# Identification of linear networks with latent nodes

Yingbo Zhao and Jorge Cortés

***Abstract*— We study identification of linear networks under the assumption that only a subset of all the nodes in the network can be observed. The observable nodes are called manifest nodes and they form the manifest subnetwork. The unobservable nodes are called latent nodes and the number of latent nodes is unknown. We explore the possibility of identifying the transfer function of the manifest subnetwork and whether an interaction between two manifest nodes is direct or mediated by latent nodes. In particular, we show that if the external inputs are injected into a linear network only through the manifest nodes, then there exists an auto-regressive model whose transfer function is arbitrarily close to the transfer function of the manifest subnetwork in the $H_\infty$ norm sense. Moreover, we prove that the least-squares method provides consistent estimate of the auto-regressive model using the measured states of the observed nodes. Finally, we show that if the latent subnetwork is acyclic, then the transfer function of the manifest subnetwork can be perfectly identified using the least-squares auto-regressive method. Various examples illustrate our results.**

## I. Introduction

Complex networks play an essential part in many areas of modern science and engineering, such as biology [1], [2], physics [3], finance [4], and the smart grid [5]. A complex network typically consists of many dynamical subsystems or nodes that interact with each other. The state of a node evolves over time according to its internal dynamics as well as the interactions with its neighbors. While much emphasis has been put on the synthesis and analysis of coordination algorithms where the interaction topology is either given or the design objective itself, there is a need to develop concepts and techniques to address the identification of unknown topologies from measured data, especially when not all the nodes in the network can be directly manipulated or measured.

*Literature review:* The recent interest on network topology identification is motivated by the need to better identify the increasingly complex interactions in large-scale networks and understand their role in driving network behavior. Our work here is in particular inspired by the wide use of auto-regressive (AR) models to analyze brain data via Granger causality and the study of effective connectivity among different areas of the brain, see e.g., [6], [7], [8]. Broadly speaking, the objective in network identification is to determine the causal relationships among the nodes, that is, the direction and strength of the interaction between them. It is common to model a complex network using a directed graph, where interactions among neighboring nodes are represented by directed edges whose weights reflect interaction strength. Based on the cross-power spectral densities of the network response to wide-sense stationary noises, [9] proposes algorithms to identify the topology of directed linear time-invariant networks that require a series of node-knockout experiments. Also using a node-knockout procedure, [10] considers the complete characterization of the interaction topology in consensus-type networks. Under the assumption that every node is manipulable and large number of experiments are allowed, [3] presents a method to infer the topology of a network from its stable response dynamics. [11] considers the topology identification for tree networks. Without the ability to manipulate every node and perform a large number of experiments, it is in general difficult to accurately identify the network topology. In fact, [12] shows that, once the transfer function from controlled inputs to measured outputs has been perfectly identified, it is not possible to identify even the Boolean structure of a network without additional information. As a result, the focus has been put on particular realizations of the network that explain the measured data, such as the sparsest realization, sometimes with a design parameter to manage the trade-off between model accuracy and sparsity, see e.g. [1], [13]. Furthermore, in many cases, it is not possible to obtain measurements from all the nodes or even the existence of some nodes, called latent, might be unknown [14]. [15] proposes a method to identify the latent nodes and the network topology under assumptions that the network is a polytree and the degree of each latent node is at least three with out-degree of at least two.

*Statement of contributions:* We assume that one can only stimulate and observe a subset of all the nodes, termed manifest, in a linear time-invariant (LTI) network with unknown topology. We also assume that the total number of nodes is unknown, i.e., latent nodes might be present. The objective is to identify the transfer function of the manifest subnetwork (the *manifest transfer function*), which is a submatrix of the transfer function (matrix) of the entire network. While most of the existing topology identification algorithms are restrained to acyclic graphs, our results allow cycles in the network topology. Under the assumption that all the latent nodes are passive, that is, there are no external inputs acting directly on them, our first contribution is showing that there exists a class of AR models whose transfer functions converge (exponentially in the $H_\infty$ norm) to the manifest transfer function as the order of the AR model increases. An advantage of the AR

Yingbo Zhao and Jorge Cortés are with the Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093. Emails: {yiz326,cortes}@ucsd.edu

model is that it distinguishes direct interaction between a pair of manifest nodes from indirect interaction mediated by latent nodes. This provides more information about the network structure than the manifest transfer function matrix and may be helpful in latent node identification. Next, we describe the least-squares auto-regressive method to estimate the AR model. Our second contribution shows that the least-squares auto-regressive method obtains an arbitrarily small $H_\infty$ norm error as the length of data and the model order grow. In fact, once the order of the AR model candidates exceeds a certain threshold, the $H_\infty$ norm error decays exponentially. We also show that if the latent subnetwork is acyclic, then the method achieves perfect identification of the manifest transfer function. Simulations on ring networks and Erdős–Rényi random illustrate our results. For reasons of space, all proofs are omitted and will appear elsewhere.

