Quantifying the robustness of power networks against initial failure

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Abstract— **This paper proposes a notion of robustness metric for a power network in terms of the minimal power disturbance required to cause an initial failure. We allow multiple disturbances to act upon different load-side nodes. We formulate the computation of the robustness metric in terms of various equivalent optimization problems with affine inequality constraints. To obtain these results, our analysis identifies a polyhedral cone determined by the topology of the power network whose extreme rays determine the affine inequality constraints. We employ the Double Description method to calculate these extreme rays and discuss how to properly initialize it. Simulations on an IEEE 9-bus power network illustrate our results.**

I. Introduction

The vulnerability or robustness analysis of electrical power network under voltage and frequency fluctuation plays a crucial role in power grid management. Despite the careful design of power networks, large blackouts still happen due to various factors, including relatively small power flow deviations after disconnecting some transmission lines and loss of frequency synchronization leading to cascading failures. There is a need for metrics and methods that can serve to quantify the robustness of power networks at steady state and help determine why and how failure could happen. The availability of such tools is of great practical value and this motivates us in the study here of a robustness metric for linearized AC power networks under transmission and source constraints.

Literature review: Examples of power blackouts include the 2003 blackout in-between the Northeastern America and Canada, and the 2006 blackout in Europe [1], [2]. Robustness of complex network is a topic of much recent interest, see e.g. [3], [4], [5]. In the context of power networks, some works [6], [7] have employed concepts from control theory, such as Lyapunov functions and Nyquist criterion, to study the stability, stability margins, and robustness in both AC and DC power networks. One recent research direction [8] focuses on identifying the set of most vulnerable lines in a power network which may lead to a given amount of load shedding to guarantee the existence of a feasible solution to the power flow equation. A related line of research [9] employs reactive power flow equations to establish necessary condition for voltage collapse. In the investigation of cascading failures in power systems, see [10] and references therein, it is common to provide a model for what constitutes a failure in a transmission line or a bus, and then quantify a posteriori the robustness of the power network by observing the remaining demand after the end of the cascading failure normalized by the original demand. The work [11] carries out an interdependence analysis of the physical power network and its communication network, and shows how this interdependence may lead to cascading failures, which motivates the design of a control policy to improve its robustness. In general, there is a lack of metrics that quantify a priori (e.g., before a cascading failure occurs) the robustness of power networks. In this work we specifically focus on quantifying the cost to trigger an initial failure as a result of disruptions in transmission lines and/or source nodes.

Statement of contributions: We consider a linearized AC power system model composed by sources, loads, and transmission lines. We define a failure in the power network as the non-existence of solution to the power flow equations with capacity constraints on source injections and transmission lines. Our first contribution is the introduction of a metric to measure the robustness of the power network against failure given multiple disturbances injected at the load side. The value of the robustness metric is defined as the minimal power injection under a pre-defined measurement function of load-side disturbance injection that is able to cause a failure. We allow for power disturbances from different load-side nodes, reflecting the fact that in reality demands of various power consumers may deviate from their nominal values, and the collective power fluctuation from different nodes can jointly initiate a failure. Our second contribution concerns the computation of the robustness metric in terms of a set of equivalent optimization problems under affine inequality constraints. We rely on tools from algebraic graph theory and polyhedral cone theory to obtain these equivalent formulations. Specifically, our analysis identifies a polyhedral cone which is determined by the topology of the power network and whose extreme rays determine the affine inequality constraints. We employ the Double Description method to calculate these extreme rays and identify a relaxed polyhedral cone to properly initialize the algorithm. Various simulations illustrate the results. For space reasons, all proofs are omitted and will appear elsewhere.

Organization: Section II introduces notations and preliminaries. Section III presents the power system model and the problem statement. Section IV shows how the computation of the proposed robustness metric can be cast as several equivalent optimization problems under affine inequality constraints. Finally, Section V provides simulation results. We gather our conclusions and ideas for future work in Section VI.

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II. Preliminaries

This section introduces basic notions from algebraic graph theory, inequality constraint systems, and polyhedral cones.

A. Notation

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers, respectively. Let $\mathbb{R}^n_{\geqslant}$ (resp. $\mathbb{R}^n_{>}$) denote the set of n-dimensional vectors that are element-wise non-negative (resp. non-negative with at least one positive element). Variables are assumed to belong to the Euclidean space if not specified otherwise. Denote by $a \leqslant b$ $(<, \geqslant, >)$ the element-wise set of inequalities for vectors a and b. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the vector of ones and zeros with dimension n, respectively. For $p, q \in \mathbb{N}$, let $]p,q[= \{x \in \mathbb{N} \mid p \leqslant x \leqslant q\}$. We denote the cardinality of a set σ by $|\sigma|$.

