Iterative bidding in electricity markets: rationality and robustness

Ashish Cherukuri   Jorge Cortés

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Abstract—This paper studies an electricity market consisting of an independent system operator (ISO) and a group of generators. The goal is to solve the DC optimal power flow (DC-OPF) problem: have the generators collectively meet the power demand while minimizing the aggregate generation cost and respecting line flow limits in the network. The ISO by itself cannot solve the DC-OPF problem as generators are strategic and do not share their cost functions. Instead, each generator submits to the ISO a bid, consisting of the price per unit of electricity at which it is willing to provide power. Based on the bids, the ISO decides how much production to allocate to each generator to minimize the total payment while meeting the load and satisfying the line limits. We provide a provably correct, decentralized iterative scheme, termed BID ADJUSTMENT ALGORITHM, for the resulting Bertrand competition game. Regarding convergence, we show that the algorithm takes the generators' bids to any desired neighborhood of the efficient Nash equilibrium at a linear convergence rate. As a consequence, the optimal production of the generators converges to the optimizer of the DC-OPF problem. Regarding robustness, we show that the algorithm is robust to affine perturbations in the bid adjustment scheme and that there is no incentive for any individual generator to deviate from the algorithm by using an alternative bid update scheme. We also establish the algorithm robustness to collusion, i.e., as long as each bus with generation has a generator following the strategy, there is no incentive for any group of generators to share information with the intent of tricking the system to obtain a higher payoff.

1 INTRODUCTION

As part of the plan to integrate distributed energy resources (DERs) into the electricity grid, regulating authorities envision a hierarchical architecture where, at the lower layer, different sets of DERs coordinate their response under an aggregator and, at the upper layer, the independent system operator (ISO) interacts with the aggregators to solve the optimal power flow (OPF) problem. In this scenario, aggregators function as (virtual, large-capacity) generators, and the aggregation would allow DERs to participate into markets in which, individually, they do not have the capacity to do so. While the DERs under one aggregator can cooperate among themselves, the aggregators compete with each other in the electricity market. In this paper, we focus on the competition aspect of this vision: we study policies that individual generators, in conjunction with the ISO, can implement to solve the OPF problem while acting in a selfish and rational fashion.

Literature review: The study of competition in electricity markets is a classical topic [3], [4]. Extensively studied models are supply function, Bertrand (price) and Cournot (capacity) bidding, see [5], [6], [7], respectively, and references therein. These studies analyze the properties of the game resulting from each bidding model by determining the existence of Nash equilibrium (NE) and estimating its efficiency. Some works [8], [9], [10], [11], on the other hand, propose iterative algorithms for the players that compute the NE of the game. However, these algorithms either require generators to have some information about other generators (cost functions or bids) or assume that the demand of each generator is a continuous function of the bids. Our work does not make any such assumptions, which also rules out the possibility of using various other NE learning algorithms, such as best-response [12], fictitious play [13], and extremum seeking [14], [15]. In a related set of works [16], [17], decentralized generation planning is achieved by assuming the generators to be price-takers and designing iterative schemes based on dual-decomposition [18]. The work [19] has extensively surveyed the requirements on information exchange, computational complexity, and physical implementability of various practical pricing mechanisms (both iterative and non-iterative) assuming DERs are price-takers. In our work, however, we consider a strategic scenario where generators bid into the market and are hence price-setters. The work [20] proposes an iterative auction algorithm for a market where both generators and consumers are strategic but does not provide convergence guarantees for the generated bid sequences. The paper [21], closer in spirit to our work, proposes an iterative method for the generators to find the NE assuming they do not have any information about each other. At each iteration, the generators send to the ISO the gradient of their cost functions at a certain generation value and the ISO then adjusts these generation values so that social welfare is maximized. An important difference between this setup and ours is that we do not need generators to share gradient information with the ISO.

Our electricity market game belongs to the broader class of multi-leader-single-follower games [22], [23]. The

Nash equilibria of such games can be thought of as optimizers of mathematical programs with equilibrium constraints (MPEC) [24], that are traditionally solved in a centralized manner [25]. The work [26] provides a distributed method to find the equilibria of an MPEC problem but requires the follower's (the ISO in our case) optimization to have a unique solution for each action of the leaders (the generators). This is in general not the case for electricity markets. Our work broadly relates to the recent developments in the area of "learning in games", see e.g., [27], [28], and references therein. Learning mechanisms proposed in there do not apply directly to the electricity market setting as they do not consider network constraints for allocation of goods. Finally, our work has close connections with the growing interest in the design of provably correct distributed algorithms for the cooperative solution of economic dispatch, see [29], [30], and references therein.

Statement of contributions: This work aims to explore the feasibility of introducing iterative bidding, where social welfare can be maximized while agents are being selfish, as a new market-clearing mechanism for electricity markets. Given the novelty of this task, we make a number of assumptions on the proposed model which simplify the exposition while still allowing us to obtain meaningful results. We believe that the framework presented here is generalizable and will pave way for a more thorough analysis of iterative bidding in electricity markets.

Our *setup* considers the inelastic electricity market game where the ISO seeks the production levels that solve the DC optimal power flow (DC-OPF) problem for a group of strategic generators which do not share their cost functions. Each generator submits a bid to the ISO specifying the price per unit of electricity at which the generator is willing to provide power. Given these bids, the ISO decides the production of each generator so that the cost is minimized, loads are met, and line constraints are satisfied. The resulting Bertrand competition model defines the game among the generators, where the actions are the bids and the payoffs are the profits. We define the concept of the efficient Nash equilibrium, that is, the NE at which the generators are willing to produce the amount that corresponds to the optimizer of the DC-OPF problem.

Our first contribution gives two set of conditions that ensure existence and uniqueness, respectively, of an efficient NE for the inelastic electricity market game.

Our second contribution is the design of the BID ADJUSTMENT ALGORITHM along with its correctness analysis. This algorithm can be understood as "learning via repeated play", where generators are "myopically selfish", changing their bid at each iteration with the aim of maximizing their own payoff. Along the execution, the only information available to the generators is their bid and the amount of generation that the ISO request from them. In particular, generators are not aware of the number of other generators, their costs, bids, or payoffs. We show that this decentralized iterative scheme is guar-

anteed to take the generators' bids to any neighborhood of the unique efficient NE at a linear rate provided stepsizes are chosen appropriately. We also provide the DETERMINE GENERATION procedure to allow the ISO to find an approximate optimizer to the DC-OPF problem once bids have converged to a neighborhood of the efficient NE.

Our third contribution analyzes the robustness properties of the BID ADJUSTMENT ALGORITHM. Specifically, we establish that the convergence is not affected by affine disturbances, thus showing that deviations in stepsizes by the generators can be handled gracefully. Additionally, we show that there is no incentive for any individual generator to deviate from the algorithm. Finally, we also show that, if at each generator bus there is at least one generator running the BID ADJUSTMENT ALGORITHM, then there is no incentive for other generators connected to the network to not follow the algorithm. These properties provide a sound justification for why generators would adopt this iterative bid adjustment scheme[1].

2 PROBLEM STATEMENT

Consider an electrical power network with $N_b \in \mathbb{Z}_{\geq 1}$ buses operating under the DC power flow model. The physical interconnection between the buses is given by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes correspond to buses and edges to physical power lines. The direction for each edge represents the convention of positive power flow. The power flow on the line $(i, j) \in \mathcal{E}$ is $z_{ij} \in \mathbb{R}$. Each line $(i, j) \in \mathcal{E}$ has a limit on the power flowing through it (in either direction), represented by $\overline{z}_{ij} > 0$. The voltage phase angle at bus $i \in [N_b]$ is $\theta_i \in \mathbb{R}$. Assume that each bus $i \in [N_b]$ is connected to $n_i \in \mathbb{Z}_{\geq 0}$ strategic generators. We let $N = \sum_{i=1}^{N_b} n_i$ be the total number of generators and assign them a unique identity in $[N]$. Let the set of generators at node i be $G_i \subset [N]$ (this set is empty if there are no generators connected to bus i). The power demand at bus i is denoted by $y_i \geq 0$ and is assumed to be fixed and known to the Independent System Operator (ISO) that acts as the central regulating authority. The total demand is $y_{\text{total}} = \sum_{i=1}^{N_b} y_i$. The cost $f_n(x_n)$ of generating $x_n \in \mathbb{R}_{\geq 0}$ amount of power by the n-th generator is given by

$$f_n(x) = a_n x^2 + c_n x, \tag{1}$$

where $a_n > 0$, $c_n \geq 0$. Given $x = (x_1, \ldots, x_N) \in \mathbb{R}_{\geq 0}^N$, the aggregate cost is $\sum_{n=1}^N f_n(x_n)$. The *dc optimal power flow*

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problem (DC-OPF) [31] consists of

$$\underset{(x,z,\theta)}{\text{minimize}} \quad \sum_{n=1}^{N} f_n(x_n), \tag{2a}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{N}_i^+} z_{ij} - \sum_{j\in\mathcal{N}_i^-} z_{ij} = \sum_{n\in G_i} x_n - y_i, \ \forall i, \tag{2b}$$

$$-\overline{z}_{ij} \le z_{ij} \le \overline{z}_{ij}, \ \forall(i,j), \tag{2c}$$

$$z_{ij} = \gamma_{ij}(\theta_i - \theta_j), \ \forall(i,j), \tag{2d}$$

$$x \ge \mathbf{0}_N, \tag{2e}$$

This problem finds the generation profile that meets the load at each bus (ensured by (2b)), respects the line constraints (due to (2c)) where flows are related to voltage phase angles by (2d), and minimizes the total cost (given by the objective function (2a)). All constraints are treated as hard throughout the paper. Here, for each line (i,j), γ_{ij} represents the line susceptance. In (2b) we make the convention that if $G_i = \emptyset$, then the first term on the right-hand side is zero. We assume that (2) is feasible. Since the individual costs are quadratic, the optimizers of the problem have a unique generation profile [32], which we denote as x^*. Thus, we denote the set of optimizers as $\{x^*\} \times \mathcal{Z}^* \times \Theta^*$, where $\mathcal{Z}^* \subset \mathbb{R}^{|\mathcal{E}|}$ and $\Theta^* \subset \mathbb{R}^{N_b}$ are the set of flow vectors and voltage phase angles that satisfy (2) given generation x^*.

The goal for the ISO is to solve (2). The ISO can interact with the generators, whereas each generator can only communicate with the ISO and is not aware of the number of other generators participating in the market and their respective cost functions, or the load at its own bus. While the ISO knows the loads and the limits on the power lines, it does not have any information about the cost functions of the generators. Therefore, power allocation is decided following a bidding process, resulting into a game-theoretic formulation. Instead of sharing their cost with the ISO, the generators bid the price per unit of power that they are willing to provide the power at. This price-based bidding is well known in the economics literature as Bertrand competition [33, Chapter 12]. Specifically, generator n bids the cost per unit power $b_n \in \mathbb{R}_{\ge 0}$ and, when convenient, we denote the bids of all other generators except n by $b_{-n} = (b_1, \ldots, b_{n-1}, b_{n+1}, \ldots, b_N)$. Given the bids $b = (b_1, \ldots, b_N) \in \mathbb{R}_{\ge 0}^N$, the ISO solves the following *strategic dc optimal power flow problem* (S-DC-OPF)

$$\underset{(x,z,\theta)}{\text{minimize}} \quad \sum_{n=1}^{N} b_n x_n, \tag{3a}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{N}_i^+} z_{ij} - \sum_{j\in\mathcal{N}_i^-} z_{ij} = \sum_{n\in G_i} x_n - y_i, \ \forall i, \tag{3b}$$

$$-\overline{z}_{ij} \le z_{ij} \le \overline{z}_{ij}, \ \forall(i,j), \tag{3c}$$

$$z_{ij} = \gamma_{ij}(\theta_i - \theta_j), \ \forall(i,j), \tag{3d}$$

$$x \ge \mathbf{0}_N. \tag{3e}$$

The difference between (3) and (2) is the objective function which is linear in the former and nonlinear, convex

in the latter. The ISO solves (3) once all the bids are gathered. Let $(x^{\text{opt}}(b), z^{\text{opt}}(b), \theta^{\text{opt}}(b))$ be the optimizer of (3) that the ISO selects (note that there might not be a unique optimizer) given bids b. This determines the power requested from each generator, given by the vector $x^{\text{opt}}(b)$. Knowing this process, the objective of each generator n is to bid a quantity $b_n \ge 0$ that maximizes its payoff $u_n : \mathbb{R}_{\ge 0}^2 \to \mathbb{R}$,

$$u_n(b_n, x_n^{\text{opt}}(b)) = b_n x_n^{\text{opt}}(b) - f_n(x_n^{\text{opt}}(b)), \tag{4}$$

where $x_n^{\text{opt}}(b)$ is the n-th component of the optimizer $x^{\text{opt}}(b)$.

Definition 2.1. *(Inelastic electricity market game): The* inelastic electricity market *game is defined by the following*

(i) *Players: the set of generators $[N]$,*
(ii) *Action: for each player n, the bid $b_n \in \mathbb{R}_{\ge 0}$,*
(iii) *Payoff: for each player n, the payoff u_n in (4).*

Wherever convenient, for any $n \in [N]$, we use interchangeably the notation b and (b_n, b_{-n}), as well as, $x^{\text{opt}}(b)$ and $x^{\text{opt}}(b_n, b_{-n})$. Note that the payoff of the players is not only defined by the bids of other players but also by the optimizer of (3) that the ISO selects. For this reason, the definition of the pure NE for the game described below is slightly different from the standard one, see e.g. [34].

