (the first time after S_j when the criterion is positive) until a successful reception occurs at S_{j+1}, for each $j \in \mathbb{N}_0$. The rest of the paper is devoted to characterize the system evolution under this policy.

IV. Convergence and performance analysis

In this section, we analyze the closed-loop system under the event-triggered transmission policy $\mathcal{T}_{\mathcal{E}}$.

A. Analysis of system evolution under the nominal policy

Here, we characterize the evolution of the system when operating under the nominal transmission policy. This characterization is key later to help us provide performance guarantees of the event-triggered transmission policy. The following result gives a closed-form expression for G_k^D as a function of I_k.

Lemma 4.1: (Closed-form expression for the performance-evaluation function). The performance-evaluation function G_k^D is well defined and takes the form

$$\begin{aligned} G_k^D = p\Big[& g_D(\bar{a}^2)x_k^2 + 2\big(g_D(a\bar{a}) - g_D(\bar{a}^2)\big)x_k e_k \\ & + \big(g_D(a^2) - 2g_D(a\bar{a}) + g_D(\bar{a}^2)\big)e_k^2 \\ & + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right) - g_D(c^2)z_k \\ & - \left(\frac{B}{p} - c^{2q_k^D} g_D(c^2) z_k\right)(1-p)^{q_k^D}\Big], \end{aligned}$$

where

$$g_D(b) \triangleq \frac{b^D}{1-b(1-p)}, \quad \overline{M} \triangleq \frac{M}{a^2-1}, \quad z_k \triangleq c^{2(k-S_{j_k})}x_{S_{j_k}}^2,$$

$$q_k^D \triangleq \max\left\{0, \left\lceil \frac{\log\left(\frac{x_{S_{j_k}}^2}{B}\right)}{\log(1/c^2)} \right\rceil - (k - S_{j_k}) - D\right\}. \tag{13}$$

Proof: Since G_k^D is defined as an infinite series, we first focus on computing expressions for each of its summands.

Noting the fact that $r_i = 0$ for $i \in (S_{j_k}, S_{j_k+1})_{\mathbb{Z}}$, we can iterate over (6) to see that

$$x_{k+s} = a^s x_k + \sum_{i=0}^{s-1} a^{s-1-i}(L\bar{a}^i \hat{x}_k + v_{k+i})$$
$$= a^s x_k - (a^s - \bar{a}^s)\hat{x}_k + \sum_{i=0}^{s-1} a^{s-1-i} v_{k+i}$$

$$x_{k+s} = \bar{a}^s x_k + (a^s - \bar{a}^s)e_k + \sum_{i=0}^{s-1} a^{s-1-i} v_{k+i}.$$

Using this equation, we obtain

$$\mathbb{E}\left[h_{S_{j_k+1}} \mid I_k, S_{j_k+1} = k+s\right]$$
$$= \bar{a}^{2s}x_k^2 + 2\bar{a}^s(a^s - \bar{a}^s)x_k e_k + (a^{2s} - 2a^s\bar{a}^s + \bar{a}^{2s})e_k^2$$
$$+ \overline{M}(a^{2s} - 1) - \max\{c^{2s}c^{2(k-S_{j_k})}x^2_{S_{j_k}}, B\}, \quad (14)$$

where we have used the facts that the noise process is independent and identically distributed with mean zero and variance M. Substituting (14) into the definition (11) of G_k^D, we see that its computation involves various geometric series, all with the form

$$\sum_{s=D}^{\infty} b^s (1-p)^{s-D},$$

for $b = \bar{a}^2$, $b = \bar{a}a$, and $b = a^2$. Note that the sum of this series precisely corresponds to $g_D(b)$. The assumptions that $a^2(1-p) < 1$ and $\bar{a} < 1$ ensure that each of these series converges and consequently, G_k^D is well defined. Now, it remains to simplify the terms involving c and B. Note that $z_k = c^{2(k-S_{j_k})}x^2_{S_{j_k}}$ and