*Notation:* For a vector $x \in \mathbb{R}^n$, we use $x_i$ to denote its $i$-th element. Given a sequence $\{x(k)\}_{k=0}^{\infty} \subset \mathbb{R}^n$ and $j_1 \leq j_2 \in \mathbb{Z}_{\geq 0}$, we use $\{x\}_{j_1}^{j_2}$ to denote the finite sequence $\{x(j_1), x(j_1+1), \ldots, x(j_2)\}$. We omit $j_1$ if $j_1 = 0$ and denote $||\{x\}_{j_1}^{j_2}|| \triangleq \big(\sum_{k=j_1}^{j_2} x^T(k)x(k)\big)^{\frac{1}{2}}$. For a real matrix $M \in \mathbb{R}^{m\times n}$, we denote its singular values by decreasing order as $\sigma_1(M) \geq \sigma_2(M) \geq \ldots \geq \sigma_{\min(m,n)}(M) \geq 0$ and its spectral norm by $\|M\| = \sigma_1(M)$. The max norm of $M$ is defined by $\|M\|_{\max} = \max_{i,j} |M_{ij}|$. We denote the spectral radius of a square matrix $M$ by $\rho(M)$. $M$ is Schur stable if and only if $\rho(M) < 1$. The $m \times n$ matrix with all zero elements is denoted by $\mathbf{0}_{m\times n}$ and the identity matrix of dimension $n \times n$ is denoted by $I_n$. The $H_\infty$ norm of a discrete transfer function $T$ is denoted by $||T||_\infty \triangleq \sup_{-\pi\leq\omega\leq\pi}\|T(\omega)\|$. A sequence of random variables $\{x\}^\infty$ converges in probability to a random variable $X$ (denoted $\operatorname{plim}_{k\to\infty} x(k) = X$) if $\lim_{k\to\infty} \Pr(|x(k) - X| \geq \varepsilon) = 0$ for all $\varepsilon > 0$.

## II. Problem Formulation

We consider a discrete-time, linear time-invariant (LTI) network dynamics with state-space representation

$$\begin{aligned} x(k+1) &= Ax(k) + u(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time index, $x(k) \in \mathbb{R}^n$ is the network state (with $x_i(k)$ representing the state of node $i$, $i \in \{1, \ldots, n\}$), $u(k) \in \mathbb{R}^n$ is the control input (with $u_i(k)$ acting on node $i$), and $y(k) \in \mathbb{R}^m$ is the network output. Here, $A \in \mathbb{R}^{n\times n}$ is the adjacency matrix of the network characterizing the interactions among neighboring nodes and $C \in \mathbb{R}^{m\times n}$ is the observation matrix.

In the network dynamics (1), we have assumed that there is a control input at every node. Nevertheless, we do not assume all the control inputs are user-specified. In particular, we allow that $u_i(k) \equiv 0$ for all $k \in \mathbb{Z}_{\geq 0}$ or $\{u_i\}$ is a stochastic process. We have also assumed that all the nodes are of order 1, that is, $x(k+1)$ depends directly only on $x(k)$ and conditionally independent of $\{x\}^{k-1}$ given $x(k)$.

*Assumption 2.1:* The network adjacency matrix $A$ is unknown with spectral norm smaller than 1.

Note that if $A$ is symmetric, then Assumption 2.1 is equivalent to the Schur stability of $A$.

In a large-scale network, it is common that one can control and observe only a subset of the nodes due to computational constraints, measurement costs, or physical limitations. Assume that one can only control and observe $n_m \leq n$ nodes, called the *manifest nodes*, with indices $i_1, i_2, \ldots, i_{n_m}$ in the network (1), where $(i_1, i_2, \ldots, i_{n_m}, i_{n_m+1}, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$. The uncontrollable and unobservable nodes $i_{n_m+1}, \ldots, i_n$ are called the *latent nodes*. Denote the state and the input vectors of the manifest subnetwork by $x_m(k) = \big[x_{i_1}(k) \ \ldots \ x_{i_{n_m}}(k)\big]^T$ and $u_m(k) = \big[u_{i_1}(k) \ \ldots \ u_{i_{n_m}}(k)\big]^T$, respectively, and the state and input vectors of the latent subnetwork by $x_l(k)$ and $u_l(k)$, respectively, and let $y(k) = x_m(k)$. If further $u_{l,j}(k) \equiv 0$ for some $j \in \{i_{n_m+1}, \ldots, i_n\}$, then the latent node $j$ is called a *passive latent node*. With the decomposition of the nodes into manifest and latent ones, the state-space representation (1) becomes

