Integrating iterative bidding in electricity markets and frequency regulation

Tjerk Stegink, Ashish Cherukuri, Claudio De Persis, Arjan van der Schaft, and Jorge Cortés

***Abstract*— We study an electricity market consisting of an independent system operator (ISO) and a group of strategic generators. The ISO seeks to solve the optimal power dispatch problem and to regulate the frequency of the network. However, since generators do not share their cost functions, the ISO cannot solve the dispatch problem and instead engages the generators in an iterative bidding process. This consists of each generator submitting to the ISO a bid at which it is willing to provide power and receiving from the ISO a new power production setpoint calculated given the received bids and the network frequency. We analyze the stability of the interconnected system that results from the coupling between the iterative bidding scheme and the continuous-time swing dynamics of the power network and establish the convergence to the efficient Nash equilibrium and the optimal power dispatch.**

I. Introduction

Generation planning has traditionally been done hierarchically. Broadly, at the top layer cost efficiency is ensured via market clearing and at the bottom layer frequency regulation is ensured via primary/secondary control. Mostly, research on improving the performance of these two layers has developed independently. However, with the goal of integrating distributed energy resources (DERs) into the grid, there is an increasing body of research that seeks to merge the operation of these layers and synthesize procedures that simultaneously tackle market clearing and frequency stabilization.

Literature review

With the aim of bridging the gap between long-term optimization and real-time frequency control several works [1]-[4] propose an integrated design of primary, secondary, and tertiary control layers for several models of the power network/micro-grid dynamics. The works [5]-[7] use market mechanisms to determine optimal generation dispatch and to stabilize the frequency with real-time pricing. While these references assume generators to be price-takers, we consider a setting where the generators are allowed to bid in the market (and hence are price-setters). This is often referred to as Bertrand competition [8]. In our previous work [9], [10], we have shown how iterative bidding schemes for a group of strategic generators leads to convergence to efficient Nash equilibria, where the power generation levels minimize the total cost, as intended by the ISO. However, this adjustment scheme was not analyzed in conjunction with the physics of the power network, and this has the drawback that the generation setpoints can be commanded only once the market clearing mechanism has converged. This is a limitation given the high penetration of the DERs, because the time scale at which load/generation commitment changes in this scenario is possibly faster than the time it takes for the iterative bid adjustment algorithm to converge, see e.g., [11]. Instead, here we tackle both problems, market clearance and frequency regulation, together. In this regard, our work has similarities with the feedback-control strategies proposed in [12], [13] that dynamically update generation setpoints based on the current state of the network and the bidding process.

Statement of contributions

We consider an electrical power network consisting of an ISO and a group of strategic generators. The ISO seeks to ensure that the generation meets the load with minimal cost and frequency is regulated to the nominal value. Each generator seeks to maximize its individual profit and does not share its true cost with the ISO. Instead, the ISO operates an iterative market where generators bid prices at which there are willing to provide power and makes power generation assignments based on them. Our first contribution is the design of a continuous-time update law for the ISO-generator bidding process. Under this law, each generator seeks to adjust its bid to make its allocated power generation match the one that maximizes its profit, and the ISO seeks to minimize the total cost of matching generation and demand, while incorporating the local frequency error as a feedback signal for the negotiation process. Building on the identification of a Lyapunov function for the interconnection between the bidding process and the power network dynamics, we show that the proposed law leads to frequency regulation and convergence to the efficient Nash equilibrium. Our second contribution addresses the fact that, in reality, the ISO-generator bidding process is not performed continuously but rather iteratively at discrete instants of time. We consider two time scales, one (faster) for the bidding process, that incorporates at each step the frequency measurements, and another one (slower) for the updates in the power generation levels, that are sent to the power network dynamics. We refer to this implementation as time-triggered because we do not necessarily prescribe the time schedules to be periodic. We establish upper bounds on the length between

This work is supported by the NWO *Uncertainty Reduction in Smart Energy Systems (URSES)* program and the ARPA-e *Network Optimized Distributed Energy Systems (NODES)* program.

T. W. Stegink, C. De Persis and A. J. van der Schaft are with the Jan C. Willems Center for Systems and Control, University of Groningen, the Netherlands. {t.w.stegink, c.de.persis, a.j.van.der.schaft}@rug.nl

A. Cherukuri is with the Automatic Control Laboratory, ETH Zürich. cashish@control.ee.ethz.ch

J. Cortés is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego. cortes@ucsd.edu

consecutive triggering times that guarantee that the time-triggered implementation retains the convergence properties of the continuous-time update law. For reasons of space, all proofs are omitted and will appear elsewhere.

Notation: Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 1}$ be the set of real, nonnegative real, positive real, nonnegative integer, and positive integer numbers, respectively. For $m \in \mathbb{Z}_{\geq 1}$, we denote $[m] = \{1, \ldots, m\}$. Given $v \in \mathbb{R}^n, \tau \in \mathbb{R}^{n \times n}$, we write $\|v\|_\tau^2 := v^T \tau v$. The notation $\mathbb{1} \in \mathbb{R}^n$ is used for the vector whose elements are equal to 1. The Hessian of a twice-differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla^2 f$.

