Decentralized Nash equilibrium seeking by strategic generators for DC optimal power flow

Ashish Cherukuri and Jorge Cortés
Department of Mechanical and Aerospace Engineering
University of California, San Diego
{acheruku,cortes}@ucsd.edu

***Abstract*—This paper studies an electricity market consisting of an independent system operator (ISO) and a group of generators. The goal is to solve the DC optimal power flow (DC-OPF) problem: have the generators collectively meet the power demand while minimizing the aggregate generation cost and respecting line flow limits. The ISO by itself cannot solve the DC-OPF problem as the generators are strategic and do not share their cost functions. Instead, each generator submits to the ISO a bid, consisting of the price per unit of electricity at which it is willing to provide power. Based on the bids, the ISO decides how much production to allocate to each generator to minimize the total payment while meeting the load and satisfying the line limits. We provide a provably correct, decentralized iterative scheme, termed BID ADJUSTMENT ALGORITHM for the resulting Bertrand competition game. The algorithm takes the generators bids to any desired neighborhood of the efficient Nash equilibrium at a linear convergence rate. As a consequence, the optimal production of the generators converges to the optimizer of the DC-OPF problem. Our algorithm can be understood as "learning via repeated play", where generators are "myopically selfish", changing their bid at each iteration with the sole aim of maximizing their payoff.**

I. Introduction

The future power grid will have numerous different types of distributed energy resources (DERs). Robust DER integration into the grid calls for the design of architectures defining the interaction between DERs and the independent system operator (ISO) so that power generation can be planned in a tractable way. At the core of generation planning is the optimal power flow (OPF) problem, where a group of generators need to decide upon their production level so that loads can be met in a cost effective manner while network constraints are satisfied. The need to solve the OPF problem may arise at different layers of the grid architecture, from large-capacity generators competing to meet a demand dispatch event to a group of small-capacity DERs coordinating their response to meet an assigned load. In this paper, we are interested in the competition version of this problem: we study policies that individual generators, in conjunction with the ISO, can implement to solve the OPF problem while acting in a selfish and rational fashion.

Literature review: Competition in electricity markets is a classical topic of study [1], [2]. Extensively studied models are supply function, Bertrand (price) and Cournot (capacity) bidding, see [3], [4], [5], respectively, and references therein. Most of these studies focus on the properties of the game that different bidding models lead to. In particular, they analyze the existence of the Nash equilibrium of the game and estimate its efficiency. Some works [6], [7], [8], [9], on the other hand, propose iterative algorithms for the players that compute the Nash equilibrium of the game. The factors differentiating these setups include pricing mechanisms, bidding functions, nature of the demand, and the consideration of power flow constraints. However, for these algorithms to work, either the generators need to have some information about other generators (cost functions or bids) or the demand of each generator should be a continuous function of the bids. We do not make any such assumptions in this work, which in turn, also rules out the possibility of using various other Nash equilibrium learning algorithms, such as best-response [10], fictitious play [11], and extremum seeking [12]. Our electricity market game belongs to the class of multi-leader-single-follower games [13], [14]. Equilibria of such games can be thought of as optimizers of mathematical programs with equilibrium constraints (MPEC) [15], [16] that are traditionally solved in a centralized manner [17]. The work [18] presents a distributed method to find the equilibria of an MPEC problem but requires the follower's (the ISO in our case) optimization to have a unique solution for each action of the leaders (the generators). This is in general not the case for electricity markets. In a related set of works [19], [20], decentralized generation planning is achieved by assuming the generators to be price-takers and designing iterative schemes based on dual-decomposition [21]. In our work, however, we consider a strategic scenario where generators bid into the market and are hence price-setters. The work [22] proposes an iterative auction algorithm for a market where both generators and consumers are strategic but does not provide convergence guarantees for the generated bid sequences. The paper [23], closer in spirit to our work, proposes an iterative method for the generators to find the Nash equilibrium assuming they don't have any information about each other. At each iteration, the generators send to the ISO the gradient of their cost functions at a certain generation value and the ISO then adjusts these generation values so that social welfare is maximized. An important difference between this setup and ours is that we do not assume truthful bidding of gradient information by the generators. Finally, our work has close connections with the growing interest in the design of provably correct distributed algorithms for the cooperative solution of the ED problem, see [24], [25] and references therein.