Denote by I_n the n-dimension identity matrix. For a matrix $A \in \mathbb{R}^{m\times n}$, let $[A]_i$ denote the ith row of A. For a vector $a \in \mathbb{R}^n$, let $\operatorname{diag}(a)$ be the diagonal matrix whose diagonal are the elements of a. Given $\sigma \subseteq]1,n[$, A_σ is the matrix composed by rows of A indexed by σ, e.g.,

$$A_{\{1,2\}} = \begin{bmatrix} [A]_1 \\ [A]_2 \end{bmatrix}.$$

With a slight abuse of notation, we use $v \in A$ to denote that v is a column vector of the matrix A.

B. Algebraic Graph Theory

We review basic notions from algebraic graph theory from [12], [13]. An weighted undirected graph (or simply weighted graph) is a tuple $\mathscr{G} = (\mathscr{I}, \mathscr{E}, \mathscr{B})$, where $\mathscr{I} = \{1, \dots, n\}$ is the vertex set, $\mathscr{E} = \{e_1, \dots, e_m\} \subseteq \mathscr{I} \times \mathscr{I}$ is the edge set, and $\mathscr{B} = (b_1, \dots, b_m)^T$ is the weight set with $b_\alpha > 0$ representing the weight of edge e_α, for each $\alpha \in]1,m[$. A path is an ordered sequence of vertices such that any pair of consecutive vertices in the sequence is an edge of the graph. A graph is connected if there exists a path between any two vertices. For each edge $e_\alpha \in \mathscr{E}$ with vertices i, j, the orientation procedure consists of choosing either i or j to be the positive end of e_α and the other vertex to be the negative end. The incidence matrix $D = (d_{\alpha i}) \in \mathbb{R}^{m\times n}$ associated with $\mathscr{G}$ is then defined as

$$d_{\alpha i} = \begin{cases} 1 & \text{if } i \text{ is the positive end of } e_\alpha, \\ -1 & \text{if } i \text{ is the negative end of } e_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

The Laplacian matrix associated with $\mathscr{G}$ is $L = D^T b D$, where the weight matrix $b = \operatorname{diag}(\mathscr{B})$. Note that L is independent from the ordering of the edges or the orientation procedure.

C. Inequality Constraint Systems

We gather here some basic notions on inequality constraint systems following [14]. For $A \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^m$, we define the affine inequality intersection set

$$S = \{x \in \mathbb{R}^n \mid Ax > a\}.$$

For $k \in]1,m[$, let S^k denote the set obtained from S by deleting its kth inequality, i.e.,

$$S^k = \{x \in \mathbb{R}^n \mid [A]_i x > a_i,\ \forall i \in]1,m[\setminus \{k\}\}.$$

In the rest of this section, we assume that the set S is non-empty. For $k \in]1,m[$, the constraint $[A]_k x > a_k$ is redundant in S if and only if $S^k = S$. A constraint is non-redundant if it is not redundant. The following results provides a test to check whether a constraint of S is redundant.

Lemma 2.1: (Redundant constraint of affine inequality intersection set). The constraint $[A]_k > a_k$ is redundant in S if and only if

$$\min\{[A]_k x - a_k \mid x \in \mathbb{R}^n \text{ s.t. } [A]_i x \geqslant a_i,\ \forall i \in]1,m[\setminus \{k\}\} \geqslant 0.$$

A set S' is a minimal representation of the affine inequality intersection set S if it takes the form

$$S' = \{x \in \mathbb{R}^n \mid A'x > a'\},$$

with $A' \in \mathbb{R}^{m'\times n}$ and $a' \in \mathbb{R}^{m'}$, and every other set of affine inequality intersection set describing S has at least m' constraints. The set S' is a minimal representation of S if and only if it contains non-redundant constraints.

Similarly, for $E \in \mathbb{R}^{m\times n}$ and $e \in \mathbb{R}^m$, we define the affine inequality union set as

$$V = \bigcup_{i \in]1,m[} \{x \in \mathbb{R}^n \mid [E]_i x \leqslant e_i\}. \quad (1)$$

One can define the notions of (non-)redundant constraint and minimal representation of V analogously, and also obtain the following result.

Lemma 2.2: (Redundant constraint of affine inequality union set). The constraint $[E]_k x \leqslant e_k$ is redundant in V if and only if

$$\min\{[E]_k x - e_k \mid x \in \mathbb{R}^n \text{ s.t. } [E]_i x \geqslant e_i, \forall i \in]1,m[\setminus \{k\}\} \geqslant 0.$$

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Algorithm 1 describes a procedure to remove all redundant constraints from (1).