$$\begin{aligned} \begin{bmatrix} x_m(k+1) \\ x_l(k+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m(k) \\ x_l(k) \end{bmatrix} + \begin{bmatrix} u_m(k) \\ u_l(k) \end{bmatrix}, \\ y(k) &= x_m(k). \end{aligned} \tag{2}$$

In the remainder of this paper, we consider the network in the relabeled form (2) directly. This relabeling is illustrated in Figure 1.

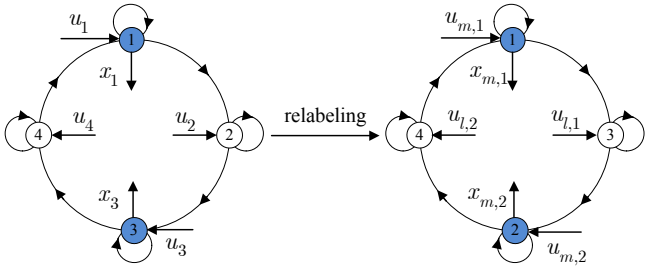


Fig. 1. Node relabeling in a ring network so that manifest and latent nodes have consecutive indices.

*Assumption 2.2:* The input process to the manifest subnetwork, $\{u_m\}$, is a zero-mean stochastic process with independent and identically distributed (i.i.d.) random vectors $u_m(k)$ with covariance $\sigma_{u_m}^2 I_{n_m}$, where $\sigma_{u_m} \in \mathbb{R}_{>0}$.

Assumption 2.2 guarantees that the input signal $\{u_m\}$ is persistently exciting of arbitrary order and its power spectral density does not vanish for any frequency $\omega$, whose variations are commonly seen in system identification [16], [9]. The zero-mean assumption can be relaxed by assuming a nonzero but known expectation $\mathbb{E}[u_m(k)]$, which corresponds to the scenario where the designer injects a deterministic stimulating signal into every manifest node that is subject to the disturbance of a

zero-mean white noise. Without loss of generality, we assume $\mathbb{E}[u_m(k)] \equiv 0$ for notational simplicity.

Given the setup above, our objective is to identify the transfer function $T_{x_m u_m}(\omega)$ of the manifest subnetwork, that is, the transfer function from $u_m$ to $x_m$, absent knowledge of the latent nodes. In the following, we define this problem formally.

*Problem 2.3: (Identification of the manifest transfer function).* Given the measured data sequence $\{y\}_1^N$, find a linear auto-regressive model of order $\tau$ ($N \gg \tau$) of the form

$$\tilde{x}_m(k+1) = \sum_{i=0}^{\tau-1} \tilde{A}_i \tilde{x}_m(k-i) + u_m(k), \tag{3}$$

such that the transfer function $T_{\tilde{x}_m u_m}$ from $u_m$ to $\tilde{x}_m$ and $T_{x_m u_m}$ from $u_m$ to $x_m$ in (1) are close in the $H_\infty$ norm, i.e., $\|T_{\tilde{x}_m u_m} - T_{x_m u_m}\|_\infty$ is small.

There are alternative methods to identify the transfer function matrix $T_{x_m u_m}$ besides the AR method in (3). For instance, one can try to estimate $T_{x_m u_m}$ directly using frequency domain approaches such as power spectral analysis [9]. However, the transfer function matrix $T_{x_m u_m}$ alone does not differentiate between direct connections (two manifest nodes interact directly) and latent connections (two manifest nodes interact via latent nodes). In contrast, we will show later that the AR model (3) provides information about the network structure by predicting whether two manifest nodes are connected directly or through some latent nodes. Further, if the latent subnetwork is acyclic, then the number of latent nodes in a path connecting two manifest nodes can be predicted.