II. Power network model and dynamics

Here we present basic concepts on the dynamics of an electrical power network. Consider a power network consisting of n buses. The network is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes $\mathcal{V} = [n]$ represent buses and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ are the transmission lines connecting the buses. Let m denote the number of edges, which are arbitrarily labeled with a unique identifier in $[m]$. The ends of each edge are arbitrary labeled with '+' and '-', so that the incidence matrix $D \in \mathbb{R}^{n \times m}$ of the resulting directed graph is

$$D_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of edge } k, \\ -1 & \text{if } i \text{ is the negative end of edge } k, \\ 0 & \text{otherwise.} \end{cases}$$

Each bus represents a control area and is assumed to have one generator and one load. Following [14], the dynamics at the buses is given by the *swing equations*,

$$\begin{aligned} \dot{\delta} &= \omega \\ M\dot{\omega} &= -D\Gamma \sin(D^T \delta) - A\omega + P_g - P_d. \end{aligned} \tag{1}$$

Here $\Gamma = \operatorname{diag}\{\gamma_1, \ldots, \gamma_m\}$, where $\gamma_k = B_{ij} V_i V_j = B_{ji} V_i V_j$ and $k \in [m]$ corresponds to the edge between nodes i and j. Table I presents a list of symbols used in the model (1). To avoid rotational symmetry issues in (1),

$\delta \in \mathbb{R}^n$	(vector of) voltage phase angles
$\omega \in \mathbb{R}^n$	frequency deviation w.r.t. the nominal frequency
$V_i \in \mathbb{R}_{>0}$	voltage magnitude at bus i
$P_d \in \mathbb{R}^n_{\geq 0}$	power load
$P_g \in \mathbb{R}^n$	power generation
$M \in \mathbb{R}^{n \times n}_{\geq 0}$	diagonal matrix of moments of inertia
$A \in \mathbb{R}^{n \times n}_{\geq 0}$	diagonal matrix of asynchronous damping constants
$B_{ij} \in \mathbb{R}_{>0}$	negative of the susceptance of transmission line (i, j)

TABLE I

PARAMETERS AND STATE VARIABLES OF SWING EQUATIONS (1).

see e.g., [15], we define the voltage phase angle differences $\varphi = D_t^T \delta \in \mathbb{R}^{n-1}$. Here $D_t \in \mathbb{R}^{(n-1) \times n}$ is the incidence matrix of a tree graph on the set of buses $\mathcal{V}$ (this could be a spanning tree of the physical network, for example). Furthermore, let $U(\varphi) = -\mathbb{1}^T \Gamma \cos(D^T D_t^{\dagger T} \varphi)$ where $D_t^\dagger = (D_t^T D_t)^{-1} D_t^T$ denotes the Moore-Penrose pseudo-inverse of D_t. By observing that $D_t D_t^\dagger D = (I - \frac{1}{n} \mathbb{1}\mathbb{1}^T) D = D$, the physical system (1) in the (φ, ω)-coordinates takes the form

$$\begin{aligned} \dot{\varphi} &= D_t^T \omega \\ M\dot{\omega} &= -D_t \nabla U(\varphi) - A\omega + P_g - P_d. \end{aligned} \tag{2}$$

In the sequel, we assume that there exists an equilibrium $(\bar{\varphi}, \bar{\omega})$ of (2) for the power generation $\bar{P}_g$ that satisfies $D^T D_t^{\dagger T} \bar{\varphi} \in (-\pi/2, \pi/2)^m$. The latter assumption is standard and often referred to as the *security constraint* [14].

III. Problem statement

In this section we describe the main ingredients of our problem statement and then explain the paper objectives. We start from the electrical power network model described in Section II and describe the elements of the ISO-generator coordination problem following the exposition of [9], [10]. Let $C_i : \mathbb{R} \to \mathbb{R}$ capture the cost incurred by generator $i \in \mathcal{V}$ in producing P_{gi} units of power,

$$C_i(P_{gi}) := \tfrac{1}{2} q_i P_{gi}^2 + c_i P_{gi}, \tag{3}$$

where $q_i > 0$ and $c_i \geq 0$. The total network cost is

$$C(P_g) := \textstyle\sum_{i \in [n]} C_i(P_{gi}) = \tfrac{1}{2} P_g^T Q P_g + c^T P_g, \tag{4}$$

with $Q = \operatorname{diag}\{q_1, \ldots, q_n\}$ and $c = (c_1, \ldots, c_n)$. Given the cost (4) and the power loads P_d satisfying $\mathbb{1}^T P_d > 0$, the ISO seeks to solve the *optimal power dispatch problem*,

$$\underset{P_g}{\text{minimize}} \quad C(P_g) \tag{5a}$$

$$\text{subject to} \quad \mathbb{1}^T P_g = \mathbb{1}^T P_d, \tag{5b}$$

and, at the same time, regulate the frequency of the physical power network. Since the function C is strongly convex, there exists a unique optimizer P_g^* of (5). However, the ISO is incapable of fulfilling the objective of finding the optimizer of (5) because generators are strategic and they do not reveal their cost functions to anyone. Instead, the ISO holds a market where generators submit their bids and dispatch decisions are made by a market clearing procedure. Here we consider price-based bidding, that is, each generator $i \in [n]$ submits the price per unit electricity $b_i \in \mathbb{R}$ at which it is willing to provide power. Given these bids, the ISO aims to find the power allocation that meets the load and that minimizes the total payment to the generators. Mathematically, given a bid $b \in \mathbb{R}^n$, the ISO solves

