Scheduling of Control Nodes for Improved Network Controllability

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***Abstract*— This paper considers the problem of controlling a linear time-invariant network by means of (possibly) time-varying set of control nodes. As control metric, we adopt the worst-case input energy to drive the network state from the origin to any point on the unit hypersphere in the state space. We provide a geometric interpretation of the controllability Gramian of networks with time-varying input matrices, and establish a connection between the controllability degree of a network and its eigenstructure. Based on the geometric structure of the controllability Gramian, we then propose a scheduling algorithm to select control nodes over time so as to improve the network controllability degree. Finally, we numerically show that, for a class of clustered networks, our algorithm improves upon the performance obtained by a constant set of control nodes, and outperforms an existing heuristic-based on column subset selection.**

I. Introduction

Complex networks emerge in diverse contexts including engineering, biology, and social science. The dynamic behavior of a complex network depends on the behavior of its dynamical subsystems or nodes that interact with each other. Controllability is a desired property, which ensures our ability to arbitrarily alter the network state in a predictable manner by means of external inputs affecting a small subset of nodes. Existing works focus on the case where the set of control nodes is constant over time. Instead, in this paper we investigate the possibility of selecting different control nodes over time, which may be advantageous to remedy spreading processes and cascading effects, while limiting the extent of external interventions.

Related work. Network controllability is an active topic of research, which is receiving considerable attention from different communities. Existing works can be classified into *qualitative* and *quantitative* studies. Qualitative approaches adopt the binary controllability notion first introduced in [1], and employ graphical and combinatorial techniques, e.g., see [2], [3], [4], [5], [6]. On the other hand, quantitative studies use a graded metric of controllability and typically leverage control-theoretic methods, e.g., see [7], [8], [9], [10], [11]. This work falls within the second class, in that it presents a novel geometric characterization of the controllability Gramian of networks with time-varying input matrices, and it describes a control node scheduling algorithm to maximize the network controllability degree.

The problem of finding an optimal placement of sensors and actuators is of utmost importance, as this choice ultimately determines the performance of a dynamic control system. Typically, sensors and actuators are selected to optimize structural properties, such as observability, controllability, and stability, and the selection is constant over time [12]. In modern systems, however, the possibility exists to schedule the use of different sensors and actuators in order to use them in a more efficient way [13], [14], [15]. In this paper we depart from the existing literature by developing an actuator scheduling algorithm based on a novel characterization of the controllability Gramian. Our technical approach builds on [16], which develops geometric interpretations of various controllability notions for continuous-time and time-invariant systems. In constrast, here we focus on developing control nodes scheduling algorithms for discrete-time systems and time-varying input matrices.

Paper contribution. The contribution of this paper is twofold. First, we derive a geometric characterization of the controllability Gramian of linear dynamic networks with time-varying input matrix. In particular, we define a parallelepiped in the state space that is tangent to the ellipsoid defined by the controllability Gramian. This parallelepiped is an explicit function of the network eigenvalues and eigenvectors, as well as the choice of input matrix. Thus, our analysis ultimately provides a quantitative link between different controllability properties of a networks and its eigenstructure. Second, we adopt the worst-case control energy as performance criteria, and study the problem of selecting an optimal set of control nodes over time. We propose a heuristic algorithm based on our geometric characterization of the controllability Gramian, and we validate its effectiveness via simulations. Our numerical studies show that, for clustered networks, our time-varying selection of control nodes provides a higher controllability degree with respect to any constant selection, and that our heuristic algorithm outperforms existing selection methods based on the selection of a subset of columns from the controllability matrix. For space reasons, proof are omitted and will appear elsewhere.

Notation. For a vector $x \in \mathbb{R}^n$, x_i denotes its i-th element. Given a sequence $\{x(k)\}_{k=0}^{\infty}$ and $j_1 \leq j_2 \in \mathbb{Z}_{\geq 0}$, we use $\{x\}_{j_1}^{j_2}$ to denote $\{x(j_1), x(j_1+1), \dots, x(j_2)\}$. We also use $\|x(k)\| = \sqrt{x^T(k)x(k)}$ and $\|\{x\}_{j_1}^{j_2}\| = \sqrt{\sum_{k=j_1}^{j_2} \|x(k)\|^2}$. Given $v_1, \dots, v_m \in \mathbb{R}^n$, we let $\mathrm{span}(v_1, \dots, v_m)$ denote the vector space in $\mathbb{R}^n$ generated by them. For $M \in \mathbb{R}^{n \times m}$, we use $\mathrm{row}_i(M)$ to denote its i-th row, and $M_i = \mathrm{col}_i(M)$ its i-th column. $\{e_i\}_{i=1}^m$ represents the canonical basis of $\mathbb{R}^m$. Given $S = \{i_1, i_2, \dots\} \subseteq \{1, \dots, m\}$, we let