*Remark 2.4: (Assumption 2.1 implies stability of subnetworks).* It is well known [17] that if $M$ is a submatrix of $A$, then $\sigma_j(M) \leq \sigma_j(A)$. Thus, Assumption 2.1 implies that $\|A_{ii}\| \leq \|A\| < 1$ for $i = 1, 2$, which further indicates $A_{11}$ and $A_{22}$ are Schur stable, since $\rho(M) \leq \|M\|$ for any $M \in \mathbb{R}^{n\times n}$. Note that if $A$ is only Schur stable then both $A_{11}$ and $A_{22}$ may be unstable. •

*Remark 2.5: (Direct versus latent interactions).* The interaction graph of the manifest subnetwork is characterized by $A_{11}$. In particular, the state of node $p$ affects the state of node $q$ *directly* if and only if the entry on the $q$-th row and the $p$-th column, denoted by $A_{11}(q,p)$, is nonzero. However, even if $A_{11}(q,p) = 0$, it is still possible that node $p$ affects node $q$ *indirectly* through some latent nodes. We show later in Remark 3.3 that identifying the parameters of the AR model in (3) distinguishes direct from indirect interactions. •

## III. Asymptotically exact identification of manifest transfer function via AR models

In this section, we show that there exists a group of AR models in the form of (3) whose transfer functions converges to $T_{x_m u_m}$ exponentially in the $H_\infty$ sense. Moreover, if the latent subnetwork is acyclic, then there exists an AR model (3) whose transfer function is exactly the manifest transfer function.

*Theorem 3.1: (AR model whose transfer function converges to the manifest transfer function).* Consider the LTI network described by (2). Assume all latent nodes are passive and

$$(1 - \|A_{11}\|)(1 - \|A_{22}\|) > \|A_{12}\| \cdot \|A_{21}\| \tag{4}$$

holds. Then, there exists $\bar{\gamma} \in \mathbb{R}_{>0}$ such that, for all $\tau \in \mathbb{Z}_{\geq 0}$, the AR model (3) with the matrix sequence in $\mathbb{R}^{n_m \times n_m}$

$$\tilde{A}_0^* = A_{11}, \quad \tilde{A}_i^* = A_{12} A_{22}^{i-1} A_{21}, \tag{5}$$

for $i \in \{1, \ldots, \tau - 1\}$, guarantees

$$\|T_{\tilde{x}_m u_m}(\omega, \tau) - T_{x_m u_m}(\omega)\|_\infty \leq \bar{\gamma} \|A_{22}\|^\tau. \tag{6}$$

Theorem 3.1 shows that if the latent nodes in a network do not receive any external inputs, then there exists a group of AR models in the form of (3) with transfer functions converging to the manifest transfer function exponentially in $H_\infty$-norm.

*Remark 3.2: (Stability degree versus interaction strength between the manifest and latent subnetworks).* The requirement (4) relates the stability degrees of the manifest and latent subnetworks and their interaction strengths. Roughly speaking, under Assumption 2.1, for networks with increasingly small stability margin, i.e., as $\rho(A_{ii})$ gets close to 1 (which implies that $\|A_{ii}\| \geq \rho(A_{ii})$ is also close to 1), the term $(1 - \|A_{11}\|)(1 - \|A_{22}\|)$ becomes increasingly small and thus $\|A_{12}\| \cdot \|A_{21}\|$ (characterizing the subnetwork interaction strength) should be increasingly small as well. •

*Remark 3.3: (Direct versus latent interactions - cont'd).* It follows immediately from (2) that

$$x_m(k+1) = \sum_{i=0}^{k} \tilde{A}_i^* x_m(k-i) + A_{12} A_{22}^k x_l(0) + u_m(k). \tag{7}$$

By virtue of (7), we can distinguish whether two manifest nodes interact directly or indirectly through latent nodes by looking at the matrix sequence $\{\tilde{A}_i^*\}$. First, the state of manifest node $p$ affects the state of manifest node $q$ directly if and only if $\tilde{A}_0^*(q,p) = A_{11}(q,p) \neq 0$. Similarly, the state of manifest node $p$ affects the state of manifest node $q$ indirectly through latent nodes iff $\tilde{A}_i^*(q,p) \neq 0$ for some $i \geq 1$. In particular, from the relation (5) one can see that the state of $p$ first affects some latent nodes (that correspond to the nonzero entries in the $p$-th column of $A_{21}$) through $A_{21}$, then propagates through the latent subnetwork (reflected by $A_{22}^{i-1}$), and finally affects $q$ through $A_{12}$. Furthermore, if the latent subnetwork is acyclic, then $\tilde{A}_i^*(q,p) \neq 0$ implies that there are exactly $i$ latent nodes in a path connecting $p$ to $q$. •

Next, we show that there exists an AR model (3) whose transfer function coincides with the manifest transfer function if the latent subnetwork is acyclic.