$$\underset{P_g}{\text{minimize}} \quad b^T P_g \tag{6a}$$

$$\text{subject to} \quad \mathbb{1}^T P_g = \mathbb{1}^T P_d. \tag{6b}$$

A fundamental difference between (5) and (6) is that the latter optimization is linear and may in general have multiple (unbounded) solutions. Let $P_g^{\text{opt}}(b)$ be the optimizer of (6) the ISO selects given bids b (note that this might neither be unique, nor bounded). Knowing this process, each generator i bids a quantity b_i to maximize its payoff

$$\Pi_i(b_i, P_{gi}^{\text{opt}}(b)) := P_{gi}^{\text{opt}} b_i - C_i(P_{gi}^{\text{opt}}(b)), \tag{7}$$

where $P_{gi}^{\text{opt}}(b)$ is the i-th component of the optimizer $P_g^{\text{opt}}(b)$. Since each generator is strategic, we analyze the market clearing, and hence the dispatch process explained above using tools from game theory [16], [17]. We define the *inelastic market game* as

- Players: the set of generators $\mathcal{V} = [n]$.
- Action: for each player i, the bid $b_i \in \mathbb{R}$.
- Payoff: for each player i, the payoff Π_i in (7).

We use interchangeably the notation $b \in \mathbb{R}^n$ and $(b_i, b_{-i}) \in \mathbb{R}^n$ for the bid vector. Here b_{-i} is the bids of all players except i. Since the payoff of generator i not only depends on the bids of the other players but also on the optimizer $P_g^{\text{opt}}(b)$ the ISO selects, we require a slightly different definition of the Nash equilibrium compared to the standard one. We define the (pure) *Nash equilibrium* of the inelastic electricity market game as the bid profile $b^* \in \mathbb{R}^n$ for which there exists an optimizer $P_g^{\text{opt}}(b^*)$ of (6) such that for each $i \in [n]$,

$$\Pi_i(b_i, P_{gi}^{\text{opt}}(b_i, b_{-i}^*)) \leq \Pi_i(b_i^*, P_{gi}^{\text{opt}}(b^*))$$

for all $b_i \neq b_i^*$ and all optimizers $P_{gi}^{\text{opt}}(b_i, b_{-i}^*)$ of (6) given bids (b_i, b_{-i}^*). We are particularly interested in bid profiles for which the optimizer of (5) is also a solution to (6). More specifically, an *efficient bid* of the inelastic electricity market is a bid $b^* \in \mathbb{R}^n$ for which an optimizer P_g^* of (5) is also an optimizer of (6) given bids $b = b^*$ and $P_{gi}^* = \arg\max_{P_{gi}}\{P_{gi}b_i^* - C_i(P_{gi})\}$ for all $i \in [n]$. A bid b^* is an *efficient Nash equilibrium* of the inelastic electricity market game if it is an efficient bid and a Nash equilibrium. At the efficient Nash equilibrium, the production that the generators are willing to provide, maximizing their profit (7), coincides with the optimal generation associated to the optimal power flow problem (5). This property justifies the study of efficient Nash equilibria. The next result states the existence and uniqueness of the efficient Nash equilibrium.

Proposition III.1 (Existence and uniqueness of efficient Nash equilibrium [10])**.** *Let (P_g^*, λ^*) be a primal-dual optimizer of* (5)*, then $b^* = \nabla C(P_g^*) = \mathbb{1}\lambda^*$ is the unique efficient Nash equilibrium of the inelastic electricity market game.*

Our objective here is to study the stability of the interconnection between the bidding process that takes place to carry out the ISO-generator coordination and the physical dynamics of the power network. In the previous work [9], [10], the focus was only on the bidding process, without regard for the power network dynamics, essentially assuming that the incorporation of the dispatch decisions into the physical dynamics happens once the market is cleared. In our forthcoming discussion, we start by proposing a continuous-time bid update scheme and analyzing its interconnection with the network dynamics. We then develop a time-triggered implementation of this interconnection that does not presume continuous bidding and retains its provably correct properties. In our design, the ISO knows the loads and frequency of the network and can interact with the generators, whereas each generator can only communicate with the ISO and is not aware of the number of other generators participating, their respective cost functions, or the load at its own bus.

IV. Continuous-time bid update scheme

In this section, we provide a continuous-time update law for both the actions of the ISO and the generators. Generators update their bids in a decentralized fashion based on the power generation setpoints received by the ISO, while the ISO determines the next power generation setpoints depending on both the generator bids and the frequency of the network. This results in a coupling with the physical dynamics of the system. For the overall interconnected system, we establish local convergence to the efficient Nash equilibrium by providing a suitable Lyapunov function.

We assume the generators update their bids according to

$$\tau_b \dot{b} = P_g - Q^{-1}b + Q^{-1}c, \tag{8a}$$

where $\tau_b \in \mathbb{R}^{n \times n}$ is a diagonal positive definite matrix. The rationale behind the update (8a) is the following: given bid b_i, generator i wants to produce the amount of power that maximizes its profit, i.e., $P_{gi}^{\text{des}} = \arg\max_{P_{gi}}\{C_i(P_{gi}) - b_i P_{gi}\} = q_i^{-1}(b_i - c_i)$. Therefore, if the ISO wants i to produce more power than its desired quantity, that is $P_{gi} > P_{gi}^{\text{des}}$, the generator increases its bid, and vice versa. Interestingly, (8a) can be seen as the continuous-time version of the discrete-time dynamics proposed in [9], [10].