Algorithm 1: Redundant constraint removal

Data: $E \in \mathbb{R}^{m\times n}, e \in \mathbb{R}^m$;
Result: E^m, e^m that constructs a minimal representation;
initialization: $E^0 = E, e^0 = e$
for $j = 1 : m$ **do**
  w^j = number of rows in E^{j-1}
  $RCC =$ $\min\{[E^{j-1}]_1 x - e_1 \mid [E^{j-1}]_i x \geqslant e_i, \forall i \in]2, w^j[\}$
  if $RCC \geqslant 0$ **then**
    Delete the first row of both E^{j-1} and e^{j-1}
  else
    Move the first row of both E^{j-1} and e^{j-1} to its corresponding last row
  end
  Let E^j and e^j be the revised E^{j-1} and e^{j-1} after deletion or removal, respectively
end

D. Polyhedral Cones

Here we present some basic results from (convex) polyhedral cones [15]. Given $G \in \mathbb{R}^{m_1 \times n}$ and $H \in \mathbb{R}^{m_2 \times n}$, the convex set

$$C = \left\{x \in \mathbb{R}^n \;\middle|\; Gx = \mathbf{0}_{m_1},\, Hx \geqslant \mathbf{0}_{m_2}\right\} \tag{2}$$

is a polyhedral cone. If $\left\{x \in \mathbb{R}^n \;\middle|\; Gx = \mathbf{0}_{m_1},\, Hx = \mathbf{0}_{m_2}\right\} = \{\mathbf{0}_n\}$, then C is called a pointed polyhedral cone. A set F is a face of C if $F \subseteq C$ and

$$F = \left\{x \in \mathbb{R}^n \;\middle|\; Gx = \mathbf{0}_{m_1}, H_{\bar{\sigma}}x = \mathbf{0}_{|\bar{\sigma}|}, H_{\sigma}x \geqslant \mathbf{0}_{|\sigma|}\right\}$$

for some $\sigma \subseteq]1,m_2[$, where $\bar{\sigma} =]1,m_2[\backslash\sigma$. The dimension of F is the number of linearly independent points in F. A non-zero vector $r \in C$ is called a ray of C. A ray r is an extreme ray of C if the set $\left\{\eta r \;\middle|\; \eta \in \mathbb{R}_{>}\right\}$ is a one-dimensional face. Throughout the paper, we identify the extreme rays r and ηr, for $\eta \in \mathbb{R}_{>}$, as the same extreme ray. With this convention, the minimal generating matrix of C, denoted as R_C, is the matrix whose columns correspond to the extreme rays of C. We let μ_C denote the number of extreme rays of C.

The following result illustrates a crucial property of pointed polyhedral cones.

Lemma 2.3: (Double description of pointed polyhedral cones). Given a pointed polyhedral cone described by (2), it holds that $C = \{\lambda^T R_C | \lambda \in \mathbb{R}^{\mu_C}_{\geqslant}\}$ and $C\backslash\{\mathbf{0}_n\} = \{\lambda^T R_C | \lambda \in \mathbb{R}^{\mu_C}_{>}\}$.

III. Problem Statement

Here we lay out the problem statement in detail. Consider a power network whose physical structure is described by a connected undirected graph $\mathscr{G} = (\mathscr{I}, \mathscr{E}, \mathscr{B})$. The n nodes correspond to the buses whereas the m edges correspond to the transmission lines. The weight b_α associated with the edge e_α corresponds to the line susceptance. For notational simplicity, we arrange the nodal indices so that the first k_1 nodes are load nodes and the last k_2 nodes are source nodes ($k_1 + k_2 = n$). For each edge arbitrarily define its positive and negative end. We use the linearized AC power flow model as described next. Denote the power injection vector by $P = \begin{bmatrix} Z^T & U^T \end{bmatrix}^T$, where $Z \in \mathbb{R}^{k_1}$ denotes the load power injection vector and $U \in \mathbb{R}^{k_2}$ stands for the source power injection vector. Both the load and source power injection vectors have lower and upper limits, represented by $Z_{\min}$ and $Z_{\max} \in \mathbb{R}^{k_1}$ for the loads, and $U^c_{\min}$ and $U^c_{\max} \in \mathbb{R}^{k_2}$ for the sources. Let $\Theta = [\theta_1, \theta_2, \ldots, \theta_n]^T$ denote the voltage angle vector, with θ_i the voltage angle at node i. Finally, we denote by $F = [F_1, F_2, \ldots, F_m]^T$ the power flow vector, with minimum $F_{\min} \in \mathbb{R}^m$ and maximum $F_{\max} \in \mathbb{R}^m$ transmission line capacity bounds. For every $\alpha \in]1,m[$, the magnitude of F_α stands for that of the power flow in edge e_α; the sign of F_α is positive if the power flows into the negative end of e_α, and is negative if the other way around. The power flow equations at steady state $(\bar{P}, \bar{\Theta}, \bar{F})$ can be written as

$$\bar{P} = L\bar{\Theta}, \tag{3a}$$

$$\bar{F} = bD\bar{\Theta}, \tag{3b}$$

where L, b, and D are the Laplacian, weight, and incidence matrix associated with the graph $\mathscr{G}$. Note that, if $(\bar{P}, \bar{\Theta}, \bar{F})$ is a steady state of the power network, $\bar{\Theta}$ (up to a translation) and $\bar{F}$ are uniquely determined by $\bar{P}$. For this reason, in the rest of the paper, we only use $\bar{P}$ to represent a steady state.