Statement of contributions: We start with the formulation of the inelastic electricity market game. In this setup, the ISO aims to determine the production levels for the generators by solving a DC version of the optimal power flow (DC-OPF) problem. Since generators are strategic, they do not share their cost functions and the ISO cannot solve the DC-OPF problem by itself. Each generator, however, submits a bid to the ISO, which specifies the price per unit of electricity at which the generator is willing to provide power. Based on the bids, the ISO decides how much production to allocate to each generator so that cost of generation is minimized, the loads are met, and the network flow constraints are satisfied. The resulting Bertrand competition model defines the game among the generators, where the actions are the bids and the payoffs are the profits. Our first contribution gives two set of conditions that ensure existence and uniqueness, respectively, of an efficient Nash equilibrium for the inelastic electricity market game. Our second and main contribution consists on the design of the BID ADJUSTMENT ALGORITHM along with its correctness analysis. This algorithm can be understood as "learning via repeated play", where at each iteration, generators act rationally and selfishly, trying to maximize their own profit. Along the execution, the only information available to the generators is their bid and the amount of generation that the ISO request from them. In particular, generators are not aware of the number of other generators, their costs, bids, or payoffs. We show that this decentralized iterative scheme is guaranteed to take the bids of the generators to any neighborhood of the unique efficient Nash equilibrium provided the stepsizes are chosen appropriately. Further, we establish that the convergence rate is linear. Simulations illustrate our results. For reasons of space, proofs are omitted and will appear elsewhere.

II. PRELIMINARIES

This section introduces the notation and basic concepts of graph theory used throughout the paper.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 1}$ be the set of real, nonnegative real, and positive integer numbers, respectively. We use the shorthand notation $[n]$ to denote the set $\{1, \dots, n\}$. The 2-norm on $\mathbb{R}^n$ is represented by $\|\cdot\|$. Let $\mathcal{B}_\delta(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \delta\}$ be the open ball centered at $x \in \mathbb{R}^n$ with radius $\delta > 0$. Given $x, y \in \mathbb{R}^n$, x_i is the i-th component of x, and $x \leq y$ denotes $x_i \leq y_i$ for $i \in [n]$. We use $\mathbf{0}_N = (0, \dots, 0) \in \mathbb{R}^N$. We let $[u]^+ = \max\{0, u\}$ for $u \in \mathbb{R}$.

Graph theory: We present notions from algebraic graph theory following [26]. A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. A path is an ordered sequence such that any ordered pair of vertices appearing consecutively is an edge. A digraph is *strongly connected* if there is a path between any pair of distinct vertices. For a digraph, $\mathcal{N}^+_{v_i} = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$ and $\mathcal{N}^-_{v_i} = \{v_j \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E}\}$ are the sets of out- and in-neighbors of v_i, respectively.

III. PROBLEM STATEMENT

Consider an electrical power network with $N_b \in \mathbb{Z}_{\geq 1}$ buses. The physical interconnection between the buses is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes correspond to buses and edges to physical power lines. The direction for each edge represents the convention of positive power flow. The power flow on the line $(i, j) \in \mathcal{E}$ is $z_{ij} \in \mathbb{R}$. Each power line $(i, j) \in \mathcal{E}$ has a limit on the power flowing through it (in either direction), represented by $\overline{z}_{ij} > 0$. Assume that each bus $i \in [N_b]$ is connected to $n_i \in \mathbb{Z}_{\geq 1}$ strategic generators and one load. We let $N = \sum_{i=1}^N n_i$ be the total number of generators and assign them a unique identity in $[N]$. Let the set of generators at node i be $G_i \subset [N]$. The power demand at bus i is denoted by $y_i \geq 0$ and is assumed to be fixed and known to the Independent System Operator (ISO) that acts as the central regulating authority. The total demand is denoted as $\overline{y} = \sum_{i=1}^{N_b} y_i$. The objective for the generators is to collectively meet this inelastic demand through a competitive bidding process in the electricity market. The cost $f_n(x_n)$ of generating $x_n \in \mathbb{R}_{\geq 0}$ amount of power by the n-th generator is given by a quadratic function

$$f_n(x) = a_n x^2 + c_n x, \tag{1}$$

where $a_n > 0$ and $c_n \geq 0$. Given a power allocation $x = (x_1, \dots, x_N) \in \mathbb{R}^N_{\geq 0}$ for the group of generators, the aggregate cost is $\sum_{n=1}^N f_n(x_n)$. The *dc optimal power flow problem* (DC-OPF) consists of

$$\begin{aligned}
\underset{(x,z)}{\text{minimize}} \quad & \sum_{n=1}^{N} f_n(x_n), && \text{(2a)}\\
\text{subject to} \quad & \sum_{j \in \mathcal{N}_i^+} z_{ij} - \sum_{j \in \mathcal{N}_i^-} z_{ij} = \sum_{n \in G_i} x_n - y_i, \ \forall i, && \text{(2b)}\\
& -\overline{z}_{ij} \leq z_{ij} \leq \overline{z}_{ij}, \ \forall (i,j), && \text{(2c)}\\
& x \geq \mathbf{0}_N. && \text{(2d)}
\end{aligned}$$

This problem finds the generation profile that meets the load at each bus (ensured by (2b)), respects the power line constraints (due to (2c)), and minimizes the total cost (represented by the objective function (2a)). We assume that (2) is feasible. Since the individual cost functions are quadratic, the optimizer of the problem, denoted (x^*, z^*), is unique [27].