Next, we provide an update law for the ISO depending on the bid $b \in \mathbb{R}^n$ and the local frequency of the power network. Based on the primal-dual dynamics associated to the problem (6), the ISO updates its actions according to

$$\begin{aligned}\tau_g \dot{P}_g &= \mathbb{1}\lambda - b + \rho\mathbb{1}\mathbb{1}^T(P_d - P_g) - \sigma^2\omega \\ \tau_\lambda \dot{\lambda} &= \mathbb{1}^T(P_d - P_g)\end{aligned} \tag{8b}$$

with parameters $\rho, \sigma, \tau_\lambda \in \mathbb{R}_{>0}$ and where $\tau_g \in \mathbb{R}^{n \times n}$ is a diagonal positive definite gain matrix. The intuition behind the dynamics (8b) is as follows. If generator i bids higher than the Lagrange multiplier λ (often referred to as the *shadow price* in the economic literature [18]) associated with the power balance constraint (6b), then the power generation (setpoint) of node i is decreased, and vice versa. We add the ρ-term to enhance the convergence rate of (8b) and we add the feedback signal $-\sigma^2\omega$ to compensate for the frequency deviations in the physical system. Interestingly, albeit we do not pursue this here, the dynamics (8) could be also implemented in a distributed way without involvement of a central regulating authority like the ISO.

While the ISO dynamics (8b) is a primal-dual dynamics of (6) (and hence, potentially unstable), we show next that the interconnection of the physical power network dynamics (2) with the bidding process (8) is nevertheless asymptotically stable. For the remainder of the paper, we assume that there exists an equilibrium $\bar{x} = \text{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{\lambda})$ of (2)-(8) such that $D^T D_t^{\dagger T}\bar{\varphi} \in (-\pi/2, \pi/2)^m$ (cf. Section II). Note that this equilibrium satisfies

$$\bar{\lambda} = \frac{\mathbb{1}^T(P_d + Q^{-1}c)}{\mathbb{1}^T Q^{-1}\mathbb{1}} > 0, \qquad \bar{\omega} = 0, \quad \bar{b} = \mathbb{1}\bar{\lambda}, \\ \bar{P}_g = Q^{-1}\mathbb{1}\bar{\lambda} - Q^{-1}c, \qquad \mathbb{1}^T P_g = \mathbb{1}^T \bar{P}_g. \tag{9}$$

In particular, at the steady state, the frequency deviation is zero, the power balance $\mathbb{1}^T\bar{P}_g = \mathbb{1}^T P_d$ is satisfied, and $\mathbb{1}\bar{\lambda} = \bar{b} = \nabla C(\bar{P}_g)$, implying that $\bar{P}_g$ is a primal optimizer of (5) and $\bar{b}$ is an efficient Nash equilibrium. Hence, at steady state the generators do not have any incentive to deviate from the equilibrium bid. Next, define the function

$$V(x) = U(\varphi) - (\varphi - \bar{\varphi})^T\nabla U(\bar{\varphi}) - U(\bar{\varphi}) + \tfrac{1}{2}\omega^T M\omega \\ + \tfrac{1}{2\sigma^2}(\|b - \bar{b}\|^2_{\tau_b} + \|P_g - \bar{P}_g\|^2_{\tau_g} + \|\lambda - \bar{\lambda}\|^2_{\tau_\lambda}), \tag{10}$$

where $x = \operatorname{col}(\varphi, \omega, b, P_g, \lambda)$. Then the closed-loop system obtained by combining (2)-(8) is compactly written as

$$\dot{x} = F(x) = \mathcal{Q}^{-1}\mathcal{A}\mathcal{Q}^{-1}\nabla V(x) \tag{11}$$

with $\mathcal{Q} = \mathcal{Q}^T = \operatorname{blockdiag}(I, M, \frac{\tau_b}{\sigma}, \frac{\tau_g}{\sigma}, \frac{\tau_\lambda}{\sigma}) > 0$ and

$$\mathcal{A} = \begin{bmatrix} 0 & D_t^T & 0 & 0 & 0 \\ -D_t & -A & 0 & \sigma I & 0 \\ 0 & 0 & -Q^{-1} & I & 0 \\ 0 & -\sigma I & -I & -\rho\mathbb{1}\mathbb{1}^T & \mathbb{1} \\ 0 & 0 & 0 & -\mathbb{1}^T & 0 \end{bmatrix}.$$

By exploiting the structure of the system, we observe that $\frac{d}{dt}V(x) = \frac{1}{2}(\nabla V(x))^T\mathcal{Q}^{-1}(\mathcal{A} + \mathcal{A}^T)\mathcal{Q}^{-1}\nabla V(x) \leq 0$. However, since $(\mathcal{A} + \mathcal{A}^T)$ is only negative semi-definite, V is not strictly decreasing along the trajectories of (11). Nevertheless, we can use the function V to construct a (strict) Lyapunov function, as the following result shows.