$$\sum_{s=D}^{\infty} \max\{c^{2s}z_k, B\}(1-p)^{s-D}$$
$$= g_D(c^2)z_k + \sum_{s=q_k^D}^{\infty} (B - c^{2(k+D-S_{j_k})}x^2_{S_{j_k}})(1-p)^{s-D}$$
$$= g_D(c^2)z_k + \sum_{s=q_k^D}^{\infty} (B - c^{2D}z_k)(1-p)^{s-D}$$

where q_k^D is the number of time steps from $k + D$ when we first have $B \geq c^{2s}z_k$. It can be easily verified that q_k^D is given by (13) and that the infinite series above sums up to the last term in G_k^D. Hence, the result holds. ∎

While G_k^D is useful in determining whether to transmit at time k or not, in order to analyze the evolution of the performance function h_k between successive reception times S_j and S_{j+1} we also make use of the *performance-evaluation* function defined by

$$J_k^D \triangleq \mathbb{E}_{\mathcal{T}_k^D}\left[h_{S^+_{j_k+1}} \mid I_k^+\right] \quad (15)$$
$$= \sum_{s=D}^{\infty} \mathbb{E}\left[h_{S^+_{j_k+1}} \mid I_k^+, S_{j_k+1} = k+s\right](1-p)^{s-D}p,$$

which takes a form similar to G_k^D except for the fact that in J_k^D, we condition upon the information I_k^+. In particular, we are interested in $J_{S_j}^D$ for $j \in \mathbb{N}_0$ since $J_k^D \neq G_k^D$ only if $k = S_j$ for some j. The following result gives a closed-form expression for $J_{S_j}^D$ as a function of $I_{S_j}^+$.

Lemma 4.2: (Closed-form expression for the performance-evaluation function). The performance-evaluation function $J_{S_j}^D$ is well defined and takes the form

$$J_{S_j}^D = p\Big[g_D(\bar{a}^2)x^2_{S_j} + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right) - g_D(c^2)x^2_{S_j}$$
$$- \left(\frac{B}{p} - c^{2w_k^D}g_D(c^2)x^2_{S_j}\right)(1-p)^{w^D_{S_j}}\Big],$$

where g_D is defined in (13) and

$$w_{S_j}^D \triangleq \max\left\{0, \left\lceil \frac{\log\left(\frac{x^2_{S_j}}{B}\right)}{\log(1/c^2)} \right\rceil - D\right\}. \quad (16)$$

Proof: First we define the *open-loop performance evolution* function

$$H(s, x^2_{S_j}) \triangleq \mathbb{E}\left[h_{S_j+s} \mid I^+_{S_j}, S_j + s \leq S_{j+1}\right]. \quad (17)$$

One can show that the function H can be expressed in the form

$$H(s, y) = \bar{a}^{2s}y + \overline{M}(a^{2s} - 1) - \max\{c^{2s}y, B\}. \quad (18)$$

Also, note that in (17), if the condition $S_j + s \leq S_{j+1}$ were replaced by $S_j + s = S_{j+1}$, (18) would still be true. Thus, noting that

$$J_{S_j}^D = \sum_{s=D}^{\infty} H(s, x^2_{S_j})(1-p)^{s-D}p, \quad (19)$$

we conclude the result using a similar reasoning as in the proof of Lemma 4.1. ∎

In establishing Lemma 4.2, the open-loop performance evolution function H plays a key role. This function describes the evolution of the expected value of the performance function in open loop, during the inter-reception times, conditioned upon $I^+_{S_j}$, the information available at the last reception time upon reception. Therefore, it is important to analyze its behavior. The following result states an important monotonicity property, which forms the basis for our main results.

Proposition 4.3: (Monotonicity of the open-loop performance function). There exists $B^* > 0$ such that, if $B > B^*$ and $B\log\left(\frac{c^2}{\bar{a}^2}\right) > \overline{M}\log(a^2)$, then for each $y \in \mathbb{R}_{\geq 0}$, the function $H(., y)$ has the property:

$$H(s_1, y) > 0 \implies H(s_2, y) > 0, \quad \forall s_2 \geq s_1. \quad (20)$$

Proposition 4.3 captures the monotonicity property we discussed in Section III-B. Specifically, this result says that, given the plant state is y at any reception time S_j, then there is a time s_0 such that, in the absence of receptions, the plant state is expected to satisfy the performance criterion (9) until $S_j + s_0$ and violate it on every time step thereafter.