Definition 2.2. *(Nash equilibrium): The* (pure) NE *of the inelastic electricity market is the bid profile of the group $b^* \in \mathbb{R}_{\ge 0}^N$ for which there exists an optimizer $(x^{\text{opt}}(b^*), z^{\text{opt}}(b^*), \theta^{\text{opt}}(b^*))$ of (3) that satisfies*

$$u_n(b_n, x_n^{\text{opt}}(b_n, b_{-n}^*)) \le u_n(b_n^*, x_n^{\text{opt}}(b^*)), \ \forall n \in [N], \tag{5}$$

all $b_n \in \mathbb{R}_{\ge 0}$, and all optimizers $(x^{\text{opt}}(b_n, b_{-n}^*), z^{\text{opt}}(b_n, b_{-n}), \theta^{\text{opt}}(b_n, b_{-n}))$ of (3) given bids (b_n, b_{-n}^*).

We are specifically interested in bid profiles for which the optimizer of the DC-OPF problem is also a solution to the S-DC-OPF problem. This is captured in the following definition.

Definition 2.3. *(Efficient bid): An* efficient bid *of the inelastic electricity market is a bid $b^* \in \mathbb{R}_{\ge 0}^N$ such that any optimizer (x^*, z^*, θ^*) of (2) is an optimizer of (3) given bids b^* and*

$$x_n^* = \operatorname{argmax}_{x\ge 0} b_n^* x - f_n(x), \quad \text{for all } n \in [N]. \tag{6}$$

The right-hand side of (6) is unique as costs are quadratic. Condition (6) can be interpreted as a form of incentive compatibility: at the efficient bid, the production that the generators are willing to provide, maximizing their profit, coincides with the optimal generation for the DC-OPF problem (2). Without (6), the resulting operating points would not be meaningful as generators would naturally seek to deviate from the optimizer in order to maximize their profit.

Definition 2.4. *(Efficient NE): A bid b^* is an* efficient NE *of the inelastic electricity market if it both efficient and a NE.*

The efficient NE enjoys three properties, first, no generator wants to deviate from this point unilaterally, second, the ISO can select the optimizer of the DC-OPF problem as the market clearing production at this bid, and third, generators are willing to provide this selected quantity due to (6). These properties justifies the study of efficient NE points. Note that given the efficient bid profile, there might be many solutions to (3) because the problem is linear and the ISO might not be able to find x^*. However, once the ISO knows that an efficient NE bid is submitted, it can ask the generators to also submit the desirable generation levels at that bid, which corresponds to the solution of the DC-OPF problem. We come back to this point in Section 4, where we provide the algorithm employed for iterative bidding and for determining the generation levels once the bids have converged. We assume that generators are strategic and hence price-setters during iterative bidding, while they are non-strategic and price-takers when generations levels are decided.

3 EXISTENCE AND UNIQUENESS OF EFFICIENT NE

We provide conditions for existence and uniqueness of efficient NE of the inelastic electricity market game in Section 2.

Proposition 3.1. *(Existence of efficient NE): Assume at each bus either there is more than one generator or there is none, i.e., either $n_i = 0$ or $n_i \geq 2$ for each $i \in [N_b]$. Then, there exists an efficient NE of the inelastic electricity market game.*

Proof: For convenience, we write (2b), (2c), and (2d) as

$$J_1 z - J_2 x + y = 0, \ J_3 z \leq \overline{z}_c, \ \text{and} \ J_4 \theta - z = 0,$$

respectively. Here, $J_1 \in \{0, 1, -1\}^{N_b \times |\mathcal{E}|}$ defines the interconnection of buses in the digraph $\mathcal{G}$, specifically, (n, i)-th element of J_1 is 1 if the head of some edge $(i, j) \in \mathcal{E}$ is n, this element is -1 if n is the tail of the edge, and otherwise the element is 0. The matrix $J_2 \in \{0, 1\}^{N_b \times N}$ defines the connectivity of generators to buses, that is, (i, j)-th element of J_2 is 1 if and only if j-th generator at i-th bus. Further, $J_4 \in \mathbb{R}^{|\mathcal{E}| \times N_b}$ is the incidence matrix of $(\mathcal{V}, \mathcal{E})$, where the nonzero elements in the row corresponding to edge (i, j) are γ_{ij} and $-\gamma_{ij}$ at i-th and j-th column, resp. Lastly, $J_3 = [I_{|\mathcal{E}|}, -I_{|\mathcal{E}|}]^\top$ and $\overline{z}_c = [\overline{z}, \overline{z}]^\top$. The Lagrangian of the optimization (2) is

$$L(x, z, \theta, \nu, \mu, \zeta, \lambda) = \sum_{n=1}^N f_n(x_n) + \nu^\top (J_1 z - J_2 x + y)$$
$$+ \mu^\top (J_3 z - \overline{z}_c) - \zeta^\top (J_4 \theta - z) - \lambda^\top x,$$

where $\nu \in \mathbb{R}^{N_b}$, $\mu \in \mathbb{R}^{2|\mathcal{E}|}_{\geq 0}$, $\zeta \in \mathbb{R}^{|\mathcal{E}|}$ and $\lambda \in \mathbb{R}^N_{\geq 0}$ are Lagrange multipliers corresponding to constraints (2b), (2c), (2d) and (2e), resp. Since constraints of the problem (2) are affine and the feasibility set is nonempty, the refined Slater condition is satisfied and

hence, the duality gap is zero [32]. Under this condition, a primal-dual optimizer $(x^*, z^*, \theta^*, \nu^*, \mu^*, \zeta^*, \lambda^*)$ satisfies the Karush-Kuhn-Tucker (KKT) conditions

$$\nabla f(x^*) - J_2^\top \nu^* - \lambda^* = 0, \quad (7a)$$

$$J_1^\top \nu^* + J_3^\top \mu^* = 0, \ J_1 z^* - J_2 x^* + y = 0, \quad (7b)$$

$$J_3 z^* \leq \overline{z}_c, \quad J_4 \theta^* = z^*, \quad J_4^\top \zeta^* = 0, \quad x^* \geq 0, \quad (7c)$$

$$\lambda^* \geq 0, \quad \mu^* \geq 0, \quad (7d)$$

$$(x^*)^\top \lambda^* = 0, \ \text{and} \ (\mu^*)^\top (J_3 z^* - \overline{z}_c) = 0, \quad (7e)$$

where $\nabla f(x^*) = (\nabla f_1(x_1^*), \nabla f_2(x_2^*), \dots, \nabla f_N(x_N^*))^\top$. In the rest of the proof, we show that the following bid profile, constructed from a primal-dual optimizer, is an efficient NE of the inelastic electricity market game

$$b_n^* = \begin{cases} \nu^*_{i(n)}, & \text{if } \max\{x_m^* \mid m \in G_{i(n)}\} > 0, \\ \nabla f_n(0), & \text{otherwise,} \end{cases} \quad (8)$$

where $i(n) \in [N_b]$ denotes the bus of the network to which generator n is connected to. Given the form (1) of the cost functions, we deduce $b^* \geq 0$. Moreover, from the definition of J_2, one can deduce that either all generators $n \in G_i$ have $b_n^* = \nu_i$ or all of them have $x_n^* = 0$. Next, to show that the bid b^* defined in (8) is efficient, we first establish

$$x_n^* = \operatorname{argmax}_{x \geq 0} b_n^* x - f_n(x), \ \forall n \in [N]. \quad (9)$$

For each n, consider $\max_{x \geq 0} b_n^* x - f_n(x)$. Because zero duality holds for this optimization, a point $x_o \in \mathbb{R}_{\geq 0}$ is an optimizer if and only if it satisfies the KKT conditions

$$b_n^* - \nabla f_n(x_o) + \mu_o = 0, \ \mu_o \geq 0, \ x_o \geq 0, \ \mu_o x_o = 0,$$

where μ_o is the dual optimizer. Since x_n^* satisfies the above conditions with $\mu_o = \lambda_n^*$, the expression (9) holds. To claim the efficiency of b^*, we next show that (x^*, z^*, θ^*) is one of the optimizers of (3) given bids b^*. Note that the KKT conditions for (3) are given by (7) with the term $\nabla f(x^*)$ in (7a) replaced with b^*. Also, one can show using the KKT conditions (7) and the definition of b^* given in (8) that $b^* - J_2^\top \nu^* \geq 0$. Indeed, for some bus $i \in [N_b]$, either $b_n^* = \nu_i^*$ for all $n \in G_i$ or $x_n^* = 0$ for all $n \in G_i$. For the latter, due to (7a), $b_n^* - \nu_i^* = \nabla f_n(0) - \nu_i^* = \nabla f_n(x_n^*) - \nu_i^* \geq 0$. Using these facts, we deduce $(x^*, z^*, \theta^*, \nu^*, \mu^*, \zeta^*, b^* - J_2^\top \nu^*)$ satisfies the KKT conditions for (3) and hence, (x^*, z^*, θ^*) is an optimizer of (3).

Our final step is to show the NE condition (5) for the bid profile b^*. Note that for each n, the payoff at the bid profile-optimizer pair $(b^*, x^{\mathrm{opt}}(b^*)) = (b^*, x^*)$ is nonnegative. Specifically, if $x_n^* = 0$, then $u_n(b_n^*, x_n^{\mathrm{opt}}(b^*)) = 0$. If $x_n^* > 0$, using the fact that $\nabla f_n(x) \leq b_n^*$ for all $x \in [0, x_n^*]$, we get

$$u_n(b_n^*, x_n^{\mathrm{opt}}(b^*)) = b_n^* x_n^* - f_n(x_n^*)$$
$$= \int_0^{x_n^*} \nabla(b_n^* x - f_n(x)) dx = \int_0^{x_n^*} (b_n^* - \nabla f_n(x)) dx \geq 0.$$

Now pick any generator $n \in [N]$. For bid $b_n \neq b_n^*$ we have two cases, first, $b_n > b_n^*$ and second, $b_n \leq b_n^*$.

For the first case, either (i) $x_n^* = 0$ which implies that $x^{\mathrm{opt}}(b_n, b_{-n}^*) = 0$ and so $u_n(b_n, x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) = u_n(b_n^*, x_n^*) = 0$; or (ii) $x_n^* > 0$, so all bids at bus $i(n)$ are equal, implying that n increasing its bid yields $x_n^{\mathrm{opt}}(b_n, b_{-n}^*) = 0$. That is, $u_n(b_n, x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) = 0 \leq u_n(b_n^*, x_n^*)$. For the second case,

$$u_n(b_n, x_n^{\mathrm{opt}}(b_n, b_{-n}^*)) = b_n x_n^{\mathrm{opt}}(b_n, b_{-n}^*) - f_n(x_n^{\mathrm{opt}}(b_n, b_{-n}^*))$$
$$\leq b_n^* x_n^{\mathrm{opt}}(b_n, b_{-n}^*) - f_n(x_n^{\mathrm{opt}}(b_n, b_{-n}^*))$$
$$\leq b_n^* x_n^* - f_n(x_n^*) = u_n(b_n^*, x_n^*),$$

where in the first inequality we use $b_n \leq b_n^*$ and in the second we use (9). This shows (5), concluding the proof. □

Note that the condition in Proposition 3.1 of having zero or at least two generators at each bus is reasonable in scenarios modeling power networks with hierarchical architectures of multiple actors or aggregations of multiple energy resources. In such cases, each bus could represent a control area (where there can certainly be more than one generator) or a generator itself could correspond to a subnetwork or collection of smaller generators or flexible loads. If the condition does not hold, i.e., there is a bus with a single generator, and the line capacities are such that the load at that bus can only be met by that generator, then there is possibility of market manipulation. The generator at one of these buses can set its bid arbitrarily high as no other generator can meet that load and consequently, there would not exist a NE. However, if the aggregate capacity of lines connected to the bus having a single generator is greater than the load connected to that bus, then the efficient NE exists. Next we provide a sufficient condition that ensures uniqueness of the efficient bid.

Proposition 3.2. *(Uniqueness of the efficient bid): Assume that the optimizer x^* of (2) satisfies $x_n^* > 0$ for all $n \in [N]$. Then, there exists a unique efficient bid $b^* \in \mathbb{R}_{\geq 0}^N$ of the inelastic electricity market game given by*

$$b_n^* = \nabla f_n(x_n^*) = 2a_n x_n^* + c_n, \quad \text{for all } n \in [N]. \tag{10}$$

Proof: By definition, an efficient bid $b \in \mathbb{R}_{\geq 0}^N$ satisfies

$$x_n^* = \mathrm{argmax}_{x \geq 0} b_n x - f_n(x)$$

for all n. Since $x_n^* > 0$, first-order optimality condition of the above optimization yields $b_n = \nabla f_n(x_n^*)$. This establishes (10) and hence, the uniqueness. □

The next result follows from Propositions 3.1 and 3.2.

Corollary 3.3. *(Uniqueness of the efficient NE): Assume that at each bus of the network either there is more than one generator or there is none. Further assume that the optimizer x^* of (2) satisfies $x_n^* > 0$ for all $n \in [N]$. Then, there exists a unique efficient NE of the inelastic electricity market game given by (10) for all n.*

In the rest of the paper, we assume that the sufficient conditions in Corollary 3.3 hold unless otherwise stated. Note that the definition (10) of the unique efficient NE is consistent with (8) because if $x_n^* > 0$ for all n, then

$\nabla f_{\tilde{n}}(x_{\tilde{n}}^*) = \nu_i^*$ for each bus $i \in [N_b]$ and every generator $\tilde{n} \in G_i$.

We believe that the convergence and robustness guarantees of the algorithm presented in the next section will hold as long as the efficient NE exists. However, we impose the conditions of Corollary 3.3 in order to simplify the technical exposition.