Theorem IV.1. *(Local Lyapunov function for the interconnected dynamics): Consider the interconnected dynamics* (11) *and define the function*

$$W_\epsilon(x) = V(x) + \epsilon_1(\varphi - \bar{\varphi})^T D_t^{\dagger} M\omega \\ - \epsilon_2(b - \bar{b})^T(P_g - \bar{P}_g) - \epsilon_3(\lambda - \bar{\lambda})\mathbb{1}^T(P_g - \bar{P}_g), \tag{12}$$

with parameters $\epsilon = \operatorname{col}(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{R}^3_{>0}$. *Given the equilibrium* $\bar{x} = \operatorname{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{\lambda})$ *of* (11)*, let* $\bar{\eta} = D^T D_t^{\dagger T}\bar{\varphi}$. *For* γ *such that* $\|\eta_{\max}\|_\infty < \gamma < \frac{\pi}{2}$*, define the convex set*

$$\Omega = \{x \mid D^T D_t^{\dagger T}\varphi \in [-\gamma, \gamma]^m\}. \tag{13}$$

Then there exist sufficiently small ϵ *such that* W_ϵ *is a Lyapunov function of* (11) *on* Ω. *In particular, there exist constants* $\alpha, c_1, c_2 > 0$ *such that for all* $x \in \Omega$,

$$c_1\|x - \bar{x}\|^2 \leq W_\epsilon(x) \leq c_2\|x - \bar{x}\|^2, \tag{14a}$$

$$(\nabla W_\epsilon(x))^T F(x) \leq -\alpha\|x - \bar{x}\|^2. \tag{14b}$$

Using the characterization (14) and [19, Theorem 4.10], each trajectory initialized in a compact level set contained in Ω exponentially converges to the equilibrium $\bar{x}$ corresponding to optimal power dispatch and the efficient Nash equilibrium.

V. Time-triggered implementation

In reality, the bidding process between the ISO and the generators is not performed continuously and hence, for practical purposes, one needs to think about discretizations of the continuous-time dynamics proposed in Section IV. This is the subject of this section. We consider two time scales, one (faster) for the bidding process that incorporates at each step the frequency measurements, and another one (slower) for the updates in the power generation levels that are sent to the power network dynamics. This seems sensible to avoid driving the physical dynamics with rapidly changing transient setpoints. We refer to this implementation as time-triggered because we do not necessarily prescribe the time schedules to be periodic.

A. Bid adjustment algorithm

We start with an informal description of the iterative update scheme between the ISO and the generators, and the interconnection with the dynamics of the power network.

> *[Informal description]:* At each iteration $l \in \mathbb{Z}_{\geq 0}$, ISO and generators are involved in an iterative process where, at each subiteration k, generators send a bid to the ISO. Once the ISO has obtained the bids and the network frequency measurements at time t_k^l, it computes the new generation allocations, denoted $P_g^{k+1} \in \mathbb{R}^n$, and sends the corresponding one to each generator. At the $(k{+}1)$-th subiteration, generators adjust their bid based on their previous bid and the generation allocation received from the ISO at time t_{k+1}^l. Once $k = N_l \in \mathbb{Z}_{\geq 1}$ at time $t_{N_l}^l$, the market is cleared, meaning that the bidding process is reset (i.e., $k = 0$) and the power generations in the swing equations are updated according to the current setpoints $P_g^{N_l}$.

The evolution of the frequency occurs in continuous time. To relate iteration numbers with time instances, we consider time sequences of the form $\{\{t_k^l\}_{k=0}^{N_l}\}_{l=0}^{\infty}$ for $N_l \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}_{\geq 0}\}$, satisfying

$$t_k^l - t_{k-1}^l > 0, \; t_0^{l+1} = t_{N_l}^l \qquad \forall l \in \mathbb{Z}_{\geq 0}, \forall k \in [N_l]. \tag{15}$$

Algorithm 1 describes formally the iterative updates of the generators and the ISO. For analysis purposes, we find it convenient to represent the dynamics resulting from the combination of Algorithm 1 and the network dynamics (2) as the time-triggered continuous-time system

$$\begin{aligned} \dot{\varphi}(t) &= D_t^T\omega(t) \\ M\dot{\omega}(t) &= -D_t\nabla U(\varphi(t)) - A\omega(t) + P_g(t_0^l) - P_d \\ \tau_b\dot{b}(t) &= P_g(t_k^l) - Q^{-1}b(t_k^l) - Q^{-1}c \\ \tau_g\dot{P}_g(t) &= \mathbb{1}\lambda(t_k^l) - b(t_k^l) - \sigma^2\omega(t_k^l) + \rho\mathbb{1}\mathbb{1}^T(P_d - P_g(t_k^l)) \\ \tau_\lambda\dot{\lambda}(t) &= \mathbb{1}^T(P_d - P_g(t_k^l)) \end{aligned} \tag{16}$$

for $t \in [t_k^l, t_{k+1}^l) \subset [t_0^l, t_0^{l+1}), l \in \mathbb{Z}_{\geq 0}, k \in \{0, \ldots, N_l - 1\}$. We write the system (16) compactly in the form

$$\dot{x}(t) = f(x(t)) + g(x(t_k^l)) + h(x(t_0^l)) \tag{17}$$

with

$$
\begin{aligned}
f(x) &= \mathrm{col}(D_t^T\omega, -M^{-1}(D_t\nabla U(\varphi) + A\omega + P_d), 0, 0, 0)\\
g(x) &= \mathrm{col}(0, 0, \tau_b^{-1}(P_g - Q^{-1}b - Q^{-1}c),\\
&\quad \tau_g^{-1}(\mathbb{1}\lambda - b - \sigma^2\omega + \rho\mathbb{1}\mathbb{1}^T(P_d - P_g)), \tfrac{1}{\tau_\lambda}\mathbb{1}^T(P_d - P_g))\\
h(x) &= \mathrm{col}(0, M^{-1}P_g, 0, 0, 0).
\end{aligned}
$$