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$M_S = [M_{i_1}\, M_{i_2}\, \ldots]$. Moreover, we use $M_i^T = (M^T)_i$ to denote the i-th column of M^T, while $(M_i)^T$ is a row vector obtained by transposing the i-th column of M. We denote the singular values of M in decreasing order by $\sigma_{\max}(M) = \sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_{\min(m,n)}(M) = \sigma_{\min}(M) \geq 0$ and its spectral norm by $\|M\| = \sigma_{\max}(M)$. For a square matrix $M \in \mathbb{R}^{n\times n}$, we denote its eigenvalues by $\lambda_i(M)$ and the corresponding (right) eigenvectors by $V_{M,i}$. For symmetric (square) matrices M, we use $M > 0$ (resp. $M \geq 0$) to denote that M is positive definite (resp. M is positive semidefinite). We let $\mathbf{0}_n$ and $\mathbf{0}_{m\times n}$ denote the n-vector and $m \times n$ matrix with all elements equal to zero, respectively. We let I_n denote the identity matrix of dimension $n \times n$. We let $\mathrm{diag}(M_1, \ldots, M_n)$ denote the block-diagonal matrix defined by the matrices $M_1, \ldots, M_n$.

II. Problem formulation

We consider discrete-time network dynamics with state-space representation

$$x(k+1) = Ax(k) + B(k)u(k), \tag{1}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time index, $x(k) \in \mathbb{R}^n$ is the network state (with $x_i(k)$ representing the state of node $i \in \{1, \ldots, n\}$), and $u(k) \in \mathbb{R}^m$ is the control input (with $u_i(k)$ acting on the network through the input vector $B_i(k) \in \mathbb{R}^n$). Here, $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix of the network, characterizing the interactions among neighboring nodes, and $B(k) \in \mathbb{R}^{n\times m}$ is a time-dependent input matrix. We consider the scenario where there is a finite set of input vectors (also called library) available for selection at each time step, given by the matrix $\mathcal{B} \in \mathbb{R}^{n\times \bar{m}}$, with $\bar{m} \geq m$. The selection function $\mathcal{S}$ maps the time step k to a subset $\mathcal{S}(k) = \mathcal{S}_k = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, \bar{m}\}$ of cardinality m, corresponding to the input vectors selected at time k. In other words, $B(k) = \mathcal{B}_{\mathcal{S}_k} \in \mathbb{R}^{n\times m}$. For instance, if all nodes can be controlled, then $\mathcal{B} = I_n$ and $\bar{m} = n$. If only one node can be controlled at each time step, then $m = 1$ and $\mathcal{S}_k = \{i\} \subset \{1, \ldots, n\}$, which implies that $B(k)$ is a canonical unit vector in $\mathbb{R}^n$ for all $k \in \mathbb{Z}_{\geq 0}$. This formulation also captures the time-invariant case as a special case by simply requiring the function $\mathcal{S}$ to be constant.

Throughout the paper, we assume that A is symmetric. Given a time horizon $K \in \mathbb{Z}_{\geq}$, we are interested in the question of deciding which nodes should be controlled via external inputs at each time step so that the energy required to steer the network state from the origin to any target state is as small as possible. In what follows, we provide a formal statement of the problem of interest.

Definition 2.1: ***(Reachable states)*** For the network (1), the state $x_f \in \mathbb{R}^n$ is reachable at time $K \in \mathbb{Z}_{\geq 0}$ if there exists $\{u\}_0^{K-1}$ such that $x(0) = \mathbf{0}_n$ and $x(K) = x_f$. □

Using (1), one can write the network state as a function of the input, i.e.,

$$x(K) = \sum_{k=0}^{K-1} A^{K-k-1}B(k)u(k). \tag{2}$$

From (2) it follows that the state x_f is reachable at time $K \in \mathbb{Z}_{\geq 0}$ if and only if $x_f \in \mathrm{range}(\Phi_K)$, where

$$\Phi_K \triangleq \begin{bmatrix} B(K-1) & AB(K-2) & \ldots & A^{K-1}B(0) \end{bmatrix}$$

is the controllability matrix at time K. The network is reachable at time K if all states $x_f \in \mathbb{R}^n$ are reachable at time K, and this holds if and only if $\mathrm{rank}(\Phi_K) = n$. By the Cayley-Hamilton Theorem [17], a network is reachable at time $K \geq n$ if and only if $\mathrm{rank}(\Phi_n) = n$. We focus on networks that are reachable, i.e., there exists at least one input matrix sequence $\{B(k)\}_0^{n-1}$ such that $\mathrm{rank}(\Phi_n) = n$.