We are interested in studying scenarios where additional power is required by the load nodes and the effect that this might have on the operation of the power network. Formally, given the additional load injection $Z \in \mathbb{R}^{k_1}$, let $\Omega(Z)$ denote the set of all feasible adjustments $(U, \Theta, F) \in \mathbb{R}^{k_2+n+m}$ to the power network that compensate for this additional load, i.e.,

$$\begin{bmatrix}\bar{Z} \\ \bar{U}\end{bmatrix} + \begin{bmatrix} Z \\ U \end{bmatrix} = L\Theta, \tag{4a}$$

$$F = bD\Theta, \tag{4b}$$

$$U^c_{\min} < U + \bar{U} < U^c_{\max}, \tag{4c}$$

$$U^d_{\min} < U < U^d_{\max}, \tag{4d}$$

$$F_{\min} < F < F_{\max}. \tag{4e}$$

Here, (4a)-(4b) are the power flow equations, (4c) and (4e) are the source power injection and the power flow constraints, respectively, and (4d) captures limitations on the rate of change of power generation at the sources, with lower $U^d_{\min}$ and upper $U^d_{\max} \in \mathbb{R}^{k_2}$ bounds.

With all these elements in place, we are ready to introduce a metric to measure the robustness of the power network against unexpected demands in the form of changing loads. Given a steady state $\bar{P} \in \mathbb{R}^n$, let $\gamma(\bar{P})$ be the optimal value of the following optimization problem

$(P1)$

$$\inf f(Z)$$

$$\text{s.t. } Z_{\min} \leqslant Z \leqslant Z_{\max} \tag{5a}$$

$$\Omega(Z) = \emptyset. \tag{5b}$$

Here, the function $f : \mathbb{R}^{k_1} \to \mathbb{R}$ provides a measure of the cost $f(Z)$ of adding the additional load Z to the power network. For instance, a common choice is simply $f(Z) = \sum_{i=1}^{k_1} |Z_i|$ with Z_i the ith component of Z. In the rest of the paper, we assume that the feasible set of $(P1)$ is non-empty. Intuitively, the value $\gamma(\bar{P})$ corresponds to the smallest (measured according to f) power disturbance injected at the loads that is able to cause a failure of the power network at steady state $\bar{P}$.

IV. Computation of the Robustness Metric via Optimization Problems under Affine Constraints

In this section, we show that the computation of the robustness metric, cf. problem $(P1)$, can be cast as an optimization problem under non-convex constraints, and it can be further transformed into several optimization problems under convex affine constraints. The following result characterizes the load injections for which no adjustment to the power network exist that can compensate for the additional load.

Lemma 4.1: (Constraint equivalence). For fixed $Z, \bar{P}$, $U^c_{\min}$, $U^c_{\max}$, $U^d_{\min}$, $U^d_{\max}$, $F_{\min}$ and $F_{\max}$, let $U^r_{\min} =$

$\max\left\{U^c_{\min}, U^d_{\min}+\bar{U}\right\}$ and $U^r_{\max} = \min\left\{U^c_{\max}, U^d_{\max}+\bar{U}\right\}$. The following three statements are equivalent

(i) There does not exist $\Theta \in \mathbb{R}^n$ satisfying

$$\bar{Z}+Z = \begin{bmatrix} I_{k_1} & \mathbf{0}_{k_1\times k_2} \end{bmatrix} L\Theta, \tag{6a}$$

$$U^r_{\min} < \begin{bmatrix} \mathbf{0}_{k_2\times k_1} & I_{k_2} \end{bmatrix} L\Theta < U^r_{\max}, \tag{6b}$$

$$b^{-1}F_{\min} < D\Theta < b^{-1}F_{\max}. \tag{6c}$$

(ii) There exists $Y \in \mathbb{R}^{n+k_2+2m}$ satisfying

$$QY = \mathbf{0}_n, \tag{7a}$$

$$TY \geqslant \mathbf{0}_{2k_2+2m}, \tag{7b}$$

$$\begin{bmatrix} Z+\bar{Z} \\ W \end{bmatrix}^T Y \leqslant 0, \tag{7c}$$

$$Y \neq \mathbf{0}_{n+k_2+2m}, \tag{7d}$$

where

$$Q = \begin{bmatrix} M \\ N \end{bmatrix}^T, \; M = \begin{bmatrix} I_{k_1} & \mathbf{0}_{k_1\times k_2} \end{bmatrix} L \in \mathbb{R}^{k_1\times n},$$