*Corollary 3.4: (Exact manifest transfer function matching when the latent subnetwork is acyclic).* Under the hypotheses of Theorem 3.1, further assume that the latent subnetwork is

acyclic, i.e., there exists $\tau_{22} \in \mathbb{Z}_{\geq 1}$ such that $A_{22}^{\tau_{22}} = \mathbf{0}_{n_l \times n_l}$. Then, the matrix sequence $\tilde{A}_0^*, \ldots, \tilde{A}_{\tau_{22}}^*$ in (5) ensures

$$T_{\tilde{x}_m u_m} = T_{x_m u_m}.$$

Theorem 3.1 and Corollary 3.4 show that it is possible to identify the transfer function of the manifest subnetwork without any knowledge of the passive latent nodes. However, (5) cannot be directly applied to determine the auto-regressive model because its evaluation requires knowledge of the adjacency matrix $A$ of the whole network, which is unknown. Instead, we determine the AR model using the measured data sequence $\{y\}_1^N$, as explained in the next section.

## IV. Manifest transfer function identification via least-squares

In this section, we describe the least-squares (LS) estimation method and show that the parameters resulting from it asymptotically converge in probability, as the data length $N$ and model order $\tau$ increase, to the optimal matrix sequence identified in Theorem 3.1.

### A. Least-squares AR estimation

Given a vector sequence $\{y\}_1^N \subset \mathbb{R}^{n_m}$, the problem of least-squares auto-regressive (LSAR) model estimation with order $\tau \in \mathbb{Z}_{\geq 1}$ is to find a matrix sequence $\{\hat{A}\}_0^{\tau-1}$ that minimizes the 2-norm of the residual sequence $\{e\}_\tau^{N-1}$,

$$e(k) = y(k+1) - \sum_{i=0}^{\tau-1} \hat{A}_i y(k-i). \tag{8}$$

Equation (8) can be written in a compact vector form as

$$\vec{y}_N = \hat{\mathbf{A}}_\tau \Phi_N + \vec{e}_N, \tag{9}$$

where

$$\begin{aligned}
\vec{y}_N &= \begin{bmatrix} y(\tau+1) & y(\tau+2) & \ldots & y(N) \end{bmatrix} \in \mathbb{R}^{n_m \times (N-\tau)},\\
\vec{e}_N &= \begin{bmatrix} e(\tau) & e(\tau+1) & \ldots & e(N-1) \end{bmatrix} \in \mathbb{R}^{n_m \times (N-\tau)},\\
\hat{\mathbf{A}}_\tau &= \begin{bmatrix} \hat{A}_0 & \hat{A}_1 & \ldots & \hat{A}_{\tau-1} \end{bmatrix} \in \mathbb{R}^{n_m \times n_m \tau},\\
\Phi_N &= \begin{bmatrix} y(\tau) & y(\tau+1) & \ldots & y(N-1) \\ y(\tau-1) & y(\tau) & \ldots & y(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ y(1) & y(2) & \ldots & y(N-\tau) \end{bmatrix},
\end{aligned}$$

with $\Phi_N \in \mathbb{R}^{n_m \tau \times (N-\tau)}$. We immediately obtain that

$$\begin{aligned}
\|\{e\}_\tau^{N-1}\|^2 = \operatorname{tr}(\vec{e}_N \vec{e}_N^T) &= \operatorname{tr}\big((\vec{y}_N - \hat{\mathbf{A}}_\tau \Phi_N)(\vec{y}_N - \hat{\mathbf{A}}_\tau \Phi_N)^T\big)\\
&= \operatorname{tr}\big((\hat{\mathbf{A}}_\tau - \vec{y}_N \Phi_N^T (\Phi_N \Phi_N^T)^{-1}) \Phi_N\\
&\quad \cdot \Phi_N^T (\hat{\mathbf{A}}_\tau - \vec{y}_N \Phi_N^T (\Phi_N \Phi_N^T)^{-1})^T\\
&\quad + \vec{y}_N (I_{N-\tau} - \Phi_N^T (\Phi_N \Phi_N^T)^{-1} \Phi_N) \vec{y}_N^T\big),
\end{aligned}$$

where we have added and subtracted the term $\vec{y}_N \Phi_N^T (\Phi_N \Phi_N^T)^{-1} \Phi_N \vec{y}_N^T$. The minimum for this expression is achieved for

$$\hat{\mathbf{A}}_\tau = \vec{y}_N \Phi_N^T (\Phi_N \Phi_N^T)^{-1}. \tag{10}$$

We sometimes use $\hat{\mathbf{A}}_\tau(\{y\}_1^N)$ to explicitly indicate its dependency upon the measured data sequence.