The notion of reachable state in Definition 2.1 does not take into account the input energy required to drive the network to the desired state. A theoretically feasible input sequence may require arbitrarily large energy, which in practice could make the target state not reachable. This observation justifies our focus on the states reachable using input sequences with no more than unit energy, which we refer to as *unit-energy inputs* for simplicity.

Definition 2.2: ***(Reachable set using unit-energy inputs)*** Given a sequence of input matrices $\{B(k)\}_0^{K-1}$ that ensures reachability, the set of states that are reachable at time K with unit-energy inputs is

$$\mathcal{R}(K) = \{x_f \in \mathbb{R}^n \,|\, x_f = \sum_{k=0}^{K-1} A^{K-k-1}B(k)u(k), \\ \|\{u\}_0^{K-1}\| \leq 1\}. \tag{3}$$

□

Note that $\mathcal{R}(K) \subset \mathcal{R}(K+1)$, which means that given more time, more states can be reached using the same, or less, input energy. It can be shown that the minimum-energy input that steers the state from the origin to x_f in time K is

$$u^*(k) = B^{^T}(k)(A^{^T})^{K-k-1}\mathcal{W}_K^{-1}x_f,\ k \in \{0, 1, \ldots, K-1\},$$

where

$$\mathcal{W}_K(A, \{B(k)\}_{k=0}^{K-1}) = \Phi_K\Phi_K^T \tag{4}$$

is the K-step controllability Gramian. Note that the energy of this input is $\|\{u^*\}_0^{K-1}\|^2 = x_f^T\mathcal{W}_K^{-1}(A, \{B(k)\}_{k=0}^{K-1})x_f$. For convenience, we often refer to the Gramian simply by $\mathcal{W}_K$, dropping the notational dependence on the network or the sequence of input matrices. Given a network adjacency matrix A and a library of input vectors $\mathcal{B}$, the Gramian $\mathcal{W}_K$ is only a function of the selection function $\mathcal{S}$. The reachable set using unit-energy inputs can be written as

$$\mathcal{R}(K) = \{x_f \in \mathbb{R}^n \,|\, x_f^T\mathcal{W}_K^{-1}x_f \leq 1\}. \tag{5}$$

This expression clearly shows that the set $\mathcal{R}(K)$ is a hyperellipsoid in $\mathbb{R}^n$. The shape of $\mathcal{R}(K)$ characterizes quantitatively the difficulty of steering the network state along different directions. Specifically, each axis of $\mathcal{R}(K)$ is associated to an eigenvalue $\lambda_i(\mathcal{W}_K)$ of the Gramian, has length equal to $\sqrt{\lambda_i(\mathcal{W}_K)}$, and direction given by the corresponding eigenvector. We are finally ready to formally state the objective of this paper.

Problem 2.3: ***(Improving network controllability by controlling a time-varying set of nodes)*** Given a library of input vectors $\mathcal{B} \in \mathbb{R}^{n\times \bar{m}}$, the network dynamics (1), and the time horizon $K \in \mathbb{Z}_{\geq 0}$, find a time-varying selection function $\mathcal{S}$ that maximizes $\lambda_{\min}(\mathcal{W}_K(\{\mathcal{S}_k\}_{k=0}^{K-1}))$. □

Roughly speaking, we would like the ellipsoid $\mathcal{R}(K)$ to be as close to a sphere as possible. It is worth pointing out that even if we restrict our attention to the case where the selection function is time-invariant, the optimization in Problem 2.3 is known to be NP-hard, e.g., see [5], [18]. Thus, we focus on understanding if and how a time-varying input matrix can lower the energy required to steer the network state, and on developing efficient approximate input selection algorithms.

Remark 2.4: ***(Gramian-based reachability metrics)*** Recent work has explored various controllability/reachability metrics based on the spectrum of the controllability Gramian in the context of complex networks [18], [19], [20]. The determinant $\det(\mathcal{W}_K)$ reflects the volume of $\mathcal{R}(K)$, the minimum eigenvalue $\lambda_{\min}(\mathcal{W}_K)$ characterizes the minimum input energy required in the worst case to reach a state on the unit hypersphere in the state space $\mathbb{R}^n$ in time K (while the corresponding eigenvector characterizes the direction that is most difficult to move the state towards), and the trace $\mathrm{tr}(\mathcal{W}_K)$ determines the average minimum input energy required to reach all states on the unit hypersphere in the state space. Roughly speaking, for ease of controllability, one would like all eigenvalues of $\mathcal{W}_K$ to be as large as possible. □

In [19] it is shown that symmetric networks are difficult to control, in the sense that the control energy grows exponentially when the set of inputs remains constant. To conclude this section, in the next result we show that this is also the case when the set of inputs is allowed to be time-varying.