Algorithm 1: BID ADJUSTMENT ALGORITHM

Executed by: generators $i \in [n]$ and ISO

Data : time sequence $\{\{t_k^l\}_{k=0}^{N_l}\}_{l=0}^{\infty}$; cost function C_i for each generator i; frequency deviation $\omega(t_k^l)$ at each time t_k^l and load P_d for ISO

Initialize : each generator i selects arbitrarily $b_i^0 \geq c_i$, sets $k = 0, l = 0$, and jumps to step 6; ISO selects arbitrary $P_{gi}^0 > 0, \lambda_i^0 > 0$, sets $k = 0, l = 0$ and waits for step 8

1 **while** $l \geq 0$ **do**

2   **while** $k \geq 0, k < N_l$ **do**

3   `/* For each generator i: */`

4   Receive P_{gi}^k from ISO at t_k^l; Set

5     $b_i^{k+1} = b_i^k + (t_{k+1}^l - t_k^l)\tau_{bi}^{-1}(P_{gi}^k - q_i^{-1}(b_i^k + c_i))$

6   Send b_i^{k+1} to the ISO; set $k = k + 1$

7   `/* For ISO: */`

8   Receive $b_i^k, \omega_i(t_k^l)$ from each $i \in [n]$ at t_k^l

9   Set $P_{gi}^{k+1} = P_{gi}^k + (t_{k+1}^l - t_k^l)\tau_{gi}^{-1}(\lambda^k - b_i^k - \sigma^2\omega(t_k^l) + \rho\sum_{i\in[n]}(P_{di} - P_{gi}^k))$ for all $i \in [n]$
    $\lambda^{k+1} = \lambda^k + \frac{t_{k+1}^l - t_k^l}{\tau_\lambda}\sum_{i\in[n]}(P_{di} - P_{gi}^k)$

10   Send P_{gi}^{k+1} to each $i \in [n]$, set $k = k + 1$

11   **end**

12   Set $P_{gi}(t) = P_{gi}^{N_l}$ in (2) $\forall i \in [n], \forall t \in [t_{N_l}^l, t_{N_{l+1}}^{l+1})$

13   Set $b_i^0 = b_i^{N_l}, P_{gi}^0 = P_{gi}^{N_l}, \lambda_i^0 = \lambda_i^{N_l}$ for each $i \in [n]$

14   Set $l = l + 1, k = 0$

15 **end**

Since $\sup_{\varphi\in\mathbb{R}^{n-1}} \|\nabla^2 U(\varphi)\| < \infty$ and g, h are linear functions, it follows that f, g, h are globally Lipschitz, with Lipschitz constants L_f, L_g, L_h, respectively. The dynamics (16) has a discontinuous right-hand side, and we consider solutions in the Carathéodory sense. We next establish conditions on the time sequence that guarantee solutions are well defined and retain the convergence properties of (11).

B. Sufficient condition on triggering times for stability

Here, we determine a sufficient condition on the inter-sampling times t_k^l, t_{k+1}^l and t_k^l, t_k^{l+1} for guaranteeing local asymptotic convergence of (17) to $\bar{x}$. That is, $\bar{x}$ is an equilibrium of the continuous-time system (11) or equivalently

$$\dot{x}(t) = F(x(t)) = f(x(t)) + g(x(t)) + h(x(t)). \tag{18}$$

The main idea is to use the Lyapunov function W_ϵ of (18) defined by (12) for the time-triggered system (17). Then an upper bound for the time between two successive iterations and market clearing instances respectively is determined such that W_ϵ is strictly decreasing for all time t.

Theorem V.1 (Local asymptotic stability of time-triggered implementation)**.** *Consider the time-triggered implementation* (16) *of the interconnection between the ISO-generator bidding processes and the power network dynamics. With the notation of Theorem IV.1, let*

$$
\begin{aligned}
\bar{\xi} &:= \tfrac{1}{L_f+L_g}\log\big(1 + \tfrac{\beta(L_f+L_g)}{L(L_W L_h+\beta)}\big),\\
\bar{\zeta} &:= \tfrac{1}{L_f}\log\big(1 + \tfrac{L_f(\alpha-\beta)}{L_g(LL_W+\alpha)+(\alpha-\beta)(L_f+L_g)}\big).
\end{aligned}
\tag{19}
$$

where $0 < \beta < \alpha$, $L = L_f + L_g + L_h$, *and* L_W *is the Lipschitz constant of* ∇W_ϵ. *Assume the time sequence* $\{\{t_k^l\}_{k=0}^{N_l}\}_{l=0}^{\infty}$ *satisfies, for some* $\underline{\zeta} \in (0, \bar{\zeta})$ *and* $\underline{\xi} \in (0, \bar{\xi})$,

$$\underline{\zeta} \leq t_0^{l+1} - t_0^l \leq \bar{\zeta} \quad \textit{and} \quad \underline{\xi} \leq t_k^l - t_{k-1}^l \leq \bar{\xi}, \tag{20}$$

for all $l \in \mathbb{Z}_{\geq 0}$ *and* $k \in [N_l]$. *Then,* $\bar{x}$ *is locally asymptotically stable under* (16).