$$N = \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{k_2\times k_1} & I_{k_2} \end{bmatrix} L \\ -\begin{bmatrix} \mathbf{0}_{k_2\times k_1} & I_{k_2} \end{bmatrix} L \\ D \\ -D \end{bmatrix} \in \mathbb{R}^{(2k_2+2m)\times n},$$

$$T = \begin{bmatrix} \mathbf{0}_{(2k_2+2m)\times k_1} & I_{(2k_2+2m)\times(2k_2+2m)} \end{bmatrix},$$

$$W = \begin{bmatrix} U^r_{\max} \\ -U^r_{\min} \\ b^{-1}F_{\max} \\ -b^{-1}F_{\min} \end{bmatrix} \in \mathbb{R}^{(2k_2+2m)}.$$

Lemma 4.1 shows that constraint (5b) can be equivalently transformed into (7) by introducing an auxiliary variable Y coupled with Z. The next result shows how to decouple these variables using the polyhedral cone theory reviewed in Section II.

Lemma 4.2: (Feasible set equivalence). Consider the polyhedral cone Φ defined by

$$\Phi = \left\{ Y \in \mathbb{R}^{n+k_2+2m} \mid QY = \mathbf{0}_n, TY \geqslant \mathbf{0}_{2k_2+2m} \right\}. \tag{8}$$

Then, it holds that $\Delta(R_\Phi) = \Delta(\Phi \backslash \{\mathbf{0}\})$, where

$$\Delta(\Pi) = \bigcup_{Y\in\Pi} \left\{ Z \in \mathbb{R}^{k_1} \mid \begin{bmatrix} Z+\bar{Z} \\ W \end{bmatrix}^T Y \leqslant 0 \right\},$$

and R_Φ denotes the minimal generating matrix of Φ.

Building on Lemmas 4.1 and 4.2, the following result shows that the computation of the robustness metric γ as specified in $(P1)$ can be posed as a non-convex optimization problem whose feasible set is the intersection of a box constraint and the union of a finite number of affine constraints.

Theorem 4.3: (Computation of the robustness metric as non-convex optimization problem). $(P1)$ is equivalent to the following optimization problem

$(P2)$

$$\begin{aligned} \inf \quad & f(Z) \\ \text{s.t.} \quad & Z_{\min} \leqslant Z \leqslant Z_{\max} && (9a) \\ & Z \in \Delta(R_\Phi). && (9b) \end{aligned}$$

Note that the feasible set $(P2)$ is compact. If it is also non-empty and f is continuous on it, then the infimum can be replaced by the minimum. Notice also that (9b) is composed by the union of as many affine constraints as the number μ_Φ of extreme rays. However, some of these constraints might be redundant. For instance, the IEEE 3-bus power network with 2 sources has 85 extreme rays, so (9b) contains 85 constraints; nevertheless, at least 83 of which are redundant, since in this case (9b) can at most give an upper and lower bound of Z due to the fact that the dimension of Z is only 1. In general, one can remove redundant constraints of $\Delta(R_\Phi)$ applying Algorithm 1. We refer to an extreme ray that generates a redundant (resp. non-redundant) constraint as a redundant (resp. non-redundant) extreme ray, and denote the set of non-redundant extreme rays as $\tilde{R}_\Phi$. Therefore, as reviewed in Section II, $\Delta(\tilde{R}_\Phi)$ is a minimal representation of $\Delta(R_\Phi)$, and the following result follows immediately.

Corollary 4.4: (Non-convex optimization problem without redundant constraints). The following optimization problem is equivalent to $(P2)$.

$(P3)$

$$\begin{aligned} \inf \quad & f(Z) \\ \text{s.t.} \quad & Z_{\min} \leqslant Z \leqslant Z_{\max} && (10a) \\ & Z \in \Delta(\tilde{R}_\Phi). && (10b) \end{aligned}$$

Remark 4.5: (Relationship between (non-)redundant extreme rays, network structure, steady state and capacity bounds). By definition, any element in $\tilde{R}_\Phi$ must be a column of R_Φ. We have observed in simulations, see Section V below, that the cardinality of $\tilde{R}_\Phi$ might be much smaller than that of R_Φ when the number of source nodes is large. Notice from (8) that the polyhedral cone Φ and its minimal generating matrix R_Φ only depend on the network topology and the number and location of source nodes. Therefore, one can obtain R_Φ without knowledge of $\bar{P}$, $U^r_{\max}, U^r_{\min}, F_{\max}$ and $F_{\min}$. However, from the procedure of transforming $\Delta(R_\Phi)$ into $\Delta(\tilde{R}_\Phi)$, it is clear that $\tilde{R}_\Phi$ depends on both R_Φ and the above parameters, which means that if the steady state or some bounds change, non-redundant extreme rays for the old values can become redundant for the new ones, and vice versa. •

In general, the optimization problem $(P3)$ is easier to solve than $(P2)$ due to the fewer number of affine constraints. However, the procedure to remove the redundant constraints requires knowledge of the steady state and the capacity bound of every transmission line and source node to be able to execute Algorithm 1. This presents interesting challenges for distributed optimization or real-time optimization with time-varying capacity bounds or changing steady state.