The goal for the ISO is to solve (2). The ISO can interact with the generators, whereas each generator can only communicate with the ISO and is not aware of the number of other generators participating in the market and their respective cost functions, or the load at its own bus. While the ISO knows the loads and the limits on the power lines, it does not have any information about the cost functions of the generators. Thus, power allocation is decided following a bidding process, resulting into a game-theoretic formulation. Instead of sharing their cost with the ISO, the generators bid the price per unit of power that they are willing to provide the power at. This price-based bidding is well known in the economics literature as Bertrand competition [28, Chapter 12]. Specifically, generator n bids the cost per unit power $b_n \in \mathbb{R}_{\geq 0}$. When convenient, we denote the bids of all other generators except n by $b_{-n} = (b_1, \dots, b_{n-1}, b_{n+1}, \dots, b_N)$. Given the bids $b = (b_1, \dots, b_N) \in \mathbb{R}^N_{\geq 0}$, the ISO solves the following *strategic dc optimal power flow problem* (S-DC-OPF)

$$\underset{(x,z)}{\text{minimize}} \quad \sum_{n=1}^{N} b_n x_n, \tag{3a}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{N}_i^+} z_{ij} - \sum_{j \in \mathcal{N}_i^-} z_{ij} = \sum_{n \in G_i} x_n - y_i, \ \forall i, \tag{3b}$$

$$-\bar{z}_{ij} \leq z_{ij} \leq \bar{z}_{ij}, \ \forall (i,j), \tag{3c}$$

$$x \geq \mathbf{0}_N. \tag{3d}$$

The difference between (3) and (2) is the objective function which is linear in the former and nonlinear, convex in the latter. The ISO solves (3) once all the bids are gathered. Let $(x^{\text{opt}}(b), z^{\text{opt}}(b))$ be the optimizer of (3) that the ISO selects (note that there might not be a unique optimizer) given bids b. This determines the power requested from each generator, given by the vector $x^{\text{opt}}(b)$. Knowing this process, the objective of each generator n is to bid a quantity $b_n \geq 0$ that maximizes its payoff $u_n : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$,

$$u_n(b_n, x_n^{\text{opt}}(b)) = b_n x_n^{\text{opt}}(b) - f_n(x_n^{\text{opt}}(b)), \tag{4}$$

where $x_n^{\text{opt}}(b)$ is the n-th component of the optimizer $x^{\text{opt}}(b)$.

Definition 3.1: (Inelastic electricity market game): The *inelastic electricity market* game is defined by the following

(i) Players: the set of generators $[N]$,
(ii) Action: for each player n, the bid $b_n \in \mathbb{R}_{\geq 0}$,
(iii) Payoff: for each player n, the payoff u_n in (4).

Wherever convenient, for any $n \in [N]$, we use interchangeably the notation b and (b_n, b_{-n}), as well as, $x^{\text{opt}}(b)$ and $x^{\text{opt}}(b_n, b_{-n})$. Note that the payoff of the players is not only defined by the bids of other players but also by the optimizer of (3) that the ISO selects. For this reason, the definition of the pure Nash equilibrium for the game described below is slightly different from the standard one, see e.g. [29].

Definition 3.2: (Nash equilibrium): The *(pure) Nash equilibrium* of the inelastic electricity market game is the bid profile of the group $b^* \in \mathbb{R}^N_{\geq 0}$ for which there exists an optimizer $(x^{\text{opt}}(b^*), z^{\text{opt}}(b^*))$ of the optimization (3) that satisfies

$$u_n(b_n, x_n^{\text{opt}}(b_n, b_{-n}^*)) \leq u_n(b_n^*, x_n^{\text{opt}}(b^*)), \tag{5}$$

for all $n \in [N]$, all bids $b_n \in \mathbb{R}_{\geq 0}$ with $b_n \neq b_n^*$, and all optimizers $(x^{\text{opt}}(b_n, b_{-n}^*), z^{\text{opt}}(b_n, b_{-n}))$ of (3) given the bid profile (b_n, b_{-n}^*).