The uniform lower bound $\bar{\eta}$ and $\bar{\xi}$ in (20) ensures that the solutions of the time-triggered implementation are well-defined. The result implies that convergence is guaranteed for any constant stepsize implementation, where the stepsize satisfies (20). However, constant stepsizes are not required by Theorem V.1. Another interesing observation is that the upper bounds can be calculated without requiring any information about $\bar{x}$. This is desired as (the distance of the state from) the desired equilibrium may not be known beforehand.

VI. SIMULATIONS

In this section we illustrate the convergence properties of the interconnected system (11) in continuous-time and its time-triggered implementation (16). We consider the power network depicted in Figure 1, where each node has one generator and one load. Table II shows the parameter values appearing in the dynamics (2)-(8a). The parameter values for the ISO dynamics (8b) are $\tau_g = 0.4I$, $\tau_\lambda = 1$, $\rho = 3$, $\sigma = 2$. The initial load is $P_d = \mathrm{col}(3, 1, 2, 1, 2, 3, 2)$. For this load

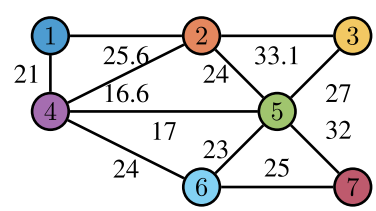

Fig. 1. A 7-node power network. Each node contains one generator and one load. The edges correspond to transmission lines and the numbers indicate the values of the line susceptances.

Parameter	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
A_i	1.6	1.22	1.38	1.42	1.6	1.22	1.38
M_i	5.22	3.98	4.49	4.22	5.22	3.98	4.49
τ_{bi}	1.11	1.48	1.22	1.24	1.11	1.48	1.22
V_i	1.0	1.2	1.1	1.05	1	1.1	1.3
q_i	1.1	1.3	0.83	0.91	0.77	1.3	1.0
c_i	1	2	3	2.5	2	1	2

TABLE II

PARAMETER VALUES OF THE BENCHMARK SYSTEM.

and generator cost functions $C_i(P_{gi}) = \frac{1}{2} q_i P_{gi}^2 + c_i P_{gi}$, the efficient Nash equilibrium is $b_i^* = 3.9929$ for all $i \in [7]$ and the optimizer of (5) amounts to

$$P_g^* = \mathrm{col}(2.69, 1.49, 1.19, 1.64, 2.59, 2.39, 1.99), \qquad (21)$$

which we calculate using the Matlab function `quadprog`. Both the continuous-time system (11) and the time-triggered system (16) are initialized at this equilibrium. At $t = 1$, the power load at node 7 is increased from 2 to 3 units, and the new efficient Nash equilibrium becomes $b_i^* = 4.1348, i \in [7]$ and the primal optimizer of (5) changes to

$$P_g^* = \mathrm{col}(2.82, 1.60, 1.36, 1.80, 2.77, 2.51, 2.14). \qquad (22)$$

Figures 2 and 3 show the evolution of the closed-loop continuous-time (11) and time-triggered (16) systems, respectively. As predicted by Theorems IV.1 and V.1, both dynamics converge to the optimal equilibrium corresponding to the efficient Nash equilibrium and optimal power dispatch.

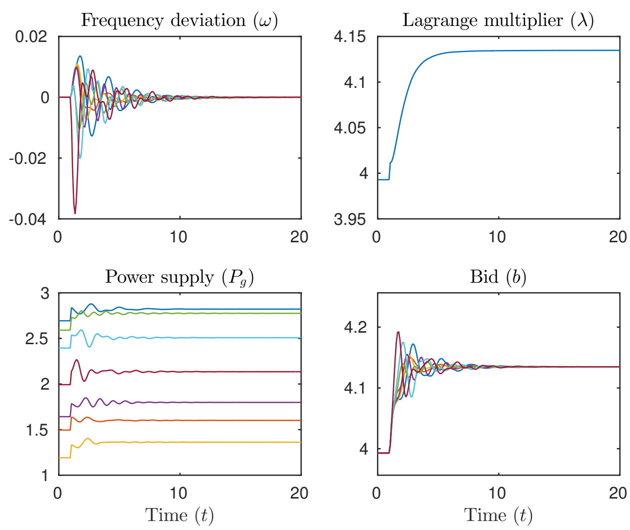

Fig. 2. Simulation of the continuous-time system (11) for the power network in Figure 1 with the parameter values in Table II. The system is initialized at steady state corresponding to (21). At $t = 1$, the load at node 7 is increased to 3. As observed in the plot, after the initial frequency drop at node 7, all the frequency deviations asymptotically converge to zero. Moreover, the bids of the generators and the Lagrange multiplier converge to the new efficient Nash equilibrium $\lambda^* = b_i^* = 4.1348, i \in [7]$ and the power production levels converge to optimal value given by (22).