We conclude this section by specifying some useful properties of the look-ahead G_k^D and the performance-evaluation J_k^D functions.

Proposition 4.4: (Properties of the look-ahead and performance-evaluation functions). Suppose $D \in \mathbb{N}$. Then, under the same hypotheses as in Proposition 4.3 the following hold:

(a) Let $\mathcal{T}$ be any transmission policy. Then, for any $k \in \mathbb{N}_0$,

$$\mathbb{E}_{\mathcal{T}}\left[G_{k+1}^D \mid I_k, r_k = 0\right] = G_k^{D+1},$$
$$\mathbb{E}_{\mathcal{T}}\left[G_{k+1}^D \mid I_k, r_k = 1\right] = J_k^{D+1}.$$

(b) Suppose

$$\left(g_D(\bar{a}^2) - g_D(c^2)\right)\frac{B}{c^{2D}} + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right) < 0. \tag{21}$$

Then $J_{S_j}^D < 0$, for any $j \in \mathbb{N}_0$.

(c) Suppose the hypothesis of (b) is true. Then, for $d \in \{1, \ldots, D\}$ and for any $j \in \mathbb{N}_0$, $J_{S_j}^d \le J_{S_j}^{d+1}$.

Proof: Regarding claim (a), we note that using the definition (11) of G_k^D, we have

$$\mathbb{E}_{\mathcal{T}}\left[G_{k+1}^D \mid I_k, r_k\right]$$
$$= \mathbb{E}_{\mathcal{T}_{k+1}^D}\left[\mathbb{E}_{\mathcal{T}_{k+1}^D}\left[h_{S_{j_{k+1}+1}} \mid I_{k+1}\right] \mid I_k, r_k\right].$$

We can change the transmission policy in the outer expectation because once I_k and r_k are given, the expectation of G_{k+1}^D is independent of the subsequent transmission policy. Now, if $r_k = 0$ then $S_{j_{k+1}} = S_{j_k}$ and hence

$$\mathbb{E}_{\mathcal{T}}\left[G_{k+1}^D \mid I_k, r_k = 0\right] = \mathbb{E}_{\mathcal{T}_{k+1}^D}\left[h_{S_{j_k+1}} \mid I_k, r_k = 0\right]$$
$$= \mathbb{E}_{\mathcal{T}_k^{D+1}}\left[h_{S_{j_k+1}} \mid I_k\right] = G_k^{D+1}.$$

On the other hand, if $r_k = 1$ then $S_{j_{k+1}} = k = S_{j_k}^+$. Thus,

$$\mathbb{E}_{\mathcal{T}}\left[G_{k+1}^D \mid I_k, r_k = 1\right] = \mathbb{E}_{\mathcal{T}_{k+1}^D}\left[h_{S_{j_k}^+ +1} \mid I_k^+\right]$$
$$= \mathbb{E}_{\mathcal{T}_k^{D+1}}\left[h_{S_{j_k}^+ +1} \mid I_k^+\right] = J_k^{D+1}.$$

To show claim (b), the main idea is to maximize $J_{S_j}^D$ over all possible values of $x_{S_j}^2$ and ensure that the result is not positive. To do this, we rely on the expression of $J_{S_j}^D$ obtained in Lemma 4.2. To maximize this expression over $x_{S_j}^2$, we split the domain of the latter into two parts

$$\mathcal{D}_1 \triangleq (0, Bc^{-2D}], \quad \mathcal{D}_2 \triangleq [Bc^{-2D}, \infty).$$