4 THE BID ADJUSTMENT ALGORITHM

Here, we introduce a decentralized NE seeking algorithm, termed BID ADJUSTMENT ALGORITHM. We show that its executions lead the generators to the unique efficient NE, and consequently, to the optimizer of the DC-OPF problem (2). Following this, we present the DETERMINE GENERATION scheme through which the ISO determines the generation levels, i.e., dispatch, for generators at the converged bid.

4.1 Algorithm description

We start with an informal description of the BID ADJUSTMENT ALGORITHM. This iterative algorithm can be interpreted as "learning via repeated play" of the inelastic electricity market game by the generators. Both ISO and generators are rational: each generator tries to maximize its profit and the ISO tries to maximize the welfare of the entities.

> *[Informal description]:* At each iteration k, generators decide on a bid and send it to the ISO. Once the ISO has obtained the bids, it computes an optimizer of the S-DC-OPF problem (3), denoted $(x^{\mathrm{opt}}(k), z^{\mathrm{opt}}(k), \theta^{\mathrm{opt}}(k))$ for convenience, and sends the corresponding production level at the optimizer to each generator. At the $(k+1)$-th iteration, generators adjust their bid based on their previous bid, the amount of produced power that maximizes their payoff for the previous bid, and the allocation of generation assigned by the ISO. The iterative process starts with the generators arbitrarily selecting initial bids that yield a positive profit.

In the BID ADJUSTMENT ALGORITHM, cf. Algorithm 1, the role of the ISO is to compute an optimizer of the S-DC-OPF problem after the bids are submitted. Generators adjust their bids at each iteration in a "myopically selfish" and rational fashion, with the aim of maximizing their payoff. Intuitively,

if n gets $x_n^{\mathrm{opt}}(k) = 0$, two things can happen: (i) n was willing to produce a positive quantity $q_n(k) > 0$ at bid $b_n(k)$ but the demand from the ISO is $x_n^{\mathrm{opt}}(k) = 0$. The rational choice for n is to decrease the bid in the next iteration and increase its chances of getting a positive payoff; (ii) n was willing to produce nothing $q_n(k) = 0$ at $b_n(k)$ and got $x_n^{\mathrm{opt}}(k) = 0$. At this point, reducing the bid will not increase the payoff as it will not be willing to produce more at a lower bid. Alternatively, increasing the bid will not make the amount that the ISO wants the generator to produce positive. Hence, the bid stays put;

if n **gets** $x_n^{\mathrm{opt}}(k) > 0$**,** then it would want to move the bid in the direction that makes its payoff higher in the next iteration, assuming that n gets a positive generation signal from the ISO in the next round. If $q_n(k) < x_n^{\mathrm{opt}}(k)$, then the demand from the ISO is more than what the generator is willing to produce, so n increases its cost, i.e., the bid. If $q_n(k) > x_n^{\mathrm{opt}}(k)$, then the demand is less than what the generator is willing to supply so n decreases its bid.

Algorithm 1: BID ADJUSTMENT ALGORITHM

Executed by: generators $n \in [N]$ and ISO

Data   : cost f_n and stepsizes $\{\beta_k\}_{k \in \mathbb{Z}_{\geq 1}}$ for each generator n, and load y for ISO

Initialize : Each generator n selects arbitrarily $b_n(1) \geq c_n$, sets $k = 1$, and jumps to step 4; ISO sets $k = 1$ and waits for step 6

1 while $k > 0$ **do**

    /* For each generator n: */

2   Receive $x_n^{\mathrm{opt}}(k-1)$ from ISO

3   Set $b_n(k) = [b_n(k-1) + \beta_k(x_n^{\mathrm{opt}}(k-1) - q_n(k-1))]^+$

4   Set $q_n(k) = \mathrm{argmax}_{q \geq 0} b_n(k)q - f_n(q)$

5   Send $b_n(k)$ to the ISO; set $k = k + 1$

    /* For ISO: */

6   Receive $b_n(k)$ from each $n \in [N]$

7   Find a solution $(x^{\mathrm{opt}}(k), z^{\mathrm{opt}}(k), \theta^{\mathrm{opt}}(k))$ to (3) given $b(k)$

8   Send $x_n^{\mathrm{opt}}(k)$ to each $n \in [N]$; set $k = k + 1$

9 end

Remark 4.1. *(Information structure and other learning approaches):* In the BID ADJUSTMENT ALGORITHM, the ISO is assumed to know beforehand the network structure, line limits, and the demand at each bus. In addition, at each round, ISO obtains the bids of all generators. On the other hand, generators have no knowledge of the number of other players, their actions, or their payoffs. The only information they have at each iteration is their own bid and the generation that the ISO requests from them. This information structure rules out the applicability of a number of NE learning methods, including best-response dynamics [12], fictitious play [13], or other gradient-based adjustments [10], all requiring some kind of information about other players. Methods that relax this requirement, such as extremum seeking used in [14], [15], rely on the payoff functions being continuous in the actions of the players, which is not the case for the inelastic electricity market game. If the generators were to obtain additional information at each round, even if indirectly about the bids submitted by others or their market clearing production, then they might employ more complex learning strategies to construct beliefs about the policies employed by others. We leave the design and convergence analysis of such learning mechanisms to future work. •

Remark 4.2. *(Stopping criteria and justification of "my-*

opically selfish" strategies): Algorithm 1 consists of an infinite number of iterations. To make it implementable, later we identify stopping criteria, see Remark 4.9, based on a parameter that the ISO selects. Since this is not known to the generators, they cannot predict when the algorithm will terminate and, hence, they do not have an incentive to play strategically to maximize their payoff in the long term. Given this, they should focus on maximizing the payoff in the next iteration, which justifies the myopically selfish perspective adopted here. •

4.2 Convergence analysis

We show that the generator bids along any execution of the BID ADJUSTMENT ALGORITHM converge to a neighborhood of the unique efficient NE. The size of the neighborhood is a decreasing function of the stepsize and can be made arbitrarily small. We first present a series of lemmas highlighting geometric properties of the bid update in Step 3 of Algorithm 1. For the reader's convenience, the proofs of these results are given in the appendix.

Lemma 4.3. *(Generator bids are lower bounded):* In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Then, $b_n(k) \geq c_n$ and for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$,

$$q_n(k) = \frac{b_n(k) - c_n}{2a_n}. \tag{11}$$

Our next result gives a different expression for the bid update (cf. Step 3) presenting a geometric perspective of the direction along which the bids are moving. We write the $k+1$-th bid as the addition of two vectors. The first one is a convex combination of the k-th bid and the efficient NE b^*. Hence, the first vector is closer to b^* as compared to the k-th bid. The second one depends on the difference between what the ISO requests from generators and the optimizer of (2). If the second term is small enough, then the bids move towards b^*.

Lemma 4.4. *(Geometric characterization of the bid update):* In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Then, we have

$$b(k+1) = b^{\mathrm{coc}}(k+1) + \beta_k(x^{\mathrm{opt}}(k) - x^*),$$

for all $k \in \mathbb{Z}_{\geq 1}$, where for each $n \in [N]$,

$$b_n^{\mathrm{coc}}(k+1) = \left(1 - \frac{\beta_k}{2a_n}\right)b_n(k) + \frac{\beta_k}{2a_n}b_n^*.$$

The next result gives a lower bound on the inner product between the direction in which the bids move and the direction towards the efficient NE.

Lemma 4.5. *(Bids move in the direction of the efficient NE):* In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Let $a_{\max} = \max_n\{a_n\}$. Then, for all $k \in \mathbb{Z}_{\geq 1}$,

$$\langle b(k+1) - b(k), b^* - b(k) \rangle \geq \frac{\beta_k}{2a_{\max}}\|b(k) - b^*\|^2. \tag{12}$$

The next result states that the distance between consecutive bids decreases as bids get closer to b^*. Together with Lemma 4.5, one can intuitively see that the bids get closer to b^* and, as they do, the bid update behaves as if the bids are reaching an equilibrium of the scheme, leading to convergence.

Lemma 4.6. *(Distance between consecutive bids is upper bounded): In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Let $a_{\min} = \min_n\{a_n\}$. Then, for all $k \in \mathbb{Z}_{\geq 1}$,*

$$\|b(k+1) - b(k)\|^2 \leq \frac{\beta_k^2}{2a_{\min}^2}\|b(k) - b^*\|^2 + 8\beta_k^2 y_{total}^2. \quad (13)$$

We are ready to present the main convergence result.

Theorem 4.7. *(Convergence of the* BID ADJUSTMENT ALGORITHM*): In Algorithm 1, let $0 < \beta_k < 2a_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Further, let $0 < r < \|b(1) - b^*\|$ and assume*

$$\alpha \leq \beta_k \leq B(r) := \frac{1}{2a_{\max}}\left(\frac{1}{2a_{\min}^2} + \frac{16y_{total}^2}{r^2}\right)^{-1}, \quad (14)$$

for all $k \in \mathbb{Z}_{\geq 1}$, for some $\alpha > 0$ (where recall that $a_{\max} = \max_n\{a_n\}$ and $a_{\min} = \min_n\{a_n\}$). Then, the following holds

(i) *there exists $l \in \mathbb{Z}_{\geq 1}$ such that $\|b(l) - b^*\| < r$ and for all $k \in [l-1]$, we have $\|b(k) - b^*\| \geq r$ with*

$$\|b(k+1) - b^*\| \leq \left(1 - \frac{\alpha}{2a_{\max}}\right)^{k/2}\|b(1) - b^*\|, \quad (15)$$

(ii) *for all $k \geq l$,*

$$\|b(k) - b^*\| \leq \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r. \quad (16)$$

Proof: Assume that $\|b(k) - b^*\| \geq r$ for some $k \in \mathbb{Z}_{\geq 1}$. Then, the upper bound on the stepsizes in the inequality (14) holds when r is replaced with $\|b(k) - b^*\|$, that is, $\beta_k \leq B(\|b(k) - b^*\|)$ for all $k \in \mathbb{Z}_{\geq 1}$. This is because $r \mapsto B(r)$ is strictly increasing in the domain $r > 0$. Proceeding with this replacement and reordering (14), we obtain

$$\beta_k\left(\frac{\|b(k) - b^*\|^2}{2a_{\min}^2} + 16y_{total}^2\right) \leq \frac{1}{2a_{\max}}\|b(k) - b^*\|^2,$$

or equivalently,

$$\frac{\beta_k}{2a_{\min}^2}\|b(k) - b^*\|^2 + 16\beta_k y_{total}^2 - \frac{1}{a_{\max}}\|b(k) - b^*\|^2$$
$$\leq -\frac{1}{2a_{\max}}\|b(k) - b^*\|^2. \quad (17)$$

Now consider the following inequalities

$$\|b(k+1) - b^*\|^2 = \|b(k+1) - b(k) + b(k) - b^*\|^2$$
$$= \|b(k+1) - b(k)\|^2 + \|b(k) - b^*\|^2$$
$$\quad + 2\langle b(k+1) - b(k), b(k) - b^*\rangle$$
$$\overset{(a)}{\leq} \frac{\beta_k^2}{2a_{\min}^2}\|b(k) - b^*\|^2 + 8\beta_k^2 y_{total}^2 + \|b(k) - b^*\|^2$$
$$\quad - \frac{\beta_k}{a_{\max}}\|b(k) - b^*\|^2 \quad (18a)$$

$$\overset{(b)}{\leq} \left(1 - \frac{\beta_k}{2a_{\max}}\right)\|b(k) - b^*\|^2, \quad (18b)$$

where in (a) we have used the bounds (12) and (13) from Lemmas 4.5 and 4.6, respectively, and the inequality (b) is implied by that in (17). Note that the inequality (17) is conservative in the sense that the term $16\beta_k y_{total}^2$ could be replaced with $8\beta_k y_{total}^2$ and the inequality (18b) would still follow. However, we opt for this conservativeness while defining the map $r \mapsto B(r)$ in (14) because it results into robustness guarantees for the algorithm as discussed in the forthcoming section. Therefore, (18b) holds whenever $\|b(k) - b^*\| \geq r$. By assumption, we have $0 < \left(1 - \frac{\beta_k}{2a_{\max}}\right) < 1$, $\|b(1) - b^*\| > r$, and $\beta_k \geq \alpha$ for all $k \in \mathbb{Z}_{\geq 1}$. Using these facts and applying (18b) recursively, we conclude part (i).