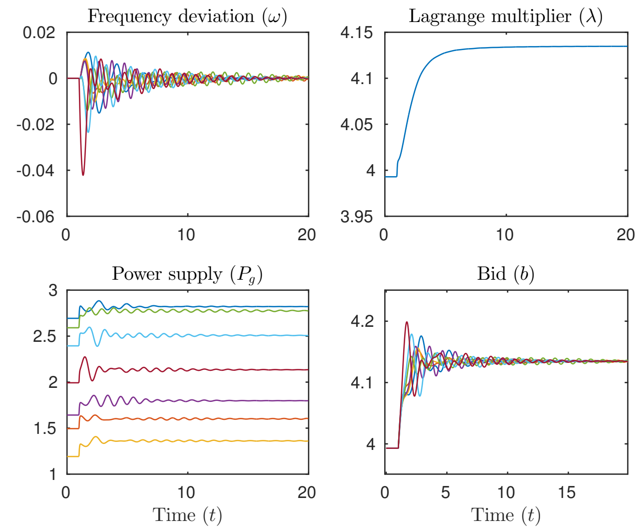

Fig. 3. Evolution of the time-triggered system (16). The same scenario and parameter values are considered as in Figure 2. We observe that the system still converges to the efficient Nash equilibrium $\lambda^* = b_i^* = 4.1348$ and the optimal generation levels (22) in case stepsizes are are given by $t_{k+1}^l - t_k^l = 0.01, t_0^{l+1} - t_0^l = 0.1$ for all $l \in \mathbb{Z}_{\geq 0}$, $k \in \{0, \ldots, 10\}$. However, because of the delay in the update process, the convergence is slower compared to the continuous-time system (11) simulated in Figure 2.

References

[1] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, "Breaking the hierarchy: Distributed control and economic optimality in microgrids," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 3, pp. 241–253, 2016.

[2] X. Zhang and A. Papachristodoulou, "A real-time control framework for smart power networks: Design methodology and stability," *Automatica*, vol. 58, pp. 43–50, 2015.

[3] S. Trip, M. Bürger, and C. De Persis, "An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages," *Automatica*, vol. 64, pp. 240–253, 2016.

[4] S. T. Cady, A. D. Domínguez-García, and C. N. Hadjicostis, "A distributed generation control architecture for islanded AC microgrids," *IEEE Transactions on Control Systems Technology*, vol. 23, no. 5, pp. 1717–1735, 2015.

[5] F. Alvarado, J. Meng, C. DeMarco, and W. Mota, "Stability analysis of interconnected power systems coupled with market dynamics," *IEEE Transactions on Power Systems*, vol. 16, no. 4, pp. 695–701, 2001.

[6] D. J. Shiltz, M. Cvetković, and A. M. Annaswamy, "An integrated dynamic market mechanism for real-time markets and frequency regulation," *IEEE Transactions on Sustainable Energy*, vol. 7, no. 2, pp. 875–885, 2016.

[7] T. Stegink, C. De Persis, and A. van der Schaft, "A unifying energy-based approach to stability of power grids with market dynamics," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 2612–2622, 2017.

[8] W. Tang and R. Jain, "Game-theoretic analysis of the nodal pricing mechanism for electricity markets," in *IEEE 52nd Annual Conference on Decision and Control (CDC),*, 2013, pp. 562–567.

[9] A. Cherukuri and J. Cortés, "Iterative bidding in electricity markets: rationality and robustness," *arXiv preprint arXiv:1702.06505*, 2017, submitted to *IEEE Transactions on Control of Network Systems*.

[10] ——, "Decentralized Nash equilibrium learning by strategic generators for economic dispatch," in *American Control Conference (ACC), 2016*. IEEE, 2016, pp. 1082–1087.

[11] L. Gan and S. H. Low, "An online gradient algorithm for optimal power flow on radial networks," *IEEE Journal on Selected Areas in Communications*, vol. 34, no. 3, pp. 625–638, 2016.

[12] Y. Zhang, M. Hong, E. Dall'Anese, S. Dhople, and Z. Xu, "Distributed controllers seeking AC optimal power flow solutions using ADMM," *IEEE Transactions on Smart Grid*, 2017, to appear.

[13] E. Dall'Anese, S. V. Dhople, and G. B. Giannakis, "Photovoltaic inverter controllers seeking ac optimal power flow solutions," *IEEE Transactions on Power Systems*, vol. 31, no. 4, pp. 2809–2823, 2016.

[14] J. Machowski, J. Bialek, and J. Bumby, *Power System Dynamics: Stability and Control*, 2nd ed. Ltd: John Wiley & Sons, 2008.

[15] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, no. 6, pp. 1539–1564, 2014.

[16] T. Başar and G. Oldser, *Dynamic Noncooperative Game Theory*. Academic Press, 1982.

[17] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA: MIT Press, 1991.

[18] S. Stoft, "Power system economics," *Journal of Energy Literature*, vol. 8, pp. 94–99, 2002.

[19] H. K. Khalil, *Nonlinear systems*. Prentice hall New Jersey, 1996, vol. 3.