We are specifically interested in bid profiles for which the optimizer of the DC-OPF problem is also a solution to the S-DC-OPF problem. This is captured in the following definition.

Definition 3.3: (Efficient bid): An *efficient bid* of the inelastic electricity market is a bid $b^* \in \mathbb{R}^N_{\geq 0}$ for which the optimizer (x^*, z^*) of (2) is also an optimizer of (3) given bids b^* and

$$x_n^* = \text{argmax}_{x \geq 0} b_n^* x - f_n(x), \tag{6}$$

for all $n \in [N]$.

Note that the right-hand side of (6) is unique as the functions are quadratic.

Definition 3.4: (Efficient Nash equilibrium): A bid b^* is an *efficient Nash equilibrium* of the inelastic electricity market game if it is an efficient bid and is a Nash equilibrium.

At the efficient Nash equilibrium, the production that the generators are willing to provide, maximizing their profit, coincides with the optimal generation for the DC-OPF problem (2). This property justifies the study of efficient Nash equilibria. Note that given the efficient bid profile, there might be many solutions to (3) because the problem is linear. As a consequence, the ISO might not be able to find the allocation x^* given the efficient bid. However, once the ISO knows that an efficient Nash equilibrium bid is submitted, it can ask the generators to also submit the desirable generation levels at that bid, which would exactly correspond to the solution of the DC-OPF problem.

IV. Existence of efficient Nash equilibrium

Here, we establish the existence of an efficient Nash equilibrium of the inelastic electricity market game described in Section III and provide a condition for its uniqueness.

Proposition 4.1: (Existence of efficient Nash equilibrium): If there is more than one generator at each bus of the network, i.e., $n_i \geq 2$ for each $i \in [N_b]$, then there exists an efficient Nash equilibrium of the inelastic electricity market game

Next we provide a sufficient condition that ensures uniqueness of the efficient bid the inelastic electricity market game.

Lemma 4.2: (Uniqueness of the efficient bid): Assume that the optimizer x^* of (2) satisfies $x_n^* > 0$ for all $n \in [N]$. Then, there exists a unique efficient bid $b^* \in \mathbb{R}^N_{\geq 0}$ of the inelastic electricity market game given by

$$b_n^* = \nabla f_n(x_n^*) = 2a_n x_n^* + c_n, \quad \text{for all } n. \tag{7}$$

From Proposition 4.1 and Lemma 4.2, we conclude the following result.

Corollary 4.3: (Uniqueness of the efficient Nash equilibrium): Assume there is more than one generator at each bus of the network and that the optimizer x^* of (2) satisfies $x_n^* > 0$ for all $n \in [N]$. Then, there exists a unique efficient Nash equilibrium of the inelastic electricity market game given by (7) for all n.

In the rest of the paper, we assume that the sufficient conditions in Corollary 4.3 hold unless otherwise stated.

V. The Bid Adjustment Algorithm and its convergence properties

In this section, we introduce a decentralized Nash equilibrium seeking algorithm, termed Bid Adjustment Algorithm. We show that its executions lead the generators to the unique efficient Nash equilibrium, and consequently, to the optimizer of the DC-OPF problem (2).

We start with an informal description of the BID ADJUSTMENT ALGORITHM. The algorithm is iterative and can be interpreted as "learning via repeated play" of the inelastic electricity market game by the generators. Both ISO and generators have bounded rationality, with each generator trying to maximize its own profit and the ISO trying to maximize the welfare of the entities.

> *[Informal description]:* At each iteration k, generators decide on a bid and send it to the ISO. Once the ISO has obtained the bids, it computes an optimizer of the S-DC-OPF problem (3) and sends the corresponding production level at the optimizer to each generator. At the $(k+1)$-th iteration, generators adjust their bid based on their previous bid, the amount of produced power that maximizes their payoff for the previous bid, and the allocation of generation assigned by the ISO. The iterative process starts with the generators arbitrarily selecting initial bids that yield a positive profit.

The BID ADJUSTMENT ALGORITHM is formally presented in Algorithm 1.