Theorem 2.5: ***(Symmetric networks remain difficult to control)*** Consider a network defined by (1) and assume that A is symmetric. Then

$$\lambda_{\min}(\mathcal{W}_K(\{\mathcal{S}_k\}_{k=0}^{K-1})) \leq \frac{\lambda_{\max}^{2\left(\left\lceil \frac{n}{m} \right\rceil -1\right)}(A)}{1-\lambda_{\max}^2(A)}, \tag{6}$$

where $|\mathcal{S}_k| = m$ for $k = 0, \ldots, K-1$.

III. Geometry of the hyperellipsoid defined by the controllability Gramian

In this section, we establish a connection between the hyperellipsoid defined by the controllability Gramian and the network eigenstructure. Inspired by [16], we find n tangent hyperplanes to $\mathcal{R}(K)$ that are expressed in terms of the eigenvectors of the network adjacency matrix A. This result is the basis for our control node selection algorithm.

Since, by assumption, A is diagonalizable, its normalized eigenvectors V_i form a basis of $\mathbb{R}^n$. Therefore, for all $k \in \mathbb{Z}_{\geq 0}$ there exists $C(k) \in \mathbb{R}^{n\times m}$ such that

$$B(k) = VC(k) = \sum_{i=1}^{n} V_i \,\mathrm{row}_i(C(k)), \tag{7}$$

where $\mathrm{row}_i(C(k)) \in \mathbb{R}^{1\times m}$ is the i-th row of $C(k)$. By substituting (7) into (2) we obtain

$$\begin{aligned} x(K) &= \sum_{i=1}^{n}\sum_{k=0}^{K-1} A^{K-k-1} V_i \,\mathrm{row}_i(C(k))u(k) \\ &= \sum_{i=1}^{n} V_i \sum_{k=0}^{K-1} \lambda_i^{K-k-1} \,\mathrm{row}_i(C(k))u(k) \\ &= \sum_{i=1}^{n} V_i \eta_i, \end{aligned} \tag{8}$$

where $\lambda_i = \lambda_i(A)$, and

$$\eta_i \triangleq \sum_{k=0}^{K-1} \lambda_i^{K-k-1} \,\mathrm{row}_i(C(k))u(k) \in \mathbb{R} \tag{9}$$

represents the i-th coordinate of $x(K)$ for the basis formed by the vectors V_i. Using the Cauchy–Schwarz inequality [17],

$$\eta_i \leq \eta_i^* \|\{u\}_0^{K-1}\| \leq \eta_i^*,$$

where

$$\eta_i^* \triangleq \sqrt{\sum_{k=0}^{K-1} \lambda_i^{2(K-k-1)} \|\,\mathrm{row}_i(C(k))\|^2}, \tag{10}$$

and, for the last inequality, $\|\{u\}_0^{K-1}\| \leq 1$. By inspection, the equality $\eta_i = \eta_i^*$ is achieved if and only if

$$u_i(k) = u_i^*(k) = (\eta_i^*)^{-1} \lambda_i^{K-k-1} \,\mathrm{row}_i(C(k))^T. \tag{11}$$

One can see from equations (8)-(11) that η_i^* is the maximal distance that one can drive the network state along the direction of the network eigenvector V_i. Therefore, the vector η^* characterizes the network controllability, both qualitatively and quantitatively. For instance, as shown in [16] for the case of a constant input matrix, if $\eta_i^* = 0$ for some $i \in \{1, \ldots, n\}$, then $\mathcal{R}(K)$ loses a dimension and the network is uncontrollable.

We continue our analysis by drawing a connection between the network controllability degree and the geometric interpretation of the controllability Gramian.

Theorem 3.1: ***(Relationship between the Gramian and the network eigenstructure)*** For each $i \in \{1, \ldots, n\}$, define the hyperplanes $\mathcal{H}_i^+, \mathcal{H}_i^-, \subseteq \mathbb{R}^n$ by

$$\begin{aligned} \mathcal{H}_i^+ &= \eta_i^* V_i + \mathrm{span}(V_1, \ldots, \overset{i}{\smile}, \ldots, V_n), \\ \mathcal{H}_i^- &= -\eta_i^* V_i + \mathrm{span}(V_1, \ldots, \overset{i}{\smile}, \ldots, V_n), \end{aligned}$$

where the symbol $\overset{i}{\smile}$ denotes that the i-th vector is removed from the list. The $2n$ hyperplanes $\mathcal{H}_1^+, \ldots, \mathcal{H}_n^+, \mathcal{H}_1^-, \ldots, \mathcal{H}_n^-$, define a parallelepiped tangent to $\mathcal{R}(K)$.