Note, from (16), that $w_k^D = 0$ for all $x_{S_j}^2 \in \mathcal{D}_1$ and $w_k^D > 0$ for all $x_{S_j}^2 \in \mathcal{D}_2$. For $x_{S_j}^2 \in \mathcal{D}_1$,

$$J_{S_j}^D = p\left[g_D(\bar{a}^2)x_{S_j}^2 + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right) - \frac{B}{p}\right],$$

which is maximized at Bc^{-2D}. Therefore, by observing

$$\frac{B}{c^{2D}}g_D(c^2) = \frac{B}{1 - c^2(1-p)} \le \frac{B}{p}$$

and using the condition (21), we see that $J_{S_j}^D < 0$ for all $x_{S_j}^2 \in \mathcal{D}_1$. Next, for $x_{S_j}^2 \in \mathcal{D}_2$, one has $w_k^D > 0$ and therefore

$$\log(c^{2w_k^D}g_D(c^2)x_{S_j}^2) = \left(D - \left\lceil\frac{\log\left(\frac{x_{S_{j_k}}^2}{B}\right)}{\log(1/c^2)}\right\rceil\right)\log(1/c^2)$$
$$+ \log(g_D(c^2)) + \log(x_{S_j}^2)$$
$$\le D\log(1/c^2) + \log(B) + \log(g_D(c^2)),$$

where we have used that $\lceil y \rceil \ge y$ for $y \in \mathbb{R}$. Taking the exponential on both sides and using the definition of g_D

$$c^{2w_k^D}g_D(c^2)x_{S_j}^2 \le \frac{B}{c^{2D}}g_D(c^2) = \frac{B}{1 - c^2(1-p)} \le \frac{B}{p}, \tag{22}$$

where we have used $c^2 \in (0,1)$ to obtain the last inequality. Therefore, we see that for all $x_{S_j}^2 \in \mathcal{D}_2$,

$$J_{S_j}^D \le p\left[\left(g_D(\bar{a}^2) - g_D(c^2)\right)x_{S_j}^2 + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right)\right],$$

in which the coefficient of $x_{S_j}^2$ is negative because $\bar{a}^2 < c^2$. Thus, the supremum of $J_{S_j}^D$ for $x_{S_j}^2 \in \mathcal{D}_2$ satisfies

$$\sup_{x_{S_j}^2 \in \mathcal{D}_2}\frac{J_{S_j}^D}{p} \le \left(g_D(\bar{a}^2) - g_D(c^2)\right)\frac{B}{c^{2D}} + \overline{M}\left(g_D(a^2) - \frac{1}{p}\right).$$

Consequently, (21) ensures that $J_{S_j}^D < 0$ for all $x_{S_j}^2 \in \mathcal{D}_2$ too.

Finally, to show claim (c), we reason as follows. Combining Proposition 4.3 with the fact, from claim (b), that $J_{S_j}^D < 0$ for all values of $x_{S_j}^2$, we use the expression (19) to conclude that $H(D, y) \le 0$ for all $y \in \mathbb{R}_{\ge 0}$. Invoking again Proposition 4.3, it follows that $H(d, x_{S_j}^2) \le 0$ for all $d \le D$. Now, note that

$$J_{S_j}^d = \sum_{s=d}^{\infty} H(s, x_{S_j}^2)(1-p)^{s-d}p$$
$$= pH(d, x_{S_j}^2) + (1-p)\sum_{s=d+1}^{\infty} H(s, x_{S_j}^2)(1-p)^{s-(d+1)}p$$
$$= pH(d, x_{S_j}^2) + (1-p)J_{S_j}^{d+1}.$$

Therefore, we conclude $J_{S_j}^d \le J_{S_j}^{d+1}$ for $d \in \{1, \ldots, D\}$. ■

B. Performance guarantees under the event-triggered policy

In this section, we characterize the performance of the system evolution operating under the event-triggered transmission policy $\mathcal{T}_{\mathcal{E}}$ defined in (12a), building on our analysis in Section IV-A. The following statement is the main result of the paper and shows that the control objective is achieved by the proposed event-triggered transmission policy.