Algorithm 1: BID ADJUSTMENT ALGORITHM

Executed by: generators $n \in [N]$ and ISO
Data : cost f_n and stepsizes $\{\beta_k\}_{k \in \mathbb{Z}_{\geq 1}}$ for each generator n, and load y for ISO
Initialize : Each generator n selects arbitrarily $b_n(1) \geq c_n$, sets $k = 1$, and jumps to step 4; ISO sets $k = 1$ and waits for step 6

1 **while** $k > 0$ **do**
  /* For each generator n: */
2 Receive $x_n^*(k-1)$ from ISO
3 Set $b_n(k) = [b_n(k-1) + \beta_k(x_n^*(k-1) - q_n(k-1))]^+$
4 Set $q_n(k) = \mathrm{argmax}_{q \geq 0} b_n(k) q - f_n(q)$
5 Send $b_n(k)$ to the ISO; set $k = k+1$
  /* For ISO: */
6 Receive $b_n(k)$ from each $n \in [N]$
7 Find a solution $(x^*(k), z^*(k))$ to (3) given bids $b(k)$
8 Send $x_n^*(k)$ to each $n \in [N]$; set $k = k+1$
9 **end**

In the BID ADJUSTMENT ALGORITHM, the role of the ISO is to compute an optimizer of the S-DC-OPF problem after the bids are submitted. The bid adjustment at each iteration is done by the generators in a "myopically selfish" and rational fashion, with the sole aim of maximizing their payoff. Roughly speaking, the algorithm prescribes that

if n **gets** $x_n^*(k) = 0$**,** two things can happen: (i) n was willing to produce a positive quantity $q_n(k) > 0$ at bid $b_n(k)$ but the demand from ISO is $x_n^*(k) = 0$. Thus, the rational choice for n would be to decrease the bid in the next iteration to increase its chances of getting positive payoff; (ii) n was willing to produce nothing $q_n(k) = 0$ at bid $b_n(k)$ and got $x_n^*(k) = 0$. At this point, reducing the bid will not increase the payoff as it will not be willing to produce more at a lower bid. On the other hand, increasing the bid will not make the amount that the ISO wants the generator to produce positive. Therefore, the bid stays put.

if n **gets** $x_n^*(k) > 0$**,** then it would want to move the bid in the direction that makes its payoff higher in the next iteration, assuming that n gets a positive generation signal from the ISO in the next round of play. If $q_n(k) < x_n^*(k)$, then the amount demanded by the ISO is more than what the generator is willing to produce, so n increases its cost, i.e., the bid. If $q_n(k) > x_n^*(k)$, then the demand is less than what the generator is willing to supply so n decreases its bid.

Remark 5.1: (Information structure and alternative learning approaches): The generators have no knowledge of the number of other players, their actions, or their payoffs. The only information available to them at each iteration is their own bid and the amount that the ISO requests from them. This information structure rules out the applicability of a number of Nash equilibrium learning methods, including best-response dynamics [10], fictitious play [11], or other gradient-based adjustments [8], all requiring some kind of information about other players. Methods that relax this requirement, such as the extremum seeking techniques used in [12], rely on the payoff functions being continuous in the actions of the players, which is not the case for the inelastic electricity market game. •

Our convergence result states that the generator bids along any execution of the BID ADJUSTMENT ALGORITHM converge to a neighborhood of the unique efficient Nash equilibrium. The size of the neighborhood is a decreasing function of the stepsize and can be made arbitrarily small.

Theorem 5.2: (Convergence of the BID ADJUSTMENT ALGORITHM*):* In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Further, let $0 < r < \|b(1) - b^*\|$ and for all $k \in \mathbb{Z}_{\geq 1}$ assume

$$\alpha \leq \beta_k \leq B(r) := \frac{1}{2a_{\max}} \Big(\frac{1}{2a_{\min}^2} + \frac{16\overline{y}^2}{r^2} \Big)^{-1}, \tag{8}$$

for some $\alpha > 0$. Then, the following holds

(i) there exists $l \in \mathbb{Z}_{\geq 1}$ such that $\|b(l) - b^*\| < r$ and for all $k \in [l-1]$, we have $\|b(k) - b^*\| \geq r$ with

$$\|b(k+1) - b^*\| \leq \Big(1 - \frac{\alpha}{2a_{\max}}\Big)^{k/2} \|b(1) - b^*\|, \tag{9}$$

(ii) for all $k \geq l$,

$$\|b(k) - b^*\| \leq \Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r. \tag{10}$$

Remark 5.3: (Convergence properties from Theorem 5.2): The assertion (i) of Theorem 5.2 implies that for any choice of $r > 0$, one can select stepsizes according to (8) so that bids reach the set $\mathcal{B}_r(b^*)$ in finite number of steps and at a linear rate. Further, once bids reach the set $\mathcal{B}_r(b^*)$, we are assured from assertion (ii) that they remain in a neighborhood of b^* where the size of the neighborhood is proportional to r (cf. (10)). In combination, the above facts mean that bids converge to any neighborhood of the efficient Nash equilibrium at a linear rate provided the stepsizes are selected appropriately. Note that as r becomes small, $B(r)$ gets small and so does α. Thus, from (9), the rate of convergence decreases as r becomes small. This presents a trade-off between the desired precision and the rate of convergence. •

