Stable interconnection of continuous-time price-bidding mechanisms with power network dynamics

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***Abstract*—We study price-based bidding mechanisms in power networks for real-time dispatch and frequency regulation. On the market side, we consider the interaction between the independent system operator (ISO) and a group of generators involved in a Bertrand game of competition. The generators seek to maximize their individual profit while the ISO aims to solve the economic dispatch problem and to regulate the frequency. Since the generators are strategic and do not share their cost functions, the ISO engages the generators in a continuous-time price-based bidding process. This results in a coupling between the ISO-generator dynamics and swing dynamics of the network. We analyze its stability, establishing frequency regulation and the convergence to the efficient Nash equilibrium and the optimal generation levels. Simulation illustrate our results.**

***Index Terms*—electricity markets, Bertrand competition, Lyapunov stability, economic dispatch, power system dynamics.**

I. Introduction

In the current power network, market clearing and frequency regulation are considered separately. The motivation of this is the time-scale separation between real-time market (operating at 15 minute intervals) and the frequency dynamics (time-scale in the order of seconds). However, with the integration of distributed energy resources (DERs) in the grid, there is an increase in fluctuation and uncertainty in power network operation. With the goal of reducing uncertainty and retaining economic efficiency, an increasing body of research focuses on merging market clearing with frequency regulation. These observations motivate our work here on the interconnection of frequency dynamics with real-time price-based bidding in a competitive electricity market.

Literature review

The integration of economic dispatch with frequency regulation has received considerable attention in the recent years. Several works have focused on merging the traditional primary, secondary and tertiary frequency control layers for several different models of power networks [1], [2], [3] and microgrids [4], [5]. Other works adopt a different approach, proposing to use market mechanisms to obtain economic efficiency and stabilize the frequency [6], [7], [8]. In particular, these references study the stability of the interconnection of a dynamic real-time pricing model coupled with a physical model of the power network. We adopt a similar approach here, except that instead of considering generators to be price-takers, we assume that the generators are allowed to bid in the market and hence, are price setters. We have considered such an instance of a Bertrand competition game among the generators in our previous work [9], [10], where we showed that an iterative bidding mechanism leads to optimal generation levels. However, this relies on the underlying assumption that the generation setpoints can be commanded only upon convergence of the algorithm. This is a limitation given the fast time-scale of load/generation commitment changes due to the high penetration of DERs. Here, instead, we tackle both market clearing and frequency regulation simultaneously by considering the ISO-generator coordination coupled with the swing equations of the power network. This is similar to the setup in our work [11], that deals with iterative bidding schemes, but does not incorporate nonnegativity constraints on the bids and the generations levels. In fact, the theoretical guarantees