The counterpart of Theorem 3.1 for continuous-time linear time-invariant systems is stated in [16] without proof. Theorem 3.1 establishes a relation between the network controllability Gramian and the eigenstructure of the network. From (10), one can see that reducing the magnitude

Algorithm 1: *Selection of a constant control set*

Input : Number of control nodes m, network matrix A, control horizon K, input library $\mathcal{B}$;
Output : Input matrix with m columns;

for *all subsets* $\{i_1, \ldots, i_m\}$ *of* $\{1, \ldots, \bar{m}\}$ **do**
  Let Choose $B(k) = \mathcal{B}_{i_1,\ldots,i_m}$ for all $k \in \{0, \ldots, K-1\}$;
  Compute $\mathcal{W}_K(i_1, \ldots, i_m) = \mathcal{W}_K(A, \{B(k)\}_{k=0}^{K-1})$ using (4);
return $\{i_1^*, \ldots, i_m^*\} = \arg\max \lambda_n(\mathcal{W}_K(i_1, \ldots, i_m))$;

of λ_i, while keeping the other eigenvalues and all the eigenvectors constant, reduces η_i^* and the network becomes more difficult to control. Moreover, given that the axes of $\mathcal{R}(K)$ are associated to the eigenvalues of the Gramian, if $\min_{i\in\{1,\ldots,n\}} \eta_i^*$ is small, then $\lambda_{\min}(\mathcal{W}_K)$ is also small.

Example 3.2: ***(Two-node network with weak interaction)*** This example illustrates how the network eigenstructure defines a tangent parallelepiped of the ellipsoid containing all reachable states using unit-energy inputs. Consider a network with two nodes described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.02 \\ 0.05 & 0.75 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$

where $u(k) \in \mathbb{R}$, and the control horizon is $K = 4$. The ellipsoid defined by $\{x \in \mathbb{R}^2 \,|\, x^T \mathcal{W}_K^{-1} x \leq 1\}$ is plotted in Figure 2 in solid blue. The eigenvectors of A are $\begin{bmatrix} -0.9812 & 0.1932 \end{bmatrix}^T$ with $\eta_1^* = 1.1538$, $\begin{bmatrix} -0.0785 & -0.9969 \end{bmatrix}^T$ with $\eta_2^* = 0.2802$ and they characterize a parallelepiped tangent to the ellipsoid, as plotted in dotted blue in Figure 1. □

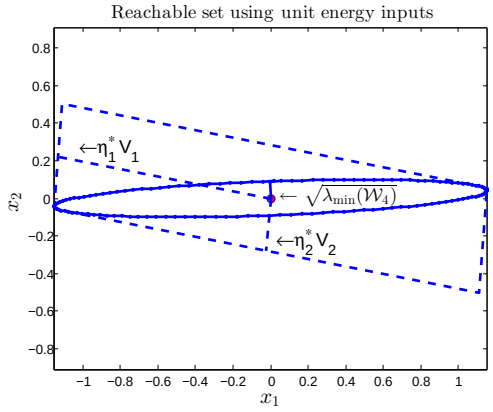

Fig. 1. Hyperellipsoid $\mathcal{R}(4)$ containing all reachable states in 4 timesteps using unit-energy inputs for a 2-node network, cf. Example 3.2. Here $\lambda_{\min}(\mathcal{W}_4) = 0.0082$.

IV. Control node selection algorithms

Here we present two greedy algorithms to select the input matrix over time, so as to provide an approximate solution to Problem 2.3. As reference benchmark, we later compare their performance against the optimal choice over all time-invariant selection functions (corresponding to controlling a constant set of nodes at every step), which we find using exhaustive search; see Algorithm 1.

Algorithm 2: *Control selection based on eigenstructure*

Input : Number of control nodes m, network matrix A, control horizon K, input library $\mathcal{B}$;
Output : Sequence of K input matrices with m columns;

Compute eigenvalues λ_i and normalized eigenvectors V_i of A;
Decompose $\mathcal{B}$ as $\mathcal{B} = V\mathcal{C} = \sum_{i=1}^n V_i \operatorname{row}_i(\mathcal{C})$;
for $j = 0 : K-1$ **do**
  $\mathcal{S}_j^* = \arg\max \min_i \sum_{k=0}^j \lambda_i^{2(K-k-1)} \|\operatorname{row}_i(C(k))\|^2$,
  where $\mathcal{S}_j \subseteq \{1, \ldots, \bar{m}\}, |\mathcal{S}_j| = m$, and $\{\mathcal{S}\}_0^{j-1} = \{\mathcal{S}^*\}_0^{j-1}$;
return $\{\mathcal{S}^*\}_0^{K-1}$;