Remark 5.4: (Stopping criteria for the ISO): Algorithm 1, as defined, is a routine with an infinite number of iterations. To make it implementable in practice, we identify here stopping criteria for the ISO. Since this is not known to the generators, they cannot predict when the algorithm will terminate and, hence do not have an incentive to play strategically to maximize their payoff in the long term. From Theorem 5.2(i) note that, as long as $\|b(k)-b^*\| > r$ and $k < l$, the distance to the efficient Nash equilibrium decreases. Hence, if $\|b(k)-b^*\| > r$ and $k < l$, then one can write

$$\begin{aligned}\|b(k+1)-b(k)\| &= \|b(k+1)-b^*+b^*-b(k)\| \\ &\geq \|b(k)-b^*\| - \|b(k+1)-b^*\| \\ &\overset{(a)}{\geq} \|b(k)-b^*\| - \Big(1-\frac{\alpha}{2a_{\max}}\Big)^{1/2}\|b(k)-b^*\| \\ &= \Big(1-\Big(1-\frac{\alpha}{2a_{\max}}\Big)^{1/2}\Big)\|b(k)-b^*\|, \qquad (11)\end{aligned}$$

where in (a) we have used the inequality

$$\|b(k+1)-b^*\| \leq \Big(1-\frac{\alpha}{2a_{\max}}\Big)^{1/2}\|b(k)-b^*\|$$

which is a property of the bid update scheme that was used to prove (9). Using the observation in (11), if the ISO has an estimate of α and $a_{\max}$, then it can design a stopping criteria based on the distance between consecutive bids. In fact, if the ISO selects $\epsilon > 0$ and stops the iteration whenever $\|b(k+1)-b(k)\| \leq \epsilon$, then it has the guarantee that either of the following is satisfied

(i) the condition $\|b(k)-b^*\| > r$ and $k < l$ is met and from (11) we get
$$\|b(k)-b^*\| \leq \epsilon\Big(1-\Big(1-\frac{\alpha}{2a_{\max}}\Big)^{1/2}\Big)^{-1}; \qquad (12)$$
(ii) $\|b(k)-b^*\| \leq r$; or
(iii) $k > l$ in which case from (10) we get
$$\|b(k)-b^*\| \leq \Big(1+\frac{B(r)}{2a_{\max}}\Big)^{1/2} r.$$

The ISO does not know the value of r; its value depends on the stepsizes that the generators select. Assuming that stepsizes are small and r is small, the ISO can adjust ϵ depending on the desired level of accuracy to get the guarantee (12) for the k-th bid. Note that for a small ϵ, the stopping criteria might never be met if the stepsizes are too big. •

VI. Simulations

We illustrate the application of the Bid Adjustment Algorithm to find the efficient Nash equilibrium for an inelastic electricity market game with 10 generators. We consider the network to have 5 buses with each bus connected to 2 generators and a load, see Figure 1. The line flow limit

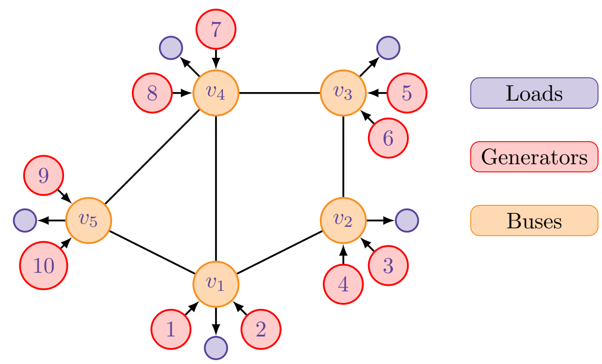


Fig. 1. Network layout with 5 buses, 10 generators, and 5 loads.

between any two buses (v_i, v_j) is 1. The loads are $y_1 = 3$, $y_2 = 10$, $y_3 = 1$, $y_4 = 4$, and $y_5 = 2$, where y_i denotes the load at bus v_i. The cost function for each generator i is $f_i(x_i) = a_i x_i^2 + c_i x_i$ where the coefficients for all the generators are given by the vectors

$$\begin{aligned} a &= (0.070, 0.095, 0.090, 0.090, 0.080, \\ &\qquad 0.075, 0.100, 0.090, 0.072, 0.080), \qquad (13) \\ c &= (7.0, 10.0, 8.5, 11.0, 10.5, 12.0, 10.0, 9.0, 11.0, 8.8). \end{aligned}$$