For part (ii), note that if $\|b(k) - b^*\| \geq r$ for some $k \geq l$, then $\|b(k+1) - b^*\| < \|b(k) - b^*\|$ by (18b). Therefore, to find an upper bound on $\|b(k) - b^*\|$ for all $k \geq l$, we only need to consider the case when $\|b(k) - b^*\| < r$. Plugging this bound in (18a) and neglecting the negative term,

$$\|b(k+1) - b^*\|^2 \leq \frac{\beta_k^2 r^2}{2a_{\min}^2} + 8\beta_k^2 y_{total}^2 + r^2. \quad (19)$$

From (14), we have $\frac{\beta_k^2 r^2}{2a_{\min}^2} + 16\beta_k^2 y_{total}^2 \leq \frac{\beta_k r^2}{2a_{\max}}$. The result now follows by upper bounding the right-hand side of (19) with the left-hand side of the above expression and then employing the bound on the stepsizes give in (14). $\qquad \square$

Remark 4.8. *(Convergence properties from Theorem 4.7):* Selecting stepsizes as per the requirements of Theorem 4.7 would amount to choosing $r > 0$ small enough so that $r < \|b(1) - b^*\|$, computing the right-hand side of (14), and selecting $\beta_k < 2a_n$ satisfying the bound (14). With this selection, the assertion (i) of Theorem 4.7 implies that bids reach the set $\mathcal{B}_r(b^*)$ in a finite number of steps and at a linear rate. Further, once bids reach the set $\mathcal{B}_r(b^*)$, assertion (ii) ensures that they remain in a neighborhood of b^*, where the size of the neighborhood is proportional to r (cf. (16)). In combination, the above facts mean that bids converge to any neighborhood of the efficient NE at a linear rate provided the stepsizes are selected appropriately. Specifically, define

$$k_m := 1 + \frac{2\log(r/\|b(1) - b^*\|)}{\log(1 - \alpha/2a_{\max})}.$$

Then, from (15), we deduce that there exists $\bar{k} \in [k_m]$ such that $\|b(\bar{k}) - b^*\| < r$. Further, $\|b(k) - b^*\| \leq \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r$ for all $k \geq k_m$. Note that as r becomes small, $B(r)$ gets small and so does α. Consequently, k_m becomes big, implying that the rate of convergence diminishes. This presents a trade-off between the desired precision and the rate of convergence. $\qquad \bullet$

Remark 4.9. *(Stopping criteria for the ISO):* From the proof of Theorem 4.7(i) note that, as long as $\|b(k) - b^*\| > r$, the distance to the efficient NE decreases. Therefore,

if $\|b(k) - b^*\| > r$ and $k < l$, then one can write

$$
\begin{aligned}
\|b(k+1) - b(k)\| &= \|b(k+1) - b^* + b^* - b(k)\| \\
&\geq \|b(k) - b^*\| - \|b(k+1) - b^*\| \\
&\stackrel{(a)}{\geq} \|b(k) - b^*\| - \left(1 - \frac{\alpha}{2a_{\max}}\right)^{1/2} \|b(k) - b^*\| \\
&= \left(1 - \left(1 - \frac{\alpha}{2a_{\max}}\right)^{1/2}\right)\|b(k) - b^*\|, \quad (20)
\end{aligned}
$$

where in (a) we have used (18b) and $\beta_k \geq \alpha$. Given this observation, if the ISO has an estimate of α and $a_{\max}$, then it can design a stopping criteria based on the distance between consecutive bids. In fact, if the ISO selects $\epsilon > 0$ and stops the iteration whenever $\|b(k+1) - b(k)\| \leq \epsilon$, then it has the guarantee that either of the following is satisfied

(i) the condition $\|b(k) - b^*\| > r$ and $k < l$ is met and from (20) we get

$$
\|b(k) - b^*\| \leq \epsilon \left(1 - \left(1 - \frac{\alpha}{2a_{\max}}\right)^{1/2}\right)^{-1}; \quad (21)
$$

(ii) $\|b(k) - b^*\| \leq r$; or
(iii) $k > l$ in which case from (16) we get

$$
\|b(k) - b^*\| \leq \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r.
$$

The ISO does not know the value of r; its value depends on the stepsizes that the generators select. Assuming that stepsizes are small, the ISO can adjust ϵ depending on the desired accuracy level to get the guarantee (21) for the k-th bid. For small ϵ, the stopping criteria might never be met if stepsizes are too big. $\bullet$

Remark 4.10. *(Scalability of* BID ADJUSTMENT ALGO- RITHM*):* At each iteration of the BID ADJUSTMENT AL- GORITHM, the computational burden for the generators is minimal and does not depend on the size of the net- work. The ISO on the other hand solves a linear program at each round. This computation is not much taxing as large-scale linear programs can be solved efficiently. Further, the rate of convergence of BID ADJUSTMENT ALGORITHM, restricted by the right-hand side of (14), does not depend on the number of generators present in the network. Finally, the communication burden for each generator is independent of the size of the network and the communication burden for the ISO scales linearly with the number of generators in the network. Due to these reasons, the BID ADJUSTMENT ALGORITHM is implementable in large-scale power networks. $\bullet$

4.3 Recovering the DC-OPF solution

Here we provide a procedure that allows the ISO to uniquely determine a dispatch decision once the iterative bidding process has stopped. The need for this arises from the observation that given $b(k)$ close to b^*, in general $x^{\mathrm{opt}}(b(k))$ might not be close to x^*, as (3) is a linear program. The procedure proposed here ensures that the generation profile of the dispatch decision is close to the optimum of the DC-OPF problem (2).

DETERMINE GENERATION: The ISO initiates the procedure after stopping the BID ADJUSTMENT ALGORITHM at some iteration $k \in \mathbb{Z}_{\geq 1}$. The procedure has two steps. *[Step 1:]* The ISO requests each generator n to provide the gener- ation level q_n^{offer} that the generator is willing to produce at bid $b_n(k)$. *[Step 2:]* The ISO projects the vector of obtained generation levels q^{offer} onto the feasibility set of the DC-OPF prob- lem (2) and this projected vector, denoted q^{disp}, forms the dispatch decision.

Note that, unlike in BID ADJUSTMENT ALGORITHM, generators are non-strategic and price-takers when DE- TERMINE GENERATION is executed. This is justified by the observation that generators know neither the ISO's stopping criteria nor the accuracy of the final bid profile of the network, thus ruling out the possibility of agents' anticipating the outcome of the game. We elaborate on this point in Remark 4.12 below.

The next result bounds the difference $q^{\mathrm{disp}} - x^*$.

Proposition 4.11. *(Bound on the dispatch decision made by* DETERMINE GENERATION*): Let k be the iteration at which the* BID ADJUSTMENT ALGORITHM *stops with bids $b(k) \in \mathbb{R}_{\geq 0}^N$. Under the assumption that in the* DETERMINE GENERATION *procedure, each generator n finds q_n^{offer} by maximizing its payoff given bid $b_n(k)$, that is, for all n,*

$$
q_n^{\mathrm{offer}} = \mathrm{argmax}_{q \geq 0} b_n(k)q - f_n(q), \quad (22)
$$

the obtained dispatch decision satisfies

$$
\|q^{\mathrm{disp}} - x^*\| \leq \frac{\|b(k) - b^*\|}{2a_{\min}}.
$$

Proof: For each $n \in [N]$, we have

$$
q_n^{\mathrm{offer}} = \mathrm{argmax}_{q \geq 0} b_n(k)q - f_n(q) = \frac{b_n(k) - c_n}{2a_n}.
$$

The last equality follows from Lemma 4.3. Further note that for each $n \in [N]$, $b^* = 2a_n x_n^* + c_n$. Therefore,

$$
\|b(k) - b^*\| = \|2\mathrm{diag}(a)(q^{\mathrm{offer}} - x^*)\| \geq 2a_{\min}\|q^{\mathrm{offer}} - x^*\|,
$$

where $\mathrm{diag}(a)$ is the diagonal matrix with diagonal a. From DETERMINE GENERATION, q^{disp} is the projection of q^{offer} to the feasibility set of the DC-OPF problem. Since the projection to a convex set is Lipschitz with constant 1 and x^* is feasible, $\|q^{\mathrm{disp}} - x^*\| \leq \|q^{\mathrm{offer}} - x^*\|$ and the result follows. $\square$

This result implies that the accuracy of the dispatch decision can be determined as a function of the final bid's accuracy.

Remark 4.12. *(Strategic interactions in* DETERMINE GEN- ERATION*):* One could possibly model the interaction be- tween generators in DETERMINE GENERATION as a game where players are generators, actions are what they offer to generate in Step 1, and utility is their profit once q^{disp} is determined in Step 2. Such a game is parameterized by the converged bid. However, generators do not know the converged bid, the stopping criteria of the ISO, or even

the network structure or the total number of generators involved, that would critically influence the projection done in Step 2. This lack of access to critical information about the game provides justification for generators acting truthfully by revealing their utility-maximizing generation level q_n^{offer} in Step 1. Note that unlike BID ADJUSTMENT ALGORITHM, the DETERMINE GENERATION is not iterative. Allowing generators to change their offer iteratively in DETERMINE GENERATION would be more conducive to having generators behave strategically, but we do not pursue this interesting direction here. •

5 ROBUSTNESS OF THE BID ADJUSTMENT ALGORITHM

Here we study the robustness properties of the BID ADJUSTMENT ALGORITHM in a variety of scenarios. We first show that the introduction of disturbances in the bid update mechanism does not destroy the algorithm convergence properties. We then study robustness against either an individual agent or colluding agents changing their strategy to get a higher payoff.

5.1 Robustness to disturbances

Here we establish the robustness properties of the BID ADJUSTMENT ALGORITHM in the presence of disturbances by characterizing its input-to-state stability (ISS) properties [35]. Let $d : \mathbb{Z}_{\geq 1} \to \mathbb{R}^N$ model the disturbance to the bid update mechanism. Such disturbances might arise from agents using different stepsizes than the prescribed one or other disruption to the prescribed bid update scheme. The resulting perturbed version of the BID ADJUSTMENT ALGORITHM can be written as the following discrete-time dynamical system

$$b(k+1) = [b(k) + \beta_k(x^{\text{opt}}(k) - q(k)) + d(k)]^+, \quad (23a)$$
$$x^{\text{opt}}(k+1) \in \text{Sol}_{\text{sopf}}(b(k+1)), \quad (23b)$$
$$q(k+1) = \text{Sol}_{\text{eff}}(b(k+1)), \quad (23c)$$

where $\text{Sol}_{\text{sopf}} : \mathbb{R}_{\geq 0}^N \rightrightarrows \mathbb{R}_{\geq 0}^N$ and $\text{Sol}_{\text{eff}} : \mathbb{R}_{\geq 0}^N \to \mathbb{R}_{\geq 0}^N$ map a bid profile to the set of optimizers of problem (3) and (6), resp. Note that Sol_{sopf} is a set-valued map since (3) is a linear program. If $d \equiv 0$, then (23) represents the k-th iteration of the BID ADJUSTMENT ALGORITHM. We next show that the perturbed version (23) retains the algorithm convergence properties provided the magnitude of the disturbance satisfies an upper bound dependent on the bid state.

Proposition 5.1. *(The* BID ADJUSTMENT ALGORITHM *is robust to perturbations in the bid update): For dynamics* (23), *let the hypotheses of Theorem 4.7 hold and assume that* $b_n(k) \geq c_n$ *for all* $n \in [N]$ *and* $k \in \mathbb{Z}_{\geq 1}$. *Let* $0 < \theta < \frac{1}{6}\left(1 - \frac{\alpha}{2a_{\max}}\right)$ *and assume* $\|d(k)\| \leq \theta\|b(k) - b^*\|$ *for all* $k \in \mathbb{Z}_{\geq 1}$. *Then, the following holds*

(i) *there exists* $l \in \mathbb{Z}_{\geq 1}$ *such that* $\|b(l) - b^*\| < r$ *and, for all* $k \in [l-1]$, *we have* $\|b(k) - b^*\| \geq r$ *with*

$$\|b(k+1) - b^*\|$$

$$\leq \left(1 - \frac{\alpha}{2a_{\max}} + 2\theta + 4\theta^2\right)^{k/2} \|b(1) - b^*\|, \quad (24)$$

(ii) *for all* $k \geq l$,

$$\|b(k) - b^*\| \leq \left(1 + \frac{B(r)}{2a_{\max}} + 2\theta + 4\theta^2\right)^{1/2} r. \quad (25)$$

Proof: Since $b_n(k) \geq c_n$, we obtain for dynamics (23), $q_n(k) = \frac{b_n(k) - c_n}{2a_n}$, for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Moreover, mimicking Lemma 4.4, we rewrite the bid update (23a) as

$$b(k+1) = b^{\text{coc}}(k+1) + \beta_k(x^{\text{opt}}(k) - x^*) + d(k), \quad (26)$$

for all $k \in \mathbb{Z}_{\geq 1}$. Using (26) and following the steps of Lemma 4.5 for dynamics (23a) we get,

$$\langle b(k+1) - b(k), b(k) - b^*\rangle \leq \langle d(k), b(k) - b^*\rangle$$
$$- \frac{\beta_k}{2a_{\max}}\|b(k) - b^*\|^2, \quad (27)$$

for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Similarly, from the reasoning of Lemma 4.6 we obtain

$$\|b(k+1) - b(k)\|^2$$
$$\leq \frac{\beta_k^2}{2a_{\min}^2}\|b(k) - b^*\|^2 + 2\left(\|\beta_k(x^{\text{opt}}(k) - x^*) + d(k))\|\right)^2$$
$$\leq \frac{\beta_k^2}{2a_{\min}^2}\|b(k) - b^*\|^2 + 4\beta_k^2\|x^{\text{opt}}(k) - x^*\|^2 + 4\|d(k)\|^2$$
$$\leq \frac{\beta_k^2}{2a_{\min}^2}\|b(k) - b^*\|^2 + 16\beta_k^2 y_{\text{total}}^2 + 4\|d(k)\|^2 \quad (28)$$

for all $k \in \mathbb{Z}_{\geq 1}$ for dynamics (23a). Employing (27) and (28), assuming $\|b(k) - b^*\| \geq r$, and writing the set of inequalities (18) with $\alpha \leq \beta_k$, we deduce the following

$$\|b(k+1) - b^*\|^2 \leq \left(1 - \frac{\alpha}{2a_{\max}}\right)\|b(k) - b^*\|^2 + 4\|d(k)\|^2$$
$$+ 2\langle d(k), b(k) - b^*\rangle. \quad (29)$$

Finally, using $\|d(k)\| \leq \theta\|b(k) - b^*\|$ we get

$$\|b(k+1) - b^*\|^2 \leq \left(1 - \frac{\alpha}{2a_{\max}} + 2\theta + 4\theta^2\right)\|b(k) - b^*\|^2. \quad (30)$$

Iteratively, we obtain (24). The bound (25) can be computed similarly as done in the proof of Theorem 4.7. $\square$

Similar to the convergence guarantees of Theorem 4.7, the above result establishes that the perturbed version of the algorithm (23) converges to a neighborhood of the efficient NE provided the stepsizes and the disturbance satisfy appropriate bounds, and that the size of this neighborhood is tunable as a function of these. The next result complements Proposition 5.1 by giving an alternative representation of robustness of (23). It establishes two properties: first, when the disturbance is bounded (not necessarily satisfying the bound of Proposition 5.1), the bids remain bounded; second, when the disturbance goes to zero, bids satisfy (16) asymptotically. Note that both these results do not follow directly from Proposition 5.1, justifying the need for a formal proof.