### B. Identification of manifest transfer function

Next, we show that as $N, \tau \to \infty$, the LS estimate in (10) converges in probability to

$$\tilde{\mathbf{A}}_\tau^* = \begin{bmatrix} \tilde{A}_0^* & \tilde{A}_1^* & \ldots & \tilde{A}_{\tau-1}^* \end{bmatrix} \in \mathbb{R}^{n_m \times n_m \tau}.$$

*Lemma 4.1: (The LS estimate $\hat{\mathbf{A}}_\tau$ converges to $\tilde{\mathbf{A}}_\tau^*$ in probability).* Consider the LTI network described in (2) with all the latent nodes passive and let (4) hold. Given the measured data sequence $\{y\}_1^N$ generated from the unknown LTI network (2) stimulated by the white noise input $\{u_m\}$, the LS estimate $\vec{\hat{A}}_\tau(\{y\}_1^N)$ in (10) satisfies

$$\|\operatorname*{plim}_{N\to\infty} \hat{\mathbf{A}}_\tau(\{y\}_1^N) - \tilde{\mathbf{A}}_\tau^*\|_{\max} \leq \beta \tau \|A_{22}\|^\tau, \tag{11}$$

where $\beta$ is a constant that depends only on the network adjacency matrix $A$ and the input power $\sigma_{u_m}^2$.

When it is clear from context, we refer to $\operatorname{plim}_{N\to\infty} \hat{A}_i(\{y\}_1^N)$ simply as $\hat{A}_i$.

*Remark 4.2: (Direct versus latent interactions – cont'd).* Lemma 4.1 shows that $\hat{A}_i$ converges to $\tilde{A}_i^*$ exponentially with the increase of $\tau$. Using the arguments in Remark 3.3, from $\hat{A}_0$ one could tell whether two manifest nodes interact directly and from $\hat{A}_i$, $i \geq 1$ that whether two manifest nodes interact indirectly through latent nodes. •

The next result shows that the $H_\infty$-norm of the transfer function from $e$ to $y$ in (8) is uniformly upper bounded with respect to the model order $\tau$. This result will be used to prove the main result in Theorem 4.4.

*Lemma 4.3: (Norm of $T_{ye}$ is uniformly upper bounded w.r.t $\tau$).* Denote the transfer function from $e$ to $y$ in (8) by $T_{ye}(\{y\}_1^N, \tau)$. There exist positive scalars $\tau_0$ and $U_{T_{ye}}^\infty$ such that for $\tau \geq \tau_0$,

$$\|\operatorname*{plim}_{N\to\infty} T_{ye}(\{y\}_1^N, \tau)\|_\infty \leq U_{T_{ye}}^\infty. \tag{12}$$

We are ready to show that the transfer function $T_{ye}$ in (8) obtained using the LSAR method converges to $T_{x_m u_m}$.

*Theorem 4.4: (The LSAR method consistently estimates the manifest transfer function using measured data).* Consider the LTI network described in (2) with all the latent nodes passive and let (4) hold. Then, there exist positive scalars $\bar{\beta}$, $\bar{\gamma}$ and $\tau_0$ such that, for $\tau \geq \tau_0$,

$$\|\operatorname*{plim}_{N\to\infty} T_{ye}(\{y\}_1^N, \tau) - T_{x_m u_m}\|_\infty \leq (\bar{\beta}\tau^2 + \bar{\gamma}) \|A_{22}\|^\tau. \tag{13}$$

As a consequence, one has that

$$\operatorname*{plim}_{N\to\infty, \tau\to\infty} T_{ye}(\{y\}_1^N, \tau) = T_{x_m u_m}.$$

Theorem 4.4 states that when the length $N$ of the measurement data is sufficiently large and the model order $\tau$ exceeds a certain threshold, the error $\|T_{ye}(\tau) - T_{x_m u_m}\|_\infty$ obtained by the LSAR method decreases exponentially with $\tau$.