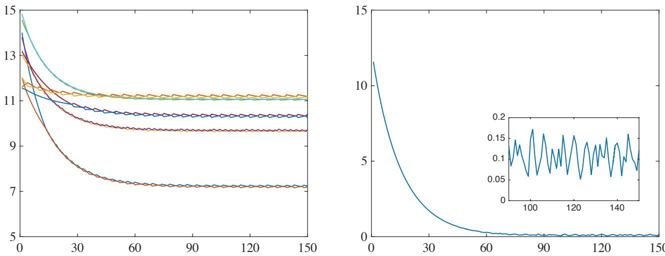
For the given costs and loads, the generation profile at the optimizer of the DC-OPF problem (2) is

$$\begin{aligned} x^* = (&1.5758, 6.4242, 6.5625, 3.4375, 3.5139 \\ &0.4861, 1.6316, 7.3684, 1.3158, 8.6842) \end{aligned}$$

and the unique efficient Nash equilibrium is

$$\begin{aligned} b^* = (&7.2206, 7.2206, 9.6812, 9.6812, 11.0622 \\ &11.0622, 10.3263, 10.3263, 11.1895, 11.1895). \qquad (14) \end{aligned}$$

Figure 2 depicts the evolution of the bids and their distance to the efficient Nash equilibrium along an execution of the Bid Adjustment Algorithm. The initial bids $b(1)$ are selected so that they satisfy the constraint $b_n(1) \geq c_n$ for all the generators $n \in [10]$. The stepsizes are chosen to be constant, $\beta_k = 0.01$ for all k, and satisfy the condition $\beta_k \leq 2a_n$. As predicted by Theorem 5.2, Figure 2 shows that the bids converge towards the efficient Nash equilibrium b^* at a linear rate and, after a finite number of steps, remain in a neighborhood of b^*. If one selects $r = 4.1$, then we get $B(r) = 0.0104$ from (8) and condition (8) is met for the chosen value of stepsizes, i.e., $\beta_k = 0.01$ for all k. Computing the right hand side of (10) using these values, we conclude that bids eventually remain in the neighborhood centered at b^* with radius 4.2052. Observe that Figure 2(b) validates this claim. In fact, this bound is quite conservative and bids actually remain in the neighborhood of radius 0.2.



(a) Bids $b(k)$ (b) Distance $\|b(k) - b^*\|$

Fig. 2. Illustration of the execution of the BID ADJUSTMENT ALGORITHM for the network in Figure 1. The cost function for each generator i is $f_i(x_i) = a_i x_i^2 + c_i x_i$, with coefficients given in (13). The load is given as $y = (3, 10, 1, 4, 2)$ where y_i is the load at bus v_i. The efficient Nash equilibrium b^* is given in (14). Plots (a) and (b) show, respectively, the evolution of the bids and their distance to b^*. The stepsizes are $\beta_k = 0.01$ for all k and the initial bids are $b(1) = (14.0051, 12.0204, 13.0238, 13.7963, 14.5636, 14.8372, 13.1889, 11.5545, 11.5972, 12.0300)$. Bids converge to and then remain in a neighborhood of the efficient Nash equilibrium.

VII. CONCLUSIONS

We have formulated an inelastic electricity market game encoding the strategic interaction between generators in a bid-based energy dispatch setting. We have established the existence and uniqueness of efficient Nash equilibria of this game. We have also designed the BID ADJUSTMENT ALGORITHM, which a strategy amenable to decentralized implementation and provably converges to a neighborhood of the efficient Nash equilibrium at a linear rate. Future work will analyze the convergence properties of the algorithm when a set of generators deviate from the proposed update scheme and study other bidding strategies, such as Cournot bidding, supply function bidding, and price-capacity bidding. Finally, we intend to examine the convergence of other learning schemes such as regret minimization in the context of electricity markets.

ACKNOWLEDGMENTS

This work was partially supported by ARPA-e Cooperative Agreement DE-AR0000695.

REFERENCES

[1] S. Stoft, *Power system economics: designing markets for electricity*. Piscataway, NJ: IEEE Press, 2002.

[2] F. Kirschen and G. Strbac, *Fundamentals of power system economics*. Sussex, England: John Wiley & Sons, 2004.

[3] R. Johari and J. N. Tsitsiklis, "Parameterized supply function bidding: Equilibrium and efficiency," *Operations Research*, vol. 59, no. 5, pp. 1079–1089, 2011.

[4] W. Tang and R. Jain, "Game-theoretic analysis of the nodal pricing mechanism for electricity markets," in *IEEE Conf. on Decision and Control*, Florence, Italy, Dec. 2013, pp. 562–567.