Proposition 5.2. *(Bounded disturbance implies bounded bids for* BID ADJUSTMENT ALGORITHM): *For dynam-*

ics (23), *let the hypotheses of Theorem 4.7 hold and assume that* $b_n(k) \geq c_n$ *for all* $n \in [N]$ *and* $k \in \mathbb{Z}_{\geq 1}$. *Let* $\|d(k)\| \leq d_{\max}$ *for all* $k \in \mathbb{Z}_{\geq 1}$ *and let* $\theta \in \left(0, \frac{1}{6}\left(1 - \frac{\alpha}{2a_{\max}}\right)\right)$. *Then, the following holds for all* $k \in \mathbb{Z}_{\geq 1}$,

$$\|b(k) - b^*\| \leq \left(1 - \frac{\alpha}{2a_{\max}} + 2\theta + 4\theta^2\right)^{k/2} \|b(1) - b^*\|$$
$$+ G(r, \theta, d_{\max}), \qquad (31)$$

where $G(r, \theta, d_{\max}) := \max\{G_1(r, d_{\max}), G_2(\theta, d_{\max})\}$ *and*

$$G_1(r, d_{\max}) := \left(\frac{B(r)r^2}{2a_{\max}} + (2d_{\max} + r)^2\right)^{1/2},$$

$$G_2(\theta, d_{\max}) := \left(2 + \frac{1}{\theta}\right)d_{\max}.$$

As a consequence, as $k \to \infty$, *if* $\|d(k)\| \to 0$, *then*

$$\max\left\{\|b(k) - b^*\|, \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r\right\} \to \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r.$$

Proof: We first show that if for some $k \in \mathbb{Z}_{\geq 1}$, $\|b(k) - b^*\| \leq G(r, \theta, d_{\max})$, then $\|b(l) - b^*\| \leq G(r, \theta, d_{\max})$ for all $l \geq k$. To this end, as a first case, assume that $r \leq \|b(k) - b^*\| \leq G(r, \theta, d_{\max})$. Then, following the steps of the proof of Proposition 5.1, we arrive at (29). If $\|d(k)\| \leq \theta\|b(k) - b^*\|$, then we get the inequality (30) which implies that $\|b(k+1) - b^*\| \leq \|b(k) - b^*\| \leq G(r, \theta, d_{\max})$. On the other hand, if $\|d(k)\| > \theta\|b(k) - b^*\|$, then using this bound in (29), we get

$$\|b(k+1) - b^*\|^2 < \theta^{-2}\left(\left(1 - \frac{\alpha}{2a_{\max}}\right)\|d(k)\|^2 + 4\|d(k)\|^2\theta^2 \right.$$
$$\left. + 2\theta\|d(k)\|^2\right)$$
$$< \theta^{-2}\left(1 + 4\theta + 4\theta^2\right)\|d(k)\|^2.$$

Thus, using $\|d(k)\| \leq d_{\max}$, we get $\|b(k+1) - b^*\| < G_2(\theta, d_{\max}) \leq G(r, \theta, d_{\max})$. As a second case, assume $\|b(k) - b^*\| < r$. Note that $r < G(r, \theta, d_{\max})$, and so $\|b(k) - b^*\| < G(r, \theta, d_{\max})$. For this case, using $\|b(k) - b^*\| < r$ and inequalities (27) and (28), we get as in (18a) that

$$\|b(k+1) - b^*\|^2 \leq \frac{\beta_k^2 r^2}{2a_{\min}^2} + 16\beta_k^2 y_{\text{total}}^2 + 4\|d(k)\|^2 + r^2$$
$$+ 2r\|d(k)\|.$$

Now applying bounds $\|d(k)\| \leq d_{\max}$ and $\beta_k \leq B(r)$, we obtain $\|b(k+1) - b^*\| \leq G_1(r, d_{\max})$. Hence, we arrive at the conclusion that if $\|b(k) - b^*\| \leq G(r, \theta, d_{\max})$, then $\|b(l) - b^*\| \leq G(r, \theta, d_{\max})$ for all $l \geq k$.

Consider now the case when for some $k \in \mathbb{Z}_{\geq 1}$, $\|b(k) - b^*\| > G(r, \theta, d_{\max})$. By definition of $G(r, \theta, d_{\max})$, this implies that $\|b(k) - b^*\| > r$ and $\|b(k) - b^*\| > \frac{d(k)}{\theta}$. Therefore, from the proof of Proposition 5.1, we arrive at (30). Finally, combining the reasoning of the two cases when $\|b(k) - b^*\|$ is greater than or less than or equal to $G(r, \theta, d_{\max})$, we obtain the inequality (31). The final limit for the case when $\|d(k)\| \to 0$ follows from that fact that as $k \to \infty$, the first term of (31) converges to zero and as $d_{\max}$ tends to zero, $G(r, \theta, d_{\max})$ tends to $\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r$. $\qquad \square$

One can observe from (31) that the limiting behavior of the bids depend on the magnitude of r and $d_{\max}$: if r is designed to be small enough and if $d_{\max}$ is small enough, or this bound becomes small as the algorithm iterates, then the bids do converge to a small neighborhood of b^*. As an aside, in the theory of ISS for discrete-time dynamical systems [35], one typically would conclude Proposition 5.2 from Proposition 5.1. However, the traditional ISS results require asymptotic convergence of the unperturbed dynamics, (i.e., dynamics (23) with $d \equiv 0$) to a point. This is not the case here and hence, we provide a formal proof.

Remark 5.3. *(*Bid Adjustment Algorithm *is robust to variation in stepsizes):* In practice, given that generators are competing and do not share information with each other, it is conceivable that they do not agree on a common stepsize. Propositions 5.1 and 5.2 provide a way to quantify the performance of the algorithm when the stepsizes are different. Specifically, let β_k, $k \in \mathbb{Z}_{\geq 1}$, denote a common set of stepsizes for all generators that satisfies the hypotheses of Theorem 4.7 and hence, guarantees the convergence properties outlined therein. Assume that each generator selects a different stepsize at each iteration, denoted as $\beta_{k,n}$, $k \in \mathbb{Z}_{\geq 1}$, for generator n. Then, the bid iteration in Step 3 of the Bid Adjustment Algorithm can be written as (23) where now

$$d_n(k) = (\beta_{k,n} - \beta_k)(x_n^{\text{opt}}(k) - q_n(k))$$

for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Now if the variation in stepsizes, i.e., the quantity $\beta_{k,n} - \beta_k$, is bounded above by a particular function of the distance of the bid-state to the efficient NE, then the linear convergence and the ultimate bound is guaranteed by Proposition 5.1. On the other hand, if the variation in stepsizes do not depend on the state but are bounded then, then the bids still converge asymptotically to a neighborhood of the efficient NE by Proposition 5.2. The assumption of $b_n(k) \geq c_n$ for all n and k holds whenever the stepsizes are positive for all agents at all times (cf. Lemma 4.3). •

5.2 Robustness to deviation in bid update

We illustrate here another aspect of robustness of the Bid Adjustment Algorithm by establishing that, if all generators follow the bid update scheme, then there is no incentive for any generator to deviate from it. We next formalize these notions. Assume that all generators, except $\tilde{n} \in [N]$, follow the Bid Adjustment Algorithm, and that $\tilde{n}$ follows an arbitrary strategy to update its bids. Then, one can write the Bid Adjustment Algorithm under this deviation as

$$b_{-\tilde{n}}(k+1) = [b_{-\tilde{n}}(k) + \beta_k(x_{-\tilde{n}}^{\text{opt}}(k) - q_{-\tilde{n}}(k))]^+, \quad (32a)$$

$$b_{\tilde{n}}(k+1) = \mathcal{H}_{\tilde{n}}^{(k)}\left(\{b_{\tilde{n}}(t), x_{\tilde{n}}^{\text{opt}}(t), q_{\tilde{n}}(t)\}_{t=1}^k\right), \quad (32b)$$

$$x^{\text{opt}}(k+1) \in \text{Sol}_{\text{sopf}}(b(k+1)), \quad (32c)$$

$$q(k+1) \in \text{Sol}_{\text{eff}}(b(k+1)), \quad (32d)$$

where the maps $\{\mathcal{H}_{\tilde{n}}^{(k)} : \mathbb{R}_{\geq 0}^{3k} \to \mathbb{R}_{\geq 0}\}_{k=1}^{\infty}$ represent the update scheme of $\tilde{n}$ at iterations $1, 2, \ldots$ Recall that the subscript $-\tilde{n}$ denotes the vector without the component corresponding to the generator $\tilde{n}$. Note that (32b) implies that at each iteration k, the generator $\tilde{n}$ only knows the bids it made and the quantities the ISO demanded from it up until iteration k.

We next introduce the notion of "incentive to deviate" from the BID ADJUSTMENT ALGORITHM for the generator $\tilde{n}$. A natural way to quantify incentives for a generator is in terms of the payoff (4): a generator has an incentive to deviate if this would bring in a higher payoff, when the ISO stops the iteration, than not deviating. This is formalized below.

Definition 5.4. *(Incentive to deviate from* BID ADJUST-MENT ALGORITHM*): Let $r > 0$ and assume that the stepsizes for any execution of* (32) *satisfy the hypotheses of Theorem 4.7. Then, the generator $\tilde{n} \in [N]$ has an* incentive to deviate *from the* BID ADJUSTMENT ALGORITHM *if there exists an execution of* (32) *and $l \in \mathbb{Z}_{\geq 1}$ such that*

$$u_{\tilde{n}}(b_{\tilde{n}}(k), x_{\tilde{n}}^{\mathrm{opt}}(k)) > u_{\tilde{n}}^{\max}, \tag{33}$$

for all $k \geq l$, where

$$u_{\tilde{n}}^{\max} := \max\Big\{ u_{\tilde{n}}(b_{\tilde{n}}, x_{\tilde{n}}^{\mathrm{opt}}(b)) \Big| \|b - b^*\| \leq \Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{\frac{1}{2}} r$$
$$\text{and } x^{\mathrm{opt}}(b) \in \mathrm{Sol}_{\mathrm{sopf}}(b) \Big\}. \tag{34}$$

Here, an execution of (32) *is a trajectory $\mathbb{Z}_{\geq 1} \ni k \mapsto (b(k), x^{\mathrm{opt}}(k))$ starting at some $b(1)$ satisfying $b_n(1) \geq c_n$ for all $n \in [N]$ and following the update scheme of* (32).

In the above definition, recall the short-hand notation $x^{\mathrm{opt}}(k)$ for $x^{\mathrm{opt}}(b(k))$. Equation (33) implies that the generator $\tilde{n}$ has an incentive to deviate if, after a finite number of iterations, it is guaranteed a higher payoff than what it might eventually get if it follows the BID ADJUSTMENT ALGORITHM. This captures the fact that the generator does not know when the ISO might stop the bid and hence it would deviate only when it is guaranteed to get a higher payoff after a finite number of steps. The next result shows that there is no incentive to deviate from the BID ADJUSTMENT ALGORITHM.

Proposition 5.5. *(Robustness to deviation from* BID AD-JUSTMENT ALGORITHM*): For dynamics* (32), *let the hypotheses of Theorem 4.7 hold and assume that $b_n(k) \geq c_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Also, assume that at each iteration $k \in \mathbb{Z}_{\geq 1}$, the ISO selects a solution $x^{\mathrm{opt}}(k) \in \mathrm{Sol}_{\mathrm{sopf}}(b(k))$ that is a vertex of the feasibility set of the problem* (3) *given bids $b(k)$. Then, no generator has an incentive to deviate from the* BID ADJUSTMENT ALGORITHM.

Proof: We reason by contradiction. Assume that a generator $\tilde{n}$ has an incentive to deviate from the BID ADJUSTMENT ALGORITHM. That is, there exists an execution of (32) and $l \in \mathbb{Z}_{\geq 1}$ such that (33) holds for all $k \geq l$. By definition,

$$u_{\tilde{n}}^{\max} \geq b_{\tilde{n}}^* x_{\tilde{n}}^* - f_{\tilde{n}}(x_{\tilde{n}}^*). \tag{35}$$

Now consider the map

$$\mathbb{R}_{\geq 0} \ni b \mapsto g_{\tilde{n}}(b) := \max\{bq - f_{\tilde{n}}(q) \mid q \geq 0\}.$$

From (6), we get $g_{\tilde{n}}(b_{\tilde{n}}^*) = b_{\tilde{n}}^* x_{\tilde{n}}^* - f_{\tilde{n}}(x_{\tilde{n}}^*)$. Further, using (1), one can show that this map is continuous, strictly increasing in the domain $b \geq c_{\tilde{n}}$, and $g_{\tilde{n}}(b) \to \infty$ as $b \to \infty$. These facts along with (35) imply that there exists a unique $b_{\tilde{n}}^{\max} \geq b_{\tilde{n}}^*$ such that $g_{\tilde{n}}(b_{\tilde{n}}^{\max}) = u_{\tilde{n}}^{\max}$, $g_{\tilde{n}}(b) > u_{\tilde{n}}^{\max}$ for all $b > b_{\tilde{n}}^{\max}$, and $g_{\tilde{n}}(b) < u_{\tilde{n}}^{\max}$ for all $c_{\tilde{n}} \leq b < b_{\tilde{n}}^{\max}$. Then, (33) reads as

$$u_{\tilde{n}}(b_{\tilde{n}}(k), x_{\tilde{n}}^{\mathrm{opt}}(k)) > g_{\tilde{n}}(b_{\tilde{n}}^{\max}), \tag{36}$$

for all $k \geq l$. From the above expression, we deduce that $b_{\tilde{n}}(k) \geq b_{\tilde{n}}^{\max}$ for all $k \geq l$. Indeed otherwise, there exists $\tilde{k} \geq l$ such that $b_{\tilde{n}}(\tilde{k}) < b_{\tilde{n}}^{\max}$. This further implies that

$$u_{\tilde{n}}(b_{\tilde{n}}(\tilde{k}), x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k})) = b_{\tilde{n}}(\tilde{k}) x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k}) - f_{\tilde{n}}(x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k}))$$
$$\leq g_{\tilde{n}}(b_{\tilde{n}}(\tilde{k})) < g_{\tilde{n}}(b_{\tilde{n}}^{\max}),$$

contradicting (36). In the above expression, the first inequality follows from the definition of $g_{\tilde{n}}$ and the second follows from the fact that $g_{\tilde{n}}$ is strictly increasing.