Theorem 4.5: (Event-triggered policy meets the control objective). Suppose $D \in \mathbb{N}$ and that (21) is satisfied. Then, under the same hypotheses as in Proposition 4.3, the event-triggered policy $\mathcal{T}_{\mathcal{E}}$ guarantees that $\mathbb{E}_{\mathcal{T}_{\mathcal{E}}}\left[h_k \mid I^+_{S_{j_k}}\right] \leq 0$ for all $k \in \mathbb{N}$.

A consequence of Theorem 4.5 along with Lemma 3.1 is that the event-triggered policy $\mathcal{T}_{\mathcal{E}}$ guarantees

$$\mathbb{E}_{\mathcal{T}_{\mathcal{E}}}\left[x_k^2 \mid I_0^+\right] \leq \max\{c^{2k}x_0^2, B\}, \quad \forall k \in \mathbb{N}_0,$$

the original control objective. In other words, the proposed event-triggered transmission policy guarantees that the expected value of x_k^2 converges at an exponential rate to its ultimate bound of B.

Next, we study the efficiency of the proposed event-triggered transmission policy in terms of the fraction of the number of time steps at which transmissions occur. Thus, for any stopping time K, we introduce the *expected transmission fraction*

$$\mathcal{F}_0^K \triangleq \frac{\mathbb{E}_{\mathcal{T}_{\mathcal{E}}}\left[\sum_{k=1}^{K} \mathbf{1}_{\{t_k=1\}} \mid I_0^+\right]}{\mathbb{E}_{\mathcal{T}_{\mathcal{E}}}\left[K \mid I_0^+\right]}. \tag{23}$$

This corresponds to the expected fraction of time steps from 1 to K at which transmissions occur. Note that K might be a random variable itself, which justifies the expectation operation taken in the denominator. The following result provides an upper-bound on this expected transmission fraction. We omit the proof due to space constraints.

Proposition 4.6: (Upper-bound on the expected transmission fraction). Suppose (21) is satisfied with $D+\mathcal{B}$ in place of D, for some $\mathcal{B} \in \mathbb{N}_0$. Then

$$\mathcal{F}_0^\infty \leq \frac{1}{1+\mathcal{B}p}.$$

Note that an expected transmission fraction of 1 corresponds to a transmission occurring at every time step. Thus, Proposition 4.6 states that the number of transmissions under the event-triggered policy $\mathcal{T}_{\mathcal{E}}$ is guaranteed to be less than that of a time-triggered policy, which transmits on every time step (and therefore has transmission fraction of exactly 1).

Remark 4.7: (Time- versus event-triggered transmission policy). Note that (21) is an assumption on the tolerance of the system and the control objective to a certain minimum length (in a statistical sense) of inter-reception times. Thus, it is conceivable that a time-triggered transmission policy exists with period larger than 1 (i.e., with a transmission fraction less than 1) that also achieves the control objective. Although we do not do it here, we plan to construct such a time-triggered transmission policy and compare its transmission fraction with that of the event-triggered policy $\mathcal{T}_{\mathcal{E}}$. In any case, note that a time-triggered implementation determines the transmission times a priori, while the event-triggered implementation determines them online, in a feedback fashion. The latter therefore renders the system more robust to uncertainties in the knowledge of the system parameters, noise and packet drop distributions. •

V. Simulations

Here we present simulation results for the system evolution under the event-triggered transmission policy $\mathcal{T}_{\mathcal{E}}$. We consider the dynamics (6) with the following parameters,

$$a = 1.1,\ p = 0.8,\ M = 1,\ c = 0.98,\ \bar{a} = 0.95c,$$
$$B = 14.96,\ x(0) = 20B.$$

The process noise is drawn from a Gaussian distribution, with covariance M. We computed the critical value B^* in Proposition 4.3 to be 12.47. We performed simulations for 1000 realizations of process noise and packet drops, all starting from the same initial condition. Then, for each time step k, we computed the empirical mean of the various quantities. This is illustrated in Figures 1 and 2. We performed simulations with $D = 1$ and $D = 3$, and in each case $D + \mathcal{B} = 3$. Figure 1 shows that the control objective (7) is satisfied, as guaranteed by Theorem 4.5. For $D = 3$, one can see that the control objective is met more conservatively, which is consistent with the interpretation of the transmission policies given in Section III-B. Figure 2 shows the empirical running transmission fractions for $D = 3$ and $D = 1$, as well as the upper bound on the transmission fraction $\mathcal{F}_0^\infty$ in the case of $D = 1$ obtained in Proposition 4.6. In the case of $D = 3$, this quantity is 1. As expected, the conservativeness of the implementation with $D = 3$ is reflected in a higher transmission fraction.