A. Greedy node selection motivated by the geometric representation of the controllability Gramian

Theorem 3.1 shows that, when $\min_i \eta_i^*$ is small, the smallest Gramian eigenvalue $\lambda_{\min}(\mathcal{W}_K)$ is also small. Thus, a heuristic procedure to select the input matrix is based on the maximization of $\min_i \eta_i^*$. From a geometric viewpoint, instead of making $\mathcal{R}(K)$ close to a hypersphere, our heuristic procedure attempts to make its tangent parallelepiped close to a cube. Our greedy procedure is in Algorithm 2. Note that, at each time k, the algorithm solves

$$\max_{\mathcal{S}_k, \{\mathcal{S}_i = \mathcal{S}_i^*\}_{i=0}^{k-1}, \{\mathcal{S}_i = \varnothing\}_{i=k+1}^{K-1}} \min_i \eta_i^*(\{\mathcal{S}_k\}_{k=0}^{K-1}),$$

where $\mathcal{S}_k^* = \arg\max_{\mathcal{S}_k, \{\mathcal{S}_i = \mathcal{S}_i^*\}_{i=0}^{k-1}, \{\mathcal{S}_i = \varnothing\}_{i=k+1}^{K-1}} \min_i \eta_i^*$ and $\mathcal{B}_\varnothing = \mathbf{0}_n$. That is, at every timestep k, we maximize $\min_i \eta_i^*(\mathcal{S}_k)$ by choosing exhaustively a set of m control nodes, denoted by $\mathcal{S}_k$, out of $\bar{m}$ candidates.

B. Greedy node selection via column subset selection

The problem of selecting a time-varying set of control nodes can be reformulated as a special case of selecting a subset of columns from the controllability matrix obtained by considering all possible input channels. In particular, columns (and hence input matrices) should be selected so that the reduced matrix is spectrally similar to the original full size matrix. The problem of column subset selection has been studied extensively due to its importance in data analysis [21], [22], [23], [24]. For our problem, consider the controllability matrix candidate

$$\Phi_K = \begin{bmatrix} \mathcal{B} & A\mathcal{B} & \ldots & A^{K-1}\mathcal{B} \end{bmatrix} \in \mathbb{R}^{n \times \bar{m}K}, \qquad (12)$$

the objective is to find a selection function $\{\mathcal{S}_k\}_{k=0}^{K-1}$, such that the minimum singular value of the submatrix

$$\Phi_{K,\{\mathcal{S}_k\}_{k=0}^{K-1}} = \begin{bmatrix} \mathcal{B}_{\mathcal{S}_{K-1}} & A\mathcal{B}_{\mathcal{S}_{K-2}} & \ldots & A^{K-1}\mathcal{B}_{\mathcal{S}_0} \end{bmatrix}$$

is maximized (recall that $\sigma_{\min}^2(\Phi_{K,\{\mathcal{S}_k\}_{k=0}^{K-1}}) = \lambda_{\min}(\mathcal{W}_K(\{\mathcal{S}_k\}_{k=0}^{K-1}))$). Note that this column subset selection problem is a special type of the general one, because in each $n \times \bar{m}$ block, exactly m columns must be selected.

The general column subset selection problem is known to be NP-hard [24]. The greedy algorithm in [21] removes one column (denoted by $\phi_i \in \mathbb{R}^n$) from Φ_K at every step $i = 1, \ldots, \bar{m}K - mK$ while minimizing

$$\operatorname{tr}[(\Phi_{\mathcal{S}_{i-1}} \Phi_{\mathcal{S}_{i-1}}^T - \phi_i \phi_i^T)^{-1}] = \operatorname{tr}[(\Phi_{\mathcal{S}_i} \Phi_{\mathcal{S}_i}^T)^{-1}]$$

Algorithm 3: *Control selection based on column removal*

Input : Number of control nodes m, network matrix A, control horizon K, input library $\mathcal{B}$;
Output : Sequence of K input matrices with m columns;