The above reasoning has helped us establish that $b_{\tilde{n}}(k) \geq b_{\tilde{n}}^{\max} \geq b_{\tilde{n}}^*$ for all $k \geq l$. Note that $x_{\tilde{n}}^{\mathrm{opt}}(k) > 0$ for all $k \geq l$ because otherwise $u_{\tilde{n}}(b_{\tilde{n}}(k), x_{\tilde{n}}^{\mathrm{opt}}(k)) = 0$ and (36) gets violated. By assumption, there exists at least one more generator connected to the bus $i(\tilde{n})$ to which $\tilde{n}$ is connected to. For now assume that there is only one other generator $\bar{n} \in [N]$ connected to $i(\tilde{n})$. Since for all $k \geq l$, $x^{\mathrm{opt}}(k)$ is a solution of (3), from the fact that $x_{\tilde{n}}^{\mathrm{opt}}(k) > 0$, we deduce $b_{\bar{n}}(k) \geq b_{\tilde{n}}(k) \geq b_{\tilde{n}}^{\max}$, for all $k \geq l$. Now let

$$q_{\bar{n}}^{\max} := \inf_{b \geq b_{\tilde{n}}^{\max}} \mathrm{argmax}\{bq - f_{\bar{n}}(q) \mid q \geq 0\}.$$

Note that $q_{\bar{n}}^{\max} > 0$ because of the facts: (i) $b_{\tilde{n}}^* = b_{\tilde{n}}^* \leq b_{\tilde{n}}^{\max}$; (ii) $\mathrm{argmax}\{b_{\tilde{n}}^* q - f_{\bar{n}}(q) \mid q \geq 0\} = x_{\tilde{n}}^* > 0$; and (iii) $b \mapsto \mathrm{argmax}\{bq - f_{\bar{n}}(q) \mid q \geq 0\}$ is nondecreasing. Since $b_{\bar{n}}(k) \geq b_{\tilde{n}}^{\max}$ for all $k \geq l$, we obtain $q_{\bar{n}}(k) \geq q_{\bar{n}}^{\max}$ for all $k \geq l$ (see Step 4 of the BID ADJUSTMENT ALGORITHM for the definition of $q_{\bar{n}}(k)$). Thus, if $b_{\bar{n}}(k) > b_{\tilde{n}}(k)$ for some $k \geq l$, then $x_{\bar{n}}^{\mathrm{opt}}(k) = 0$ (because $x^{\mathrm{opt}}(k)$ is an optimizer of (3) given bids $b(k)$) and $b_{\tilde{n}}(k) > b_{\tilde{n}}^{\max}$. As a consequence,

$$b_{\bar{n}}(k+1) = b_{\bar{n}}(k) - \beta_k q_{\bar{n}}(k) \leq b_{\bar{n}}(k) - \alpha q_{\bar{n}}^{\max}. \tag{37}$$

Therefore, if $b_{\bar{n}}(k) > b_{\tilde{n}}(k)$ for some $k \geq l$, then from (37) we deduce that there exists a finite $\tilde{k} > k$ such that, either $b_{\bar{n}}(\tilde{k}) < b_{\tilde{n}}(\tilde{k})$ or $b_{\bar{n}}(\tilde{k}) = b_{\tilde{n}}(\tilde{k})$. In the former case, $u_{\tilde{n}}(b_{\tilde{n}}(\tilde{k}), x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k})) = 0$ as $x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k}) = 0$. This contradicts (33). In the latter case, two further cases arise based on the vertex solution that the ISO selects at $\tilde{k}$. In the first one, we get $x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k}) = 0$ implying $u_{\tilde{n}}(b_{\tilde{n}}(\tilde{k}), x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k})) = 0$ and contradicting (33). In the second one, we obtain $x_{\bar{n}}^{\mathrm{opt}}(k) = 0$, implying $b_{\bar{n}}(k+1) < b_{\tilde{n}}(k+1)$. This further yields $u_{\tilde{n}}(b_{\tilde{n}}(\tilde{k}+1), x_{\tilde{n}}^{\mathrm{opt}}(\tilde{k}+1)) = 0$, thereby, contradicting (33). Finally, if there are other generators connected to $i(\tilde{n})$ that follow the BID ADJUSTMENT ALGORITHM, then one can carry out the same reasoning as done above

and show that we contradict (33). This completes the proof. □

Remark 5.6. *(Generalization of Proposition 5.5):* In the proof of Proposition 5.5, we have not used at any point that the generators connected at buses other than the one that $\tilde{n}$ is connected follow the BID ADJUSTMENT ALGORITHM. In fact, independently of how such generators update their bids, the BID ADJUSTMENT ALGORITHM ensures that $\tilde{n}$ does not have any incentive to deviate. This is a useful property which we use later when studying robustness to collusion. •

Remark 5.7. *(Other notions of "incentive to deviate"):* In Definition 5.4, one can impose the condition of higher payoff (33) to hold for all executions of (32). If this condition holds, then the generator has an even stronger incentive to deviate from the BID ADJUSTMENT ALGORITHM. However, by Proposition 5.5, there does not exist such strong incentive to deviate. This is because the result shows that there does not exist any execution of (32) for which (33) holds. Alternatively, one can replace the condition (33) in Definition 5.4 with the requirement that there exists an execution of (32) along which

$$\limsup_{k\to\infty} u_{\tilde{n}}(b_{\tilde{n}}(k), x_{\tilde{n}}^{\mathrm{opt}}(k)) > u_{\tilde{n}}^{\max} \tag{38}$$

holds. This inequality means that there exists an execution of (32) in which the generator $\tilde{n}$ gets a higher payoff than $u_{\tilde{n}}^{\max}$ infinitely often. Since the ISO can stop the iterations at any time, the generator is not guaranteed a higher payoff, but the possibility is still there. We conjecture that the BID ADJUSTMENT ALGORITHM is not robust to this notion of weak incentive to deviate. However, the obfuscation of the stopping criteria by the ISO makes such a weak incentive not enough for a rational generator to deviate. •

5.3 Robustness to collusion

Here we study the robustness of the BID ADJUSTMENT ALGORITHM against collusion. Collusion refers to the action of a set of generators to share among themselves information about their bids and generation demands by the ISO, with the goal of getting a higher profit, possibly by deviating from the bid update scheme. The following makes this notion formal.

Definition 5.8. *(Collusion between generators): A group of generators $\mathcal{J} \subset [N]$ form a collusion if at each iteration $k \in \mathbb{Z}_{\geq 1}$ of the algorithm, each generator $n \in \mathcal{J}$,*

   (i) *has the information*

$$\mathcal{I}_k := \{(b_r(t), x_r^{\mathrm{opt}}(t)) \mid r \in \mathcal{J}, t \in [k]\}, \text{ and}$$

   (ii) *determines its next bid $b_n(k + 1)$ based on the information $\mathcal{I}_k$, not necessarily following the update scheme (Step 3) of the BID ADJUSTMENT ALGORITHM.*

An iteration of the BID ADJUSTMENT ALGORITHM under a collusion between a group of generators $\mathcal{J} \subset [N]$ is given by the following dynamics

$$b_n(k + 1) = [b_n(k) + \beta_k(x_n^{\mathrm{opt}}(k) - q_n(k))]^+, \forall n \notin \mathcal{J}, \tag{39a}$$

$$b_n(k + 1) = \mathcal{H}_n^{(k)}\Big(\mathcal{I}_k, \{q_n(t)\}_{t=1}^k\Big), \forall n \in \mathcal{J} \tag{39b}$$

$$x^{\mathrm{opt}}(k + 1) \in \mathrm{Sol}_{\mathrm{sopf}}(b(k + 1)), \tag{39c}$$

$$q(k + 1) \in \mathrm{Sol}_{\mathrm{eff}}(b(k + 1)), \tag{39d}$$

where maps $\{\mathcal{H}_n^{(k)} : \mathbb{R}_{\geq 0}^{(2|\mathcal{J}|+1)k} \to \mathbb{R}_{\geq 0}\}_{n \in \mathcal{J}, k=1,2,\ldots}$ represent the update scheme of generators in collusion. Notice that for each generator n, the quantity $q_n(k)$, for all $k \in \mathbb{Z}_{\geq 1}$, is part of its private information, irrespective of the fact that n belongs to $\mathcal{J}$ or not. Next, we define what it means for the group of generators $\mathcal{J}$ to have an incentive to collude.

Definition 5.9. *(Incentive to collude): Let $r > 0$ and assume that the stepsizes for any execution of (39) satisfy the hypotheses of Theorem 4.7. Then, the group of generators $\mathcal{J}$ has an* incentive to collude *under the* BID ADJUSTMENT ALGORITHM *if there exists an execution of (39), a generator $\tilde{n} \in \mathcal{J}$, and $l \in \mathbb{Z}_{\geq 1}$ such that*

$$u_{\tilde{n}}(b_{\tilde{n}}(k), x_{\tilde{n}}^{\mathrm{opt}}(k)) > u_{\tilde{n}}^{\max}, \tag{40}$$

for all $k \geq l$, where $u_{\tilde{n}}^{\max}$ is defined in (34). An execution of (39) is defined analogously as in Definition 5.4.

This notion essentially says that there is an incentive to collude for the generators in $\mathcal{J}$ if there exists at least one execution of (39) along which at least one generator in $\mathcal{J}$ gets a higher payoff after finite number of steps. The next result shows that no group of generators has an incentive to collude provided there is at least one generator at each bus with generation that follows the BID ADJUSTMENT ALGORITHM.

Proposition 5.10. *(Robustness to collusion under the* BID ADJUSTMENT ALGORITHM*): For dynamics (39), let the hypotheses of Theorem 4.7 hold and assume that $b_n(k) \geq c_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Also, assume that at each iteration $k \in \mathbb{Z}_{\geq 1}$, the ISO selects a solution $x^{\mathrm{opt}}(k) \in \mathrm{Sol}_{\mathrm{sopf}}(b(k))$ that is a vertex of the feasibility set of (3) given bids $b(k)$. Assume that at each bus that has generators connected to it, there exists at least one generator that follows the update scheme of the* BID ADJUSTMENT ALGORITHM*. Denote these generators by $\mathcal{K} \subset [N]$. Then, there is no incentive to collude for any group of generators contained in $[N] \setminus \mathcal{K}$.*

Proof: Let $\mathcal{J} \subset [N] \setminus \mathcal{K}$ be a group of generators that form a collusion. Assume first Scenario 1 where each generator in $\mathcal{J}$ is connected to a different bus. By hypotheses, there exists at least one other generator following the BID ADJUSTMENT ALGORITHM at the bus where a generator in $\mathcal{J}$ is connected to. Thus, mimicking the proof of Proposition 5.5 (cf. Remark 5.6), at each bus, no generator has an incentive to deviate from the BID ADJUSTMENT ALGORITHM. By Definition 5.4, this implies that there does not exist any execution of (39) for which (40) holds for any generator in $\mathcal{J}$. Hence,

generators in $\mathcal{J}$ do not have an incentive to collude.

Next, consider Scenario 2, where at least a bus, say $i \in [N_b]$, has more than one generator from $\mathcal{J}$, that is, $\mathcal{J}_i := G_i \cap \mathcal{J}$ has cardinality larger than or equal to 2. Let $\bar{n} \in G_i$ be the generator at i that follows BID ADJUSTMENT ALGORITHM. For the sake of contradiction, assume the existence of a generator $\tilde{n} \in \mathcal{J}_i$ for which (40) holds for some execution of (39). Since the ISO selects a vertex solution at each iteration $k \in \mathbb{Z}_{\geq 1}$, we deduce that for all $k \geq l$, all other generators in $\mathcal{J}_i$ get zero production signal from the ISO, i.e., $x_n^{\mathrm{opt}}(k) = 0$ for all $n \in \mathcal{J}_i \setminus \{\tilde{n}\}$ and $k \geq l$. Therefore, for the purpose of analysis, one can neglect the generators in $\mathcal{J}_i \setminus \{\tilde{n}\}$ and assume that only $\tilde{n}$ and $\bar{n}$ are connected to i. Again, mimicking the proof of Proposition 5.5, we deduce that $\tilde{n}$ does not have an incentive to deviate and so (40) does not hold, a contradiction. Since i is arbitrary, generators in $\mathcal{J}$ do not have an incentive to collude either. $\square$

An incentive to collude can be defined in other ways. One possibility is to say there is incentive to collude if every generator in the collusion gets a higher payoff after a finite number of steps. Proposition 5.10 shows that, under the assumed hypotheses, such a scenario does not occur as there is not even a single generator that gets a higher payoff after a finite number of iterations. Another possible notion is that there is incentive to collude if the aggregate payoff of the colluding generators is higher after a finite number of steps, as formalized next.