*Remark 4.5: (It is more difficult to identify the manifest subnetwork when the latent subnetwork is less stable).* One can show that the threshold $\tau_0$ in Theorem 4.4 decreases with the size of $\|A_{22}\|$. Consequently, a smaller $\|A_{22}\|$ makes $\tau_0$ smaller. On the other hand, a less stable latent subnetwork

makes $\|A_{22}\| \geq \rho(A_{22})$ closer to 1 and the corresponding $\tau_0$ will be larger, which further requires the order of the AR model to be higher. •

### C. Exact identification for acyclic latent subnetworks

In this section, we show that if the latent subnetwork is acyclic, then the transfer function of the manifest subnetwork can be perfectly identified using the LSAR method. First, we show that a stronger result than Lemma 4.1 holds in that $\hat{\mathbf{A}}_\tau(\{y\}_1^N)$ converges to $\tilde{\mathbf{A}}_\tau^*$ in the mean-square sense as $N \to \infty$ for any $\tau \geq \tau_{22} + 1$.

*Lemma 4.6: (The LS estimate $\hat{\mathbf{A}}_\tau$ converges to $\tilde{\mathbf{A}}_\tau^*$ in mean square if the latent subnetwork is acyclic).* Consider the LTI network described in (2) with all the latent nodes passive, let (4) hold and further assume that the latent subnetwork is acyclic ($A_{22}^{\tau_{22}} = \mathbf{0}_{n_l \times n_l}$). For any $\tau \geq \tau_{22} + 1$ the LS estimate $\hat{\mathbf{A}}_\tau(\{y\}_1^N)$ in (10) satisfies

$$\lim_{N\to\infty} \mathbb{E}[(\hat{\mathbf{A}}_\tau(\{y\}_1^N) - \tilde{\mathbf{A}}_\tau^*)^T (\hat{\mathbf{A}}_\tau(\{y\}_1^N) - \tilde{\mathbf{A}}_\tau^*)] = \mathbf{0}_{n_m\tau \times n_m\tau}.$$

The manifest transfer function can be perfectly identified using the LSAR method if the latent subnetwork is acyclic.

*Theorem 4.7: (Exact manifest transfer function identification when the latent subnetwork is acyclic).* Consider the setup in Lemma 4.6. For any $\tau \geq \tau_{22} + 1$ the LS estimate $\hat{\mathbf{A}}_\tau(\{y\}_1^N)$ in (10) guarantees

$$\operatorname*{plim}_{N\to\infty} T_{ye}(\{y\}_1^N, \tau) = T_{x_m u_m}.$$

## V. Simulations

Here, we illustrate our results in two examples. For both of them, the input signal $\{u_m\}_\tau^{N-1}$ is a white Gaussian process with unit variance.

*Example 5.1: (Ring network).* Consider a ring network as the one described in Figure 1 with 40 nodes. All edges, including self-loops, have the same weight $\alpha = 0.25$. The nodes with indices 1 and 3 are assumed to be manifest and the rest 38 nodes are latent. The identification error $\|T_{ye} - T_{x_m u_m}\|_\infty$ of the LSAR model is shown in Fig. 2 for different lengths of measured data and different model orders. In Fig. 3, we fix the length of measured data $N = 10^5$ and compare the error of the LSAR model with the error $\|T_{\tilde{x}_m u_m} - T_{x_m u_m}\|_\infty$ of the ideal AR model in Theorem 3.1. Note that the AR model in Theorem 3.1 is not practical because it requires the knowledge of $A$, and we use its performance merely for comparison purposes. We make the following observations:

(i) When the measured data length $N$ is too small, increasing the AR model order $\tau$ does not provide better estimation of the manifest transfer function. Similarly, when the model order $\tau$ is too low, increasing the data length $N$ is not helpful, either.

(ii) When $N$ and $\tau$ increase simultaneously, the LSAR method provides good estimation of the manifest transfer function without any knowledge of the latent nodes, as predicted by our results.

(iii) The performance of the LSAR model is fairly close to the performance of the ideal model in Theorem 3.1. •

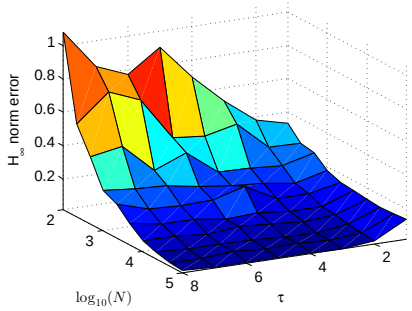


Fig. 2. Illustration of the $H_\infty$ norm error of the LSAR method with respect to the length $N$ of measured data and model order $\tau$ for the ring network in Example 5.1. Performance improves with the increase of $N$ and $\tau$.