[5] D. Cai, S. Bose, and A. Wierman, "On the role of a market maker in networked cournot competition," 2016, preprint available at http://boses.ece.illinois.edu.

[6] T. Li and M. Shahidehpour, "Strategic bidding of transmission-constrained GENCO's with incomplete information," *IEEE Transactions on Power Systems*, vol. 20, no. 1, pp. 437–447, 2005.

[7] X. Hu and D. Ralph, "Using EPECs to model bilevel games in restructured electricity markets with locational prices," *Operations Research*, vol. 55, no. 5, pp. 809–827, 2007.

[8] G. I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, *Nonlinear oligopolies: stability and bifurcations*. New York: Springer, 2010.

[9] U. Nadav and G. Piliouras, "No regret learning in oligopolies: Cournot vs bertrand," in *Algorithmic Game Theory*, ser. Lecture Notes in Computer Science, S. Kontogiannis, E. Koutsoupias, and P. G. Spirakis, Eds. New York: Springer, 2010, vol. 6386, pp. 300–311.

[10] E. N. Barron, R. Goebel, and R. R. Jensen, "Best response dynamics for continuous games," *Proceeding of the American Mathematical Society*, vol. 138, no. 3, pp. 1069–1083, 2010.

[11] D. Fudenberg and D. K. Levine, *The Theory of Learning in Games*. Cambridge, MA: MIT Press, 1998.

[12] P. Frihauf, M. Krstić, and T. Başar, "Nash equilibrium seeking in noncooperative games," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1192–1207, 2012.

[13] S. Leyffer and T. Munson, "Solving multi-leader-common-follower games," *Optimization Methods and Software*, vol. 25, no. 4, pp. 601–623, 2010.

[14] J.-S. Pang and M. Fukushima, "Quasi-variational inequalities, generalized nash equilbria, and multi-leader-follower games," *Computational Management Science*, vol. 2, no. 1, pp. 21–56, 2005.

[15] Z.-Q. Luo, J.-S. Pang, and D. Ralph, *Mathematical programs with equilibrium constraints*. Cambridge, UK: Cambridge University Press, 1996.

[16] H. Scheel and S. Scholtes, "Mathematical programs with complementarity constraints: stationarity, optimality, and sensitivity," *Mathematics of Operations Research*, vol. 25, no. 1, pp. 1–22, 2000.

[17] M. C. Ferris, S. P. Dirkse, and A. Meeraus, "Mathematical programs with equilibrium constraints: automatic reformulation and solution via constrained optimization," in *Frontiers in Applied General Equilibrium Modeling*. Cambridge, UK: Cambridge University Press, 2005, pp. 67–94.

[18] D. Zhang and G.-H. Lin, "Bilevel direct search method for leader-follower problems and application in health insurance," *Computers & Operations Research*, vol. 41, pp. 359–373, 2014.

[19] J. Warrington, P. Goulart, S. Mariéthoz, and M. Morari, "A market mechanism for solving multi-period optimal power flow exactly on ac networks with mixed participants," in *American Control Conference*, Montréal, Canada, June 2012, pp. 3101–3107.

[20] D. J. Shiltz, M. Cvetković, and A. M. Annaswamy, "An integrated dynamic market mechanism for real-time markets and frequency regulation," *IEEE Transactions on Sustainable Energy*, vol. 7, no. 2, pp. 875–885, 2016.

[21] D. P. Palomar and M. Chiang, "Alternative distributed algorithms for network utility maximization: Framework and applications," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2254–2269, 2007.

[22] M. N. Faqiry and S. Das, "Double-sided energy auction in microgrid: equilibrium under price anticipation," *IEEE Access*, vol. 4, pp. 3794–3805, 2016.

[23] S. Pu and A. Garcia, "Iterative mechanisms for electricity markets," 2016, available at https://arxiv.org/abs/1608.08987.

[24] A. Cherukuri and J. Cortés, "Initialization-free distributed coordination for economic dispatch under varying loads and generator commitment," *Automatica*, vol. 74, pp. 183–193, 2016.

[25] S. T. Cady, A. D. Domínguez-García, and C. N. Hadjicostis, "A distributed generation control architecture for islanded AC microgrids," *IEEE Transactions on Control Systems Technology*, vol. 23, no. 5, pp. 1717–1735, 2015.

[26] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009, electronically available at http://coordinationbook.info.

[27] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[28] A. Mas-Colell, M. Whinston, and J. Green, *Microeconomic Theory*. Oxford, UK: Oxford University Press, 1995.

[29] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA: MIT Press, 1991.