Definition 5.11. *(Incentive to collude –aggregate payoff): Let $r > 0$ and assume that the stepsizes for any execution of (39) satisfy the hypotheses of Theorem 4.7. Then, the group of generators $\mathcal{J}$ has an* incentive to collude *under the* BID ADJUSTMENT ALGORITHM *if there exists an execution of (39) and $l \in \mathbb{Z}_{\geq 1}$ such that*

$$\sum_{n \in \mathcal{J}} u_n(b_n(k), x_n^{\mathrm{opt}}(k)) > u_{\mathcal{J}}^{\max}, \qquad (41)$$

for all $k \geq l$, where

$$u_{\mathcal{J}}^{\max} := \max\{\sum_{n \in \mathcal{J}} u_n(b_n, x_n^{\mathrm{opt}}(b)) \mid \|b - b^*\|$$
$$\leq \left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r \ \ and \ \ x^{\mathrm{opt}}(b) \in \mathrm{Sol}_{\mathrm{sopf}}(b)\}. \quad (42)$$

In general, we do not know if BID ADJUSTMENT ALGORITHM prevents collusion under this notion of incentive under the hypotheses of Proposition 5.10. The following result shows that if the bids of non-colluding generators remain sufficiently close to their efficient bids, then collusion, in the sense of Definition 5.11, can be prevented. "How close" depends on the number of colluding generators, and in that sense the result is weaker than the one established in Proposition 5.10.

Proposition 5.12. *(Robustness to collusion under the* BID ADJUSTMENT ALGORITHM– *cont'd): For dynamics (39), let the hypotheses of Theorem 4.7 hold and assume that $b_n(k) \geq c_n$ for all $n \in [N]$ and $k \in \mathbb{Z}_{\geq 1}$. Assume that at each bus that has generators connected to it, there exists* at least one generator that follows the update scheme of the BID ADJUSTMENT ALGORITHM. Denote these generators by $\mathcal{K} \subset [N]$. Then, a group of generators $\mathcal{J} \subset [N] \setminus \mathcal{K}$ have no incentive to collude, in the sense of Definition 5.11, if for all $l \in \mathbb{Z}_{\geq 1}$, there exists an integer $k_l \geq l$ for which

$$\|b_{[N] \setminus \mathcal{J}}(k_l) - b_{[N] \setminus \mathcal{J}}^*\| \leq \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r, \tag{43}$$

where $b_{[N] \setminus \mathcal{J}}(k_l)$ is the bids of generators in the set $[N] \setminus \mathcal{J}$ and $b_{[N] \setminus \mathcal{J}}^*$ is their corresponding efficient Nash equilibria.

Proof: Assume $\mathcal{J} \subset ([N] \setminus \mathcal{K})$ be the set of generators that form a collusion. Pick any generator $\tilde{n} \in \mathcal{J}$. By assumption, there exists $\bar{n} \in \mathcal{K}$ such that $\tilde{n}$ and $\bar{n}$ are connected to the same bus. Consider any index $l \in \mathbb{Z}_{\geq 1}$ and assume the integer $k_l \geq l$ be such that (43) holds. Then we have

$$b_{\bar{n}}(k_l) \leq b_{\bar{n}}^* + \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r.$$

If at iteration k_l of the bidding process the generator $\tilde{n}$ gets a positive utility, then it must hold that

$$b_{\tilde{n}}(k_l) \leq b_{\bar{n}}(k_l) \leq b_{\bar{n}}^* + \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r$$
$$= b_{\tilde{n}}^* + \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r.$$

In the last equality we have used the fact $b_{\bar{n}}^* = b_{\tilde{n}}^*$ as they are connected to the same bus and x^* is composed of strictly positive elements. The inequality obtained above for the generator $\tilde{n} \in \mathcal{J}$ can be obtained for every generator in $\mathcal{J}$. Thus, we get

$$b_{\mathcal{J}}(k_l) \leq b_{\mathcal{J}}^* + \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r.$$

Therefore, at k_l, the maximum utility that the colluding generators can obtain is

$$\max\left\{\sum_{n \in \mathcal{J}} u_n(b_n(k_l), x^{\mathrm{opt}}(b(k_l))) \ \middle|\right.$$
$$b_{\mathcal{J}}(k_l) \leq b_{\mathcal{J}}^* + \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r,$$
$$\|b_{[N] \setminus \mathcal{J}}(k_l) - b_{[N] \setminus \mathcal{J}}^*\| \leq \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r,$$
$$\left. and \ x^{\mathrm{opt}}(b(k_l)) \in \mathrm{Sol}_{\mathrm{sopf}}(b(k_l))\right\}. \quad (44)$$

The above defined quantity is equivalent to the following

$$\max\left\{\sum_{n \in \mathcal{J}} u_n(b_n(k_l), x^{\mathrm{opt}}(b(k_l))) \ \middle|\right.$$
$$\|b_{\mathcal{J}}(k_l) - b_{\mathcal{J}}^*\|_\infty \leq \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r,$$
$$\|b_{[N] \setminus \mathcal{J}}(k_l) - b_{[N] \setminus \mathcal{J}}^*\| \leq \frac{1}{\sqrt{1 + |\mathcal{J}|}}\left(1 + \frac{B(r)}{2a_{\max}}\right)^{1/2} r,$$

$$\text{and } x^{\text{opt}}(b(k_l)) \in \text{Sol}_{\text{sopf}}(b(k_l))\Big\}. \quad (45)$$

The quantities (44) and (45) are same because if $b_{\mathcal{J}}(k_l) \not\geq b_{\mathcal{J}}^* - \frac{1}{\sqrt{1+|\mathcal{J}|}}\Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r$, then one can show that the aggregate payoff of the colluding generators is less than their payoff with bids

$$\max\{b_{\mathcal{J}}(k_l), b_{\mathcal{J}}^* - \frac{1}{\sqrt{1+|\mathcal{J}|}}\Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r\},$$

where the $\max$ is done component-wise. Focusing now on the quantity in (45), note that

$$\|b_{\mathcal{J}}(k_l) - b_{\mathcal{J}}^*\|_\infty \leq \frac{1}{\sqrt{1+|\mathcal{J}|}}\Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r, \quad \text{and}$$

$$\|b_{[N]\setminus\mathcal{J}}(k_l) - b_{[N]\setminus\mathcal{J}}^*\| \leq \frac{1}{\sqrt{1+|\mathcal{J}|}}\Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r$$

implies $\|b(k_l) - b^*\| \leq \Big(1 + \frac{B(r)}{2a_{\max}}\Big)^{1/2} r$. Thus, the quantity in (45) is less than or equal to $u_{\mathcal{J}}^{\max}$ defined in (42), and (41) does not hold for iteration k_l. Since for each l there exists $k_l \geq l$ for which this happens, generators in $\mathcal{J}$ do not have an incentive to collude. $\square$

Remark 5.13. *(Limitations on robustness under generator bounds):* The robustness of the BID ADJUSTMENT ALGORITHM against deviation and collusion relies heavily on the fact that we have not considered upper bounds on the generation capacities. In the presence of such bounds, the generators might be able to push the bids and their individual utilities to a higher value based on the load at the respective bus and the capacity constraints on the lines connected to the bus. To avoid such behavior of market manipulation, either one can modify network capacities or investigate alternative allocation mechanisms that disincentivizes such behavior. •

6 SIMULATIONS

We illustrate the convergence and robustness properties of the BID ADJUSTMENT ALGORITHM using a modified IEEE 9-bus test case [36]. The traditional IEEE 9-bus system has 3 generators, at buses v_1, v_2, and v_3 and three loads at buses v_5, v_7, and v_9. In our modified test case, we have added one generator each at buses v_1, v_2 and v_3. The interconnection topology is given in Figure 1(a). The line flow limit between any two buses (v_i, v_j) is 2.5 except for three lines, (v_5, v_6), (v_3, v_6), and (v_6, v_7), for which the limits are 1.5, 3.0, and 1.5, respectively. The loads are $y_5 = 2$, $y_7 = 3$, and $y_9 = 1$, where y_i denotes the load at bus v_i. The cost function for each generator i is $f_i(x_i) = a_i x_i^2 + c_i x_i$, where the coefficients for all the generators are given by the vectors

$$a = (0.1100, 0.0950, 0.0850, 0.1000, 0.1225, 0.0750),$$
$$c = (3.5, 3.8, 1.2, 0.8, 1.0, 1.3). \quad (46)$$

For the given costs and loads, the generation profile at the optimizer of the DC-OPF problem (2) is

$$x^* = (1.4268, 0.0732, 0.2703, 2.2297, 1.8987, 1.1013),$$

and the unique efficient NE is

$$b^* = (3.8139, 3.8139, 1.2459, 1.2459, 1.4652, 1.4652). \quad (47)$$

Figure 1 depicts the evolution of the bids and their distance to the efficient NE along an execution of the BID ADJUSTMENT ALGORITHM. The initial bids $b(1)$ are selected satisfying $b_n(1) \geq c_n$ for all the generators $n \in [6]$. The stepsizes are constant, $\beta_k = 0.01$ for all k, and satisfy $\beta_k < 2a_n$. As predicted by Theorem 4.7, Figure 1 shows that the bids converge towards the efficient NE b^* at a linear rate and, after a finite number of steps, remain in a neighborhood of b^*. If one selects $r = 1.35$, then $B(r) = 0.0101$ and condition (14) holds for the stepsizes. Computing the right hand side of (16) using these values, we conclude that bids eventually remain in the neighborhood centered at b^* with radius 1.3775. Figure 1(b) validates this claim and shows that the bound is conservative, since bids remain in a neighborhood of radius 0.05.

We next illustrate the robustness properties of the BID ADJUSTMENT ALGORITHM against disturbances (cf. Section 5.1). Figure 2 considers the same setup as above but now with generators choosing a different stepsize at each iteration. These differences in stepsizes can be interpreted as a disturbance to the BID ADJUSTMENT ALGORITHM, as discussed in Remark 5.3. In Figure 2(a)-(b), the interval from which stepsizes are selected is constant, whereas in Figure 2(c)-(d) the size of this interval decays with time. In both cases, the bids converge to a neighborhood of b^* (in the latter case of decaying interval, the bids converge to a smaller neighborhood), as established in Proposition 5.2. Observe that the convergence rate in Figure 2(a)-(b) is higher than in Figure 1(a)-(b). This is because stepsizes are allowed to be large in the former. However, this higher convergence rate comes with the pitfall of loss in accuracy, cf. Remark 4.8. Hence, to retain both properties, stepsizes should be large initially and decay as iterations proceed. This is seen in Figure 2(c)-(d), where stepsizes decay over time (in expectation), yielding both high convergence rate and accuracy. Finally, Figure 3 demonstrates the robustness against collusion of the BID ADJUSTMENT ALGORITHM (cf. Section 5.3), where generators 1, 3, and 5 form a collusion. These generators may select their bids in any fashion they want: for this example, we assume a particular strategy of bid selection, explained in Figure 3. The plot shows that the utility of the colluding generators eventually becomes lower than $u_n^{\max}$ (defined in (34)). Hence, there is no incentive for collusion, as ensured by Proposition 5.10.

7 CONCLUSIONS

We have formulated an inelastic electricity market game capturing the strategic interaction between generators in a bid-based energy dispatch setting. For this game, we have established the existence and uniqueness of the efficient Nash equilibria. We have also designed the BID ADJUSTMENT ALGORITHM, which is an iterative
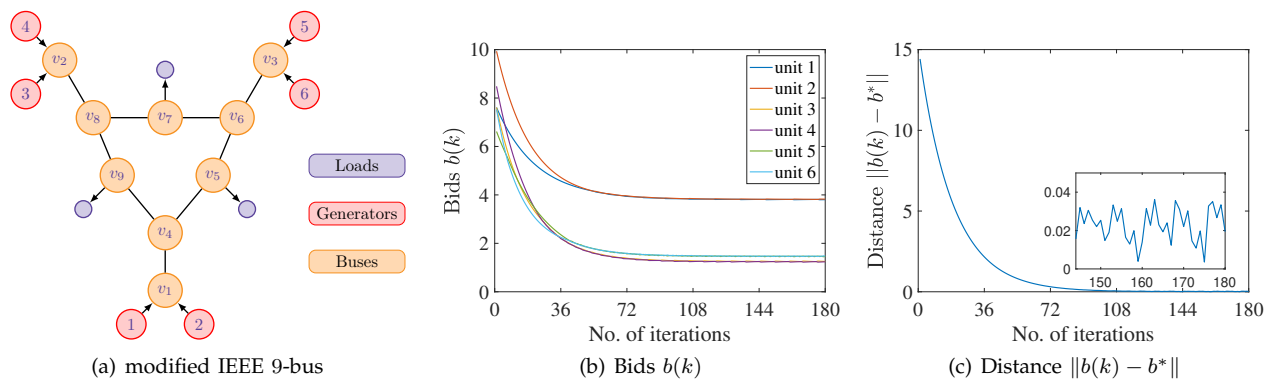
(a) modified IEEE 9-bus