Note that, in general, the feasible set of $(P2)$ (or equivalently $(P3)$) is non-convex. In fact, the set defined by the constraint (9b), as an affine inequality union set, is concave. The next result shows that the optimization problem can be decomposed into several sub-optimization problems, each with a convex and compact feasible set.

Corollary 4.6: (Computation of the robustness metric as collection of optimization problems with convex constraints).

Let $\tilde{\mu}_C$ be the cardinality of $\tilde{R}_\Phi$. For each $i \in]1, \tilde{\mu}_C[$, consider the optimization problem

$(P4i)$

$$\begin{aligned} \inf \quad & f(Z) \\ \text{s.t.} \quad & Z_{\min} \leqslant Z \leqslant Z_{\max} \\ & \begin{bmatrix} Z + \bar{Z} \\ W \end{bmatrix}^T \tilde{R}^i_\Phi \leqslant 0. \end{aligned}$$

Denote by γ_i as its optimal value, and set $\gamma_i = +\infty$ if the feasible set of the corresponding $(P4i)$ is empty. Then,

$$\gamma(\bar{P}) = \min_{i \in]1, \tilde{\mu}_C[} \{\gamma_i\}.$$

Furthermore, if f is convex within the feasible set of $(P4i)$, then the infimum can be replaced by the minimum. The above statement also holds if $\tilde{\mu}_C$ is replaced by μ_C, and $\tilde{R}^i_\Phi$ by R^i_Φ, where R^i_Φ denotes ith column of R_Φ.

We conclude this section showing how to find all extreme rays of the polyhedral cone Φ defined in (8), or equivalently, the minimal generating matrix R_Φ, using the Double Description (DD) method introduced in [15]. To initiate this method, one needs to select an pointed polyhedral cone Φ_r (termed 'relaxed') by discarding some inequality constraints from Φ, for which the minimal generating matrix R_{Φ_r} can be 'easily' computed. The DD method can then find R_Φ by some incremental algorithms. The main obstacle for the application of the DD method is the determination of the initial relaxed pointed polyhedral cone. The following result addresses this question for our specific problem.

Lemma 4.7: (Definition and properties of relaxed cone). Let $\sigma =]k_2, 2k_2 + 2m[$ and consider the relaxed polyhedral cone

$$\Phi_r = \left\{ Y \in \mathbb{R}^{n+k_2+2m} \mid QY = \mathbf{0}_n, T_\sigma Y \geqslant \mathbf{0}_{2m+k_2+1} \right\}. \quad (12)$$

Let $X = \begin{bmatrix} \bar{Q}^T & T_\sigma^T \end{bmatrix}^T$, where $\bar{Q}$ is obtained by arbitrarily remove one row from Q. Then, Φ_r is a pointed polyhedral cone whose extreme rays are the last $2m + k_2 + 1$ columns of X^{-1}.

Algorithm 2 summarizes the procedure to compute the robustness metric for a given power network.

Algorithm 2: Compute robustness metric $\gamma(\bar{P})$

Data: L, A, b, $\bar{P}$, $U^c_{\min}$, $U^c_{\max}$, $U^d_{\min}$, $U^d_{\max}$, $F_{\min}$, $F_{\max}$, $Z_{\min}$ and $Z_{\max}$ as defined in $(P1)$;

Result: robustness metric $\gamma(\bar{P})$;

1 Compute the minimal generating matrix R_{Φ_r} of the pointed polyhedral cone Φ_r in (12) using Lemma 4.7.

2 Compute the minimal generating matrix R_Φ of the pointed polyhedral cone Φ by the DD method

3 Remove all redundant constraints of $\Delta(R_\Phi)$ using Algorithm 1 to obtain $\tilde{R}_\Phi$

4 Solve optimization problem $(P4_i)$ for each $i \in]1, \tilde{\mu}_C[$, with optimal value γ_i

5 Set $\gamma(\bar{P}) = \min_{\in]1, \tilde{\mu}_C[} \{\gamma_i\}$

V. Simulations

In this section we illustrate the computation of the robustness metric on a set of networks with the IEEE-9 network topology [16] and varying numbers of source and load nodes. In doing so, we also compute the number of extreme rays and non-redundant extreme rays in each case.