1 Compute $\Phi = \begin{bmatrix} \mathcal{B} & A\mathcal{B} & \dots & A^{K-1}\mathcal{B} \end{bmatrix}$;
2 $\mathcal{S}_0 = \{1, \dots, \bar{m}K\}$;
3 Initialize counter $= \begin{bmatrix} \bar{m} & \dots & \bar{m} \end{bmatrix}$
for $j = 0 : (\bar{m} - m)K - 1$ **do**
4 Compute singular value decomposition of $\Phi_{\mathcal{S}_i} = U(i)\Sigma(i)Y(i)$;
5 $j_i = \arg\min_{r \in \mathcal{S}_i, \|Y_r(i)\|_2 < 1} \frac{\sum_{l=1}^{n} Y_{rl}(i)\sigma_l^{-1}(\Phi_{\mathcal{S}_i})}{1 - \|Y_r(i)\|_2^2}$, where $\text{counter}_{\lceil r/n \rceil}(i) \geq m$.
6 $\mathcal{S}_{i+1} = \mathcal{S}_i \setminus \{j_i\}$;
7 $\text{counter}_{\lceil j_i/n \rceil} = \text{counter}_{\lceil j_i/n \rceil} - 1$;
8 **return** Extract input matrices from $\mathcal{S}_{\bar{m}K - mK}$;

$$= \text{tr}[\Phi_{\mathcal{S}_i}^{\dagger}(\Phi_{\mathcal{S}_i}^{\dagger})^T] = \|\Phi_{\mathcal{S}_i}^{\dagger}\|_F,$$

under the assumption that the matrix never loses rank. Here $\mathcal{S}_i$ indicates the remaining $\bar{m}K - i$ columns at step i, $\mathcal{S}_0 = \{1, \dots, \bar{m}K\}$, $\Phi_{\mathcal{S}_i}^{\dagger} \triangleq \Phi_{\mathcal{S}_i}^T(\Phi_{\mathcal{S}_i}\Phi_{\mathcal{S}_i}^T)^{-1}$ is the Moore–Penrose generalized inverse of $\Phi_{\mathcal{S}_i}$. Because of the relation $\|\Phi_{\mathcal{S}_i}^{\dagger}\|_F \geq \sigma_1(\Phi_{\mathcal{S}_i}^{\dagger}) = \sigma_n^{-1}(\Phi_{\mathcal{S}_i})$, minimizing $\|\Phi_{\mathcal{S}_i}^{\dagger}\|_F$ maximizes a lower bound of $\sigma_n(\Phi_{\mathcal{S}_i})$. [24] generalizes the algorithm in [21] by intentionally ensuring that the column removed at the i-th step does not render $\Phi_{\mathcal{S}_i}$ rank deficient. To apply the algorithm in [24] to our problem, we have to add a constraint that exactly m columns need to be chosen from each block $A^k\mathcal{B}$, $k = 0, \dots, K-1$, which is ensured by the vector 'counter' in the revised algorithm given in Algorithm 3. The reader is referred to [24] for more details and discussions, including the rationale behind Steps $4 - 7$.

V. Simulations

In this section, we illustrate the performance of the control node selection algorithms proposed in Section IV in two examples, and compare it against the optimal time-invariant selection. In terms of computational complexity, one can see from their definitions that Algorithm 2 is more efficient than Algorithm 3.

Example 5.1: ***(Two-node network with weak interaction – cont'd)*** This example illustrates how Algorithm 2 can improve network controllability by making the ellipsoid containing all reachable states using unit-energy inputs closer to a sphere, based on our result in Theorem 3.1. Consider the network in Example 3.2 with time-varying input matrix:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.02 \\ 0.05 & 0.75 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + B(k)u(k),$$

where $B(k) \in \mathcal{B} = \{\begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T\}$, that is, $m = 1, \bar{m} = 2$, $u(k) \in \mathbb{R}$, and the control horizon is $K = 4$. Notice that the coupling between the two nodes is weak, so immediately one can see that it should be difficult to steer the state of the whole network by controlling any node alone. The selection function found by Algorithm 1 finds the time-invariant selection function $\mathcal{S} \equiv \{1\}$. This means that one achieves larger $\lambda_{\min}(\mathcal{W}_K)$ by controlling node 1 than controlling node 2. This makes intuitive sense, because node 1 has a larger impact on node 2 ($0.05 > 0.02$) and node 2 has smaller damping ($0.75 > 0.5$). The ellipsoid and corresponding tangent parallelepiped has been plotted in Figure 1. On the other hand, Algorithm 2 finds the time-varying selection function $\{\mathcal{S}_k\}_{k=0}^3 = \{\{1\}, \{1\}, \{2\}, \{1\}\}$, which results in $\eta_1^* = 1.0416$ and $\eta_2^* = 0.7821$. This selection makes the parallelepiped closer to a square and the ellipsoid closer to a circle, as shown in dashed and solid red, respectively, in Figure 2. As can be observed, the minimum eigenvalue of the controllability Gramian is increased by about 70 times through controlling different nodes at different time steps.□

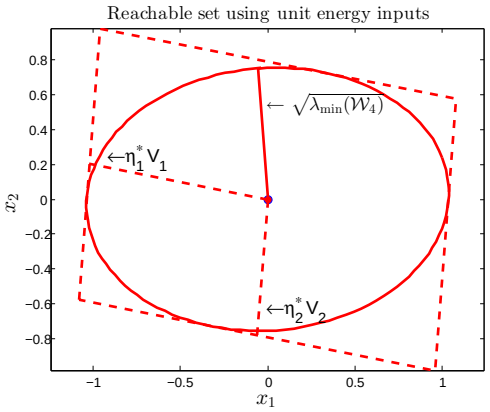

Fig. 2. Hyperellipsoid $\mathcal{R}(4)$ containing all reachable states in 4 timesteps using unit-energy inputs for a 2-node network with a time-varying set of control nodes, cf. Example 5.1. Here $\lambda_{\min}(\mathcal{W}_4) = 0.5671$.