(b) Bids $b(k)$

(c) Distance $\|b(k) - b^*\|$

Fig. 1: Execution of the BID ADJUSTMENT ALGORITHM for the modified IEEE 9-bus test case. Plot (a) shows the network layout. The cost function for each generator i is $f_i(x_i) = a_i x_i^2 + c_i x_i$, with coefficients given in (46). The load is $y_5 = 2$, $y_7 = 3$, and $y_9 = 1$. The efficient NE b^* is given in (47). Plots (b) and (c) show, respectively, the evolution of the bids and their distance to b^*. The stepsizes are $\beta_k = 0.01$ for all k and the initial bids are $b(1) = (7.6096, 9.9313, 7.6087, 8.4827, 6.6175, 7.5254)$. Bids converge to and then remain in a neighborhood of the efficient NE.
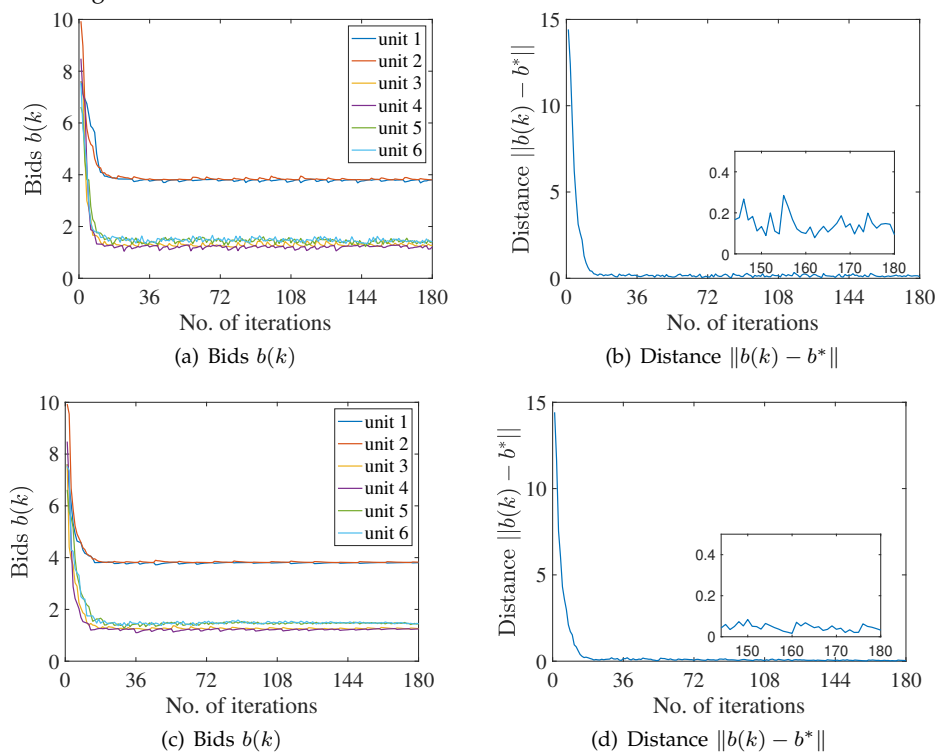

(a) Bids $b(k)$

(b) Distance $\|b(k) - b^*\|$

(c) Bids $b(k)$

(d) Distance $\|b(k) - b^*\|$

Fig. 2: Execution of the BID ADJUSTMENT ALGORITHM under different stepsize selection for the example of Figure 1. All data is the same except for the stepsizes. In plots (a) and (b), each generator at each iteration randomly selects the stepsize from the set $[0.001, 0.1]$ with uniform probability distribution. The bids still converge to a neighborhood of the efficient NE, but the size of the neighborhood is bigger than that achieved in Figure 1. In plots (c) and (d), the interval of stepsize selection decays with time to a single point 0.01. The bids now converge to the efficient NE with greater accuracy. These observations validate the robustness guarantees of Proposition 5.2.

strategy amenable to decentralized implementation that provably converges to a neighborhood of the efficient NE at a linear rate. We have characterized the robustness properties of the algorithm against disturbances, deviation in bid updates, and collusion among generators. Future work will analyze the dynamic behavior of the market under other bidding schemes, such as Cournot bidding, supply function bidding, and price-capacity bidding and under different learning schemes. We plan to generalize our setup to include generator bounds and analyze the resulting network-constrained Bertrand-

Edgeworth competition. We would also like to consider scenarios where generators seek to maximize their profit by anticipating the end of the game, and the effect that this might have on the algorithm's evolution. Finally, we wish to incorporate stochastic load demands and changing sets of generators.

APPENDIX

This appendix presents the proofs of several auxiliary results useful in establishing the convergence of the BID ADJUSTMENT ALGORITHM in Section 4.2.
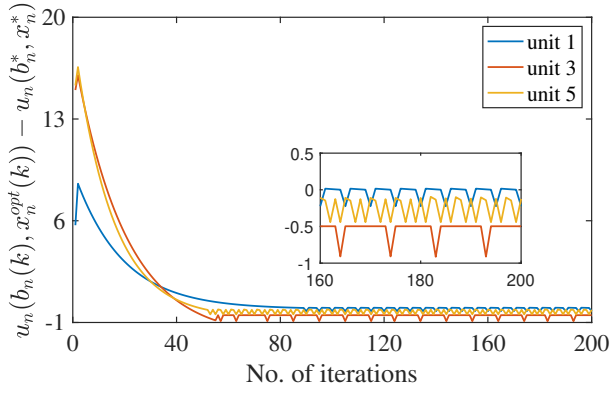
Fig. 3: Execution of the BID ADJUSTMENT ALGORITHM for the example considered in Figure 1 with generators 1, 3, and 5 forming a collusion. The initial condition is the same and the stepsize is 0.01 at each iteration for generators 2, 4, and 6. For each $n \in \{1, 3, 5\}$, at each iteration k, $b_n(k) = 0.99 * b_{n+1}(k)$ if this value is bigger than or equal to b_n^*. Otherwise, $b_n(k)$ is selected randomly from the interval $[b_n^*, b_n^* + 1]$, with uniform probability distribution. With this choice of bid, the colluding generators aim to get a positive production signal and at the same time bid high enough so as to obtain a high utility. The plot shows the evolution of the difference between the utility obtained at each iteration, $u_n(b_n(k), x_n^{\text{opt}}(k))$, and the utility at the optimal bid and generation, $u_n(b_n^*, x_n^{\text{opt}}(b^*))$ for each $n \in \{1, 3, 5\}$. This value becomes negative for all generators after a finite number of iterations. Since $u_n^{\max} > u_n(b_n^*, x_n^*)$, the example shows that (40) does not hold.

Proof of Lemma 4.3: Equation (11) follows directly from $b_n(k) \geq c_n$, so we focus on proving the latter. We proceed by induction. Note that $b_n(1) \geq c_n$ for all $n \in [N]$. Assume that $b_n(k) \geq c_n$ for some $k \in \mathbb{Z}_{\geq 1}$ and let us show $b_n(k+1) \geq c_n$. We have

$$b_n(k+1) = [b_n(k) + \beta_k(x_n^{\text{opt}}(k) - q_n(k))]^+$$
$$\overset{(a)}{\geq} [b_n(k) - \beta_k q_n(k)]^+ \overset{(b)}{=} \left[b_n(k) - \beta_k \left(\frac{b_n(k) - c_n}{2a_n} \right) \right]^+$$
$$= \left[\left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \beta_k \frac{c_n}{2a_n} \right]^+$$
$$\overset{(c)}{=} \left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \beta_k \frac{c_n}{2a_n},$$

where (a) is due to the fact that $x_n^{\text{opt}}(k) \geq 0$, (b) follows from the definition of $q_n(k)$ given the fact that $b_n(k) \geq c_n$, and (c) follows from the assumption that $\beta_k < 2a_n$ for all n (which makes both terms in the expression positive). By contradiction, assume $b_n(k+1) < c_n$. Then, $\left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \beta_k \frac{c_n}{2a_n} < c_n$, which implies that $b_n(k) < c_n$, a contradiction. $\square$

Proof of Lemma 4.4: The proof of Lemma 4.3 shows that for all n, k, the term inside the operator $[\cdot]^+$ in Step 3 of Algorithm 1 is nonnegative. Thus, the projection can be dropped and we write

$$b_n(k+1) = b_n(k) + \beta_k(x_n^{\text{opt}}(k) - q_n(k))$$
$$\overset{(a)}{=} b_n(k) + \beta_k(x_n^{\text{opt}}(k)) - \beta_k \left(\frac{b_n(k) - c_n}{2a_n} \right)$$
$$= \left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \beta_k \left(x_n^{\text{opt}}(k) + \frac{c_n}{2a_n} \right)$$

$$= \left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \beta_k(x_n^{\text{opt}}(k) - x_n^*) + \beta_k \left(x_n^* + \frac{c_n}{2a_n} \right)$$
$$\overset{(b)}{=} \left(1 - \frac{\beta_k}{2a_n} \right) b_n(k) + \frac{\beta_k}{2a_n} b_n^* + \beta_k(x_n^{\text{opt}}(k) - x_n^*).$$

In the above expression, we have used (11) in the equality (a) and (10) in the equality (b). $\square$

Proof of Lemma 4.5: Using Lemma 4.4, we write

$$\langle b(k+1) - b(k), b^* - b(k) \rangle$$
$$= \langle b(k+1) - b^{\text{coc}}(k+1), b^* - b(k) \rangle$$
$$\qquad + \langle b^{\text{coc}}(k+1) - b(k), b^* - b(k) \rangle$$
$$= \beta_k \langle x^{\text{opt}}(k) - x^*, b^* - b(k) \rangle + \sum_{n=1}^{N} \frac{\beta_k}{2a_n} (b_n^* - b_n(k))^2$$
$$\overset{(a)}{\geq} \sum_{n=1}^{N} \frac{\beta_k}{2a_n} (b_n^* - b_n(k))^2 \geq \frac{\beta_k}{2a_{\max}} \|b(k) - b^*\|^2.$$

For the inequality (a), we have used the fact that

$$\langle x^{\text{opt}}(k) - x^*, b^* - b(k) \rangle = \left(\langle x^{\text{opt}}(k), b^* \rangle - \langle x^*, b^* \rangle \right)$$
$$\qquad + \left(\langle x^*, b(k) \rangle - \langle x^{\text{opt}}(k), b(k) \rangle \right) \geq 0.$$

The last inequality follows from the fact that x^* and $x^{\text{opt}}(k)$ are the optimizers of (3) given b^* and $b(k)$, resp., making both expressions on the right-hand side nonnegative. $\square$

Proof of Lemma 4.6: Consider the following

$$\|b(k+1) - b(k)\|^2$$
$$\overset{(a)}{=} \sum_{n=1}^{N} \left(\frac{\beta_k}{2a_n} (b_n^* - b_n(k)) + \beta_k(x_n^{\text{opt}}(k) - x_n^*) \right)^2$$
$$\overset{(b)}{\leq} \sum_{n=1}^{N} 2 \left(\frac{\beta_k}{2a_n} (b_n^* - b_n(k)) \right)^2 + \sum_{n=1}^{N} 2\beta_k^2(x_n^{\text{opt}}(k) - x_n^*)^2$$
$$\overset{(c)}{\leq} \frac{\beta_k^2}{2a_{\min}^2} \|b(k) - b^*\|^2 + 2\beta_k^2 \|x^{\text{opt}}(k) - x^*\|^2. \qquad (48)$$

In the above expression, (a) follows from the expression of $b_n(k+1)$ from Lemma 4.4, (b) follows from the inequality $(x+y)^2 \leq 2(x^2 + y^2)$ for $x, y \in \mathbb{R}$, and (c) follows from the definition of $a_{\min}$. Note that

$$\|x^{\text{opt}}(k) - x^*\| \leq \sum_{n=1}^{N} |x_n^{\text{opt}}(k) - x_n^*|$$
$$\leq \sum_{n=1}^{N} |x_n^{\text{opt}}(k)| + |x_n^*| = \sum_{n=1}^{N} (x_n^{\text{opt}}(k) + x_n^*) = 2y_{\text{total}}.$$

The proof concludes by using the above bound in (48). $\square$

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Ashish Cherukuri is an Assistant Professor in Engineering and Technology Institute Groningen, University of Groningen, Netherlands. Previously, he was a Postdoctoral Researcher at the Automatic Control Laboratory, ETH Zürich, Switzerland. He received the B.Tech degree in mechanical engineering from the Indian Institute of Technology, Delhi, India, in 2008, the M.Sc degree in mechanical engineering from ETH, Zurich, Switzerland, in 2010, and the Ph.D. degree in mechanical and aerospace engineering at the University of California, San Diego, CA, USA, in 2017. His research interests include dynamical systems, game theory, distributed optimization, and power networks. He received the Excellence Scholarship in 2008 from ETH Zürich and the Focht-Powell Fellowship in 2012 from the University of California, San Diego. He was a finalist for Best Student Paper Award in the American Control Conference, 2015.

Jorge Cortés (M'02, SM'06, F'14) received the Licenciatura degree in mathematics from Universidad de Zaragoza, Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from Universidad Carlos III de Madrid, Madrid, Spain, in 2001. He held postdoctoral positions with the University of Twente, Twente, The Netherlands, and the University of Illinois at Urbana-Champaign, Urbana, IL, USA. He was an Assistant Professor with the Department of Applied Mathematics and Statistics, University of California, Santa Cruz, CA, USA, from 2004 to 2007. He is currently a Professor in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA. He is the author of Geometric, Control and Numerical Aspects of Nonholonomic Systems (Springer-Verlag, 2002) and co-author (together with F. Bullo and S. Martínez) of Distributed Control of Robotic Networks (Princeton University Press, 2009). At the IEEE Control Systems Society, he has been a Distinguished Lecturer (2010-2014), and is currently its Director of Operations and an elected member (2018-2020) of its Board of Governors. His current research interests include distributed control and optimization, network science, resource-aware control, nonsmooth analysis, reasoning and decision making under uncertainty, network neuroscience, and multi-agent coordination in robotic, power, and transportation networks.