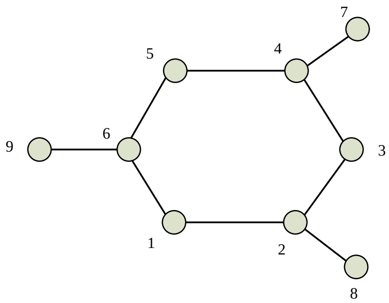

Fig. 1. Topology of IEEE 9-bus power network.

Figure 1 shows the topology structure of the IEEE 9-bus power network. We adopt the convention that the last k_2 nodes by index stand for source nodes and the rest k_1 nodes stand for load nodes, e.g., if $k_2 = 3$, then node 7, 8 and 9 are sources, and the rest are loads. Let $b = I_9$, $U^c_{\max} = -U^c_{\min} = 30 \cdot \mathbf{1}_{k_2}$, $U^d_{\max} = -U^d_{\min} = 10 \cdot \mathbf{1}_{k_2}$, $Z_{\max} = \mathbf{0}_{k_1}$, and $Z_{\min} = -10 \cdot \mathbf{1}_{k_1}$. Since the power flow capacity bound of each transmission line is the same, there is no need to number them.

In our first simulation, we vary k_2 from 1 to 8 to observe how the number of sources affects the number of extreme rays and non-redundant extreme rays. We select $\bar{Z} = -2 \cdot \mathbf{1}_{k_1}$, $\bar{U} = 2k_1/k_2 \cdot \mathbf{1}_{k_2}$ and $F_{\max} = -F_{\min} = 20 \cdot \mathbf{1}_9$. As shown in Figure 2, the number of extreme rays increases with respect to k_2, reaching up to almost 1400, but the number of non-redundant extreme rays truly involved in $(P3)$ (or equivalently, $(P4i)$), does not increase dramatically (e.g., 2 when the number of sources is 8).

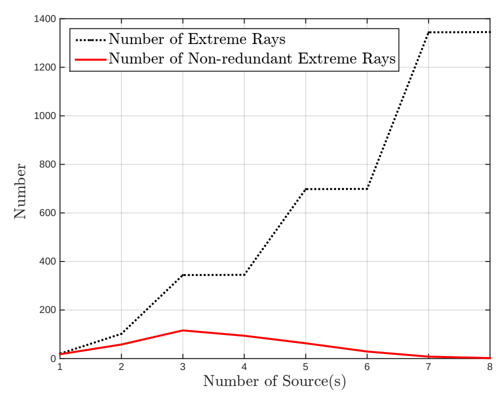

Fig. 2. Number of extreme rays and of non-redundant extreme rays with respect to number of sources.

Next we show how changing the power flow capacity bounds affect the robustness of the power network. We select

Scenario	$\bar{P}^T$	$\gamma(\bar{P})$
1	$\begin{bmatrix}-4 & -4 & -4 & -4 & -4 & -4 & -4 & 14 & 14\end{bmatrix}$	20.57
2	$\begin{bmatrix}-4 & -4 & -4 & -4 & -4 & -4 & -4 & 24 & 4\end{bmatrix}$	5.14
3	$\begin{bmatrix}-2 & -2 & -2 & -2 & -2 & -2 & -16 & 14 & 14\end{bmatrix}$	16
4	$\begin{bmatrix}-2 & -2 & -2 & -2 & -16 & -2 & -2 & 14 & 14\end{bmatrix}$	20.57

TABLE I
IMPACT OF $\bar{P}$ ON THE ROBUSTNESS METRIC.

$k_2 = 2$, $\bar{Z} = -4 \cdot \mathbf{1}_7$ and $\bar{U} = 14 \cdot \mathbf{1}_2$. The cost function is $f(Z) = \|Z\|_2^2 = \sum_{i=1}^{7}(Z_i)^2$. Let all transmission lines take the same upper and lower capacity bound with these two bounds opposite (i.e., for all $i, j \in]1, 9[$, $[F_{\max}]_i = [F_{\max}]_j$ and $[F_{\max}]_i = -[F_{\min}]_i$). Then we vary the flow capacity bound from 15 to 30 and observe the changes in $\gamma(\bar{P})$. As illustrated in Figure 3, initially $\gamma(\bar{P})$ increases if the power flow capacity increases. However, once $F_{\max} = -F_{\min}$ reaches $24 \cdot \mathbf{1}_9$, $U^r_{\max}$ and $U^r_{\min}$ become the primary cause that restricts further growth of the value $\gamma(\bar{P})$, i.e., the robustness metric is saturated with respect to higher power flow capacity.