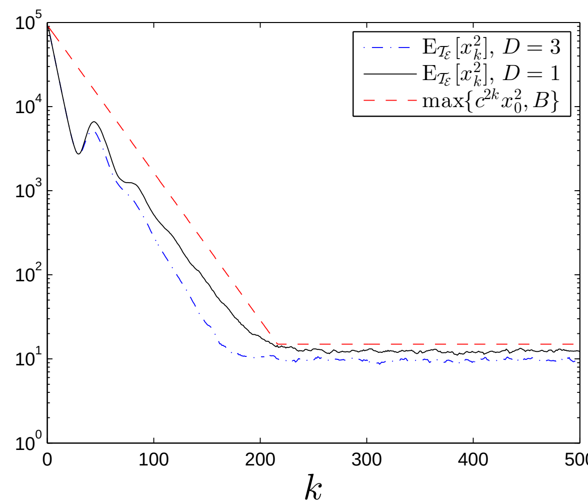

Fig. 1. Plot of the evolution of the empirical mean $\mathbb{E}_{\mathcal{T}_{\mathcal{E}}}\left[x_k^2\right]$ with $D = 3$ and $D = 1$ and the performance bound, $\max\{c^{2k}x_0^2, B\}$.

VI. Conclusions

We have designed an event-triggered transmission policy for scalar linear systems under packet drops. The control objective consists of achieving second-moment stability of the plant state with a given exponential rate of convergence

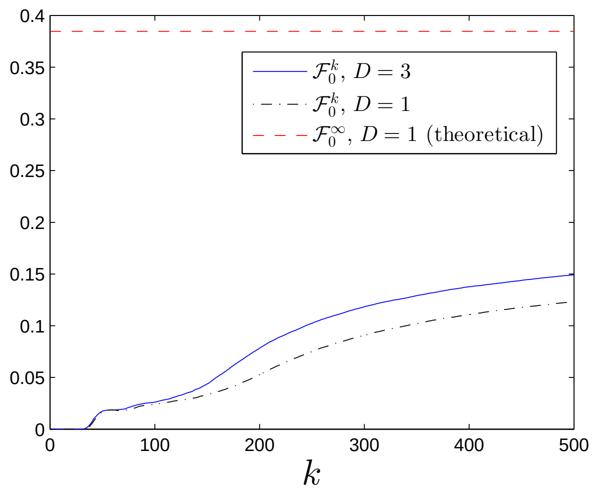

Fig. 2. Plot of the evolution of the empirical running transmission fraction $\mathcal{F}_0^k$ for $D = 3$ and $D = 1$, and the theoretical bound on the asymptotic transmission fraction $\mathcal{F}_0^\infty$ in the case of $D = 1$. For $D = 3$, the latter is 1.

to an ultimate bound in finite time. The synthesis of our policy is based on a two-step design procedure. First, we consider a nominal quasi-time-triggered policy where no transmission occurs for a given number of timesteps, and then transmissions occur on every time step thereafter. Second, we define the event-trigger policy by evaluating the expectation of the system performance at the next reception time given the current information under the nominal policy, and prescribe a transmission if this expectation does not meet the objective. We have also characterized the efficiency of our design by providing an upper bound on the fraction of the expected number of transmissions over the infinite time horizon. Future work will compare the efficiency of the proposed policy against optimal time-triggered transmission policies, extend our treatment to higher-dimensional systems, and investigate the role of quantization and information-theoretic tools to address questions about necessary and sufficient data rates.

Acknowledgments

This work was supported in part by NSF Award CNS-1446891.

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