Example 5.2: ***(Clustered random network)*** This example shows how Algorithm 2 can increase the minimum eigenvalue of the controllability Gramian of clustered networks. We consider a group of 10 random networks that are weakly interconnected. The network with index p has dynamics matrix $A_p \in \mathbb{R}^{2p \times 2p}$ and consists of p subnetworks, $p \in \{1, \dots, 10\}$. Each subnetwork has two nodes and a random dynamics matrix $A_{p,ii} \in \mathbb{R}^{2\times 2}$ with $[A_{p,ii}]_{11}$, $[A_{p,ii}]_{22}$ having a uniform distribution in $\{x \in \mathbb{R} \,|\, 0.35 < x < 0.6\}$ and $[A_{p,ii}]_{12}$, $[A_{p,ii}]_{21}$ having a uniform distribution in $\{x \in \mathbb{R} \,|\, 0 < x < 0.25\}$. Here $A_{p,ii}$'s denote the matrix blocks on the diagonal of A_p. The entries in the off-diagonal blocks of A_p have a uniform distribution in $\{x \in \mathbb{R} \,|\, 0 < x < 0.005\}$. For each network, the library $\mathcal{B}_p$ contains all the canonical vectors in the state space ($\bar{m}_p = 2p$) and $m = 1$. Figure 3 compares the performances of Algorithms 1-3. One can see that we can increase the minimum eigenvalue of the controllability Gramian dramatically by controlling a time-varying set of nodes and Algorithm 2 achieves the best performance. Figure 4 shows the control nodes selected at each time step for the network with 20 nodes ($p = 10$). □

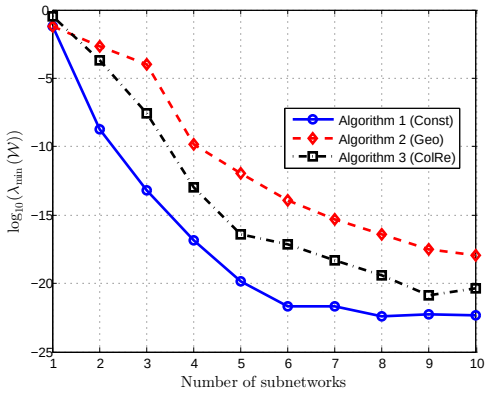

Fig. 3. Reachability metric evaluated at the input selections obtained from Algorithms 1-3 in the random network of Example 5.2.

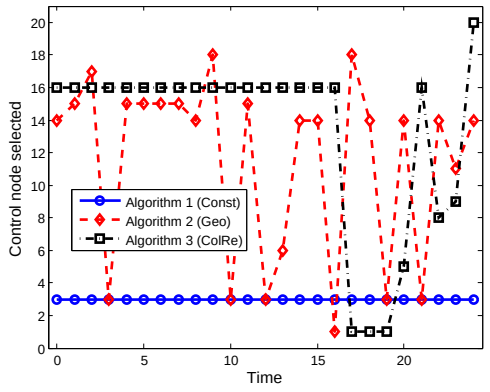

Fig. 4. Control nodes selected by Algorithms 1-3 for a clustered random network composed of 10 subnetworks with 2 nodes each, cf. Example 5.2.

VI. Conclusions

We have derived a geometric characterization of the controllability Gramian of linear discrete-time networks with time-varying input matrices. Our analysis provides a detailed link between the eigenstructure of the network and its controllability properties. by showing how the former determines a parallelepiped tangent to the ellipsoid representing the reachable states using unit-energy inputs. We have exploited this geometric characterization of the Gramian to design a scheduling algorithm for the selection of optimal control nodes over time. We have showed that, for clustered networks, our procedure outperforms any constant selection of control nodes, and existing selection algorithms based on column subset selection. Future work will include characterizing the gap between our current algorithm and the optimal selection of control nodes in complex networks, either for the worst-case target state or a specified target state, and identifying network structures for which the scheduling of control nodes can provide a significant improvement in their reachability properties.

Acknowledgments

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