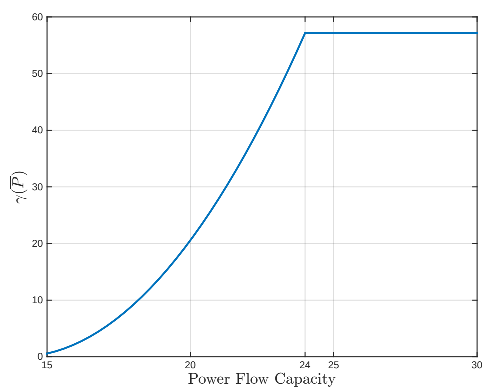

Fig. 3. Computation result of robustness metric $\gamma(\bar{P})$ with respect to different power flow capacity.

Lastly we compute the value of the robustness metric for different steady states $\bar{P}$. In this simulation, we select $k_2 = 2$, $F_{\max} = -F_{\min} = 20 \cdot \mathbf{1}_9$. We keep the total source power injection constant, but change the source and load power injection allocation across the network. Table I shows how changes in the power allocation $\bar{U}$ for source nodes and $\bar{Z}$ for load nodes can have a significant effect in the robustness of the power network. In future work, we plan to investigate ways of providing a given load profile to the power network in a way that optimizes its robustness.

VI. CONCLUSIONS

We have introduced a robustness metric to quantify the minimal power injection of load-side disturbance that can cause an initial failure in a power network constrained by transmission line and source power injection bounds. Using a special form of the Kuhn-Fourier theorem and the theory of pointed polyhedral cones, our analysis of the robustness metric has established several characterizations for its computation, including a non-convex optimization problem and the minimum of finitely many optimal values of optimization problems with affine inequality constraints. Finally, we have described a method to systematically determine all inequality constraints. Various simulations have illustrated the results. Future work will explore the formal characterization of the number of (both redundant and non-redundant) extreme rays, the efficient computation of the robustness metric under changes in the parameters, and the design of strategies to enhance power network robustness.

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REFERENCES

[1] G. A. Mass, M. Bial, and J. Fijalkowski, "Final report-system disturbance on 4 november 2006," *Union for the Coordination of Transmission of Electricity in Europe, Tech. Rep*, 2007.

[2] M. Amin, "North America's electricity infrastructure: Are we ready for more perfect storms?" *IEEE Security & Privacy*, no. 5, pp. 19–25, 2003.

[3] R. Albert, H. Jeong, and A. L. Barabási, "Error and attack tolerance of complex networks," *Nature*, vol. 406, no. 6794, pp. 378–382, 2000.

[4] M. Morohashi, A. E. Winn, M. T. Borisuk, H. Bolouri, J. Doyle, and H. Kitano, "Robustness as a measure of plausibility in models of biochemical networks," *Journal of Theoretical Biology*, vol. 216, no. 1, pp. 19–30, 2002.

[5] M. J. O. Pocock, D. M. Evans, and J. Memmott, "The robustness and restoration of a network of ecological networks," *Science*, vol. 335, no. 6071, pp. 973–977, 2012.

[6] A. Pai, *Energy Function Analysis for Power System Stability*. Springer Science & Business Media, 2012.

[7] X. Feng, J. Liu, and F. C. Lee, "Impedance specifications for stable dc distributed power systems," *IEEE Transactions on Power Electronics*, vol. 17, no. 2, pp. 157–162, 2002.

[8] A. Pinar, J. Meza, V. Donde, and B. Lesieutre, "Optimization strategies for the vulnerability analysis of the electric power grid," *SIAM Journal on Optimization*, vol. 20, no. 4, pp. 1786–1810, 2010.

[9] J. W. Simpson-Porco, F. Dörfler, and F. Bullo, "Voltage collapse in complex power grids," *Nature Communications*, vol. 7, 2016, article number: 10790.

[10] S. Soltan, D. Mazauric, and G. Zussman, "Analysis of failures in power grids," *IEEE Transactions on Control of Network Systems*, 2016, to appear.

[11] M. Parandehgheibi, E. Modiano, and D. Hay, "Mitigating cascading failures in interdependent power grids and communication networks," in *IEEE International Conference on Smart Grid Communications*, Venice, Italy, 2014, pp. 242–247.

[12] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009, electronically available at http://coordinationbook.info.

[13] N. Biggs, *Algebraic Graph Theory*, 2nd ed. Cambridge University Press, 1994.

[14] J. Telgen, "Minimal representation of convex polyhedral sets," *Journal of Optimization Theory and Applications*, vol. 38, no. 1, pp. 1–24, 1982.

[15] K. Fukuda and A. Prodon, "Double description method revisited," in *Combinatorics and Computer Science*. New York: Springer, 1996, pp. 91–111.

[16] B. Zhang, "Control and optimization of power systems with renewables: Voltage regulation and generator dispatch," Ph.D. dissertation, University of California, Berkeley, 2013.