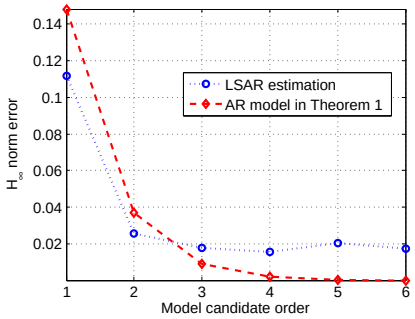


Fig. 3. Comparison of the $H_\infty$ norm errors of the LSAR method and the optimal AR model from Theorem 3.1 for the ring network in Example 5.1. The length of measured data is $N = 10^5$.

*Example 5.2: (Erdős–Rényi random network).* In this example, we use a group of 10 $G(10, 0.25)$ Erdős–Rényi random networks [18] with 3 manifest nodes to test the identification performance of the LSAR method. Specifically, every network in the group contains 10 nodes and any edge $(i, j)$, $1 \leq i, j \leq 10$ is assigned a nonzero weight with probability 0.25. The weight has a uniform distribution in $\{x \in \mathbb{R} | 0 < x < 0.35\}$. The 3 manifest nodes are chosen randomly. We consider a fixed length $N = 10^5$ of measured data and analyze the effect of varying the model order. Figure 4 illustrates the error in manifest transfer function identification for all 10 networks and shows the improvement as the model order increases. To illustrate our observations on the identification of manifest versus latent interactions, Figure 5 shows both the network with index 1 in Figure 4 and the reconstruction obtained by the LSAR method. Finally, Figure 6 shows the corresponding identification error compared against the error of the optimal AR model from Theorem 3.1. •

## VI. Conclusions

We have proposed a method to identify the transfer function of the manifest subnetwork in an LTI network, which corresponds to the transfer function from the inputs of the manifest nodes to their states. We have shown that, if all external inputs enter the network through only the manifest nodes, then the manifest transfer function can be identified with arbitrarily small $H_\infty$ norm error using

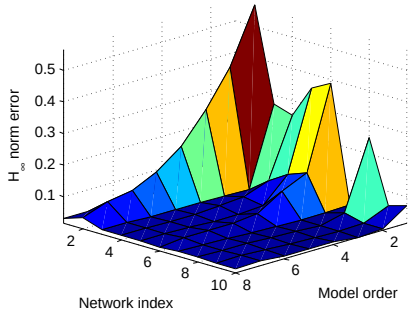


Fig. 4. Illustration of the $H_\infty$ norm error of the LSAR with respect to the length $N$ of measured data and model order $\tau$ for the group of $G(10, 0.25)$ Erdős–Rényi random networks in Example 5.2. Performance improves with the increase of model order $\tau$ for all 10 networks.

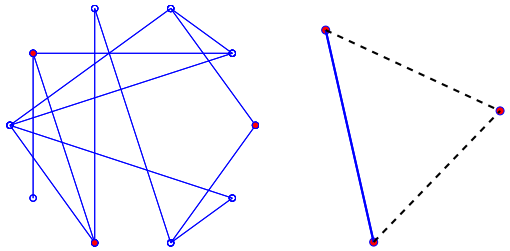


Fig. 5. (a) Erdős–Rényi random network corresponding to the network with index 1 in Figure 4. (b) reconstructed interaction graph of the manifest subnetwork using the LSAR method. Blue solid edge represents direct interaction and black dashed edge represents indirect interaction through latent nodes.

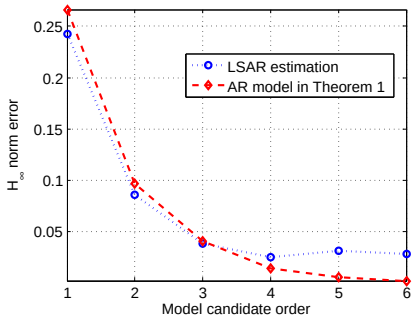


Fig. 6. Comparison of the $H_\infty$ norm errors of the AR model found using the LS method and the AR model in Theorem 3.1 for the Erdős–Rényi random network described in Fig. 5.

auto-regressive models. These models are estimated without any knowledge of the latent nodes from the measured states of the manifest nodes using least-squares. We have illustrated the results in ring networks and Erdős–Rényi random networks. Future work will investigate the sensitivity of the identification performance to latent nodes, the problem of network identification when latent nodes are not passive and in scenarios where controllable and observable nodes are separate, the impact of the network structure on the requirement relating the stability degrees of the manifest and latent subnetworks and their interaction strengths, and the application of our results to analysis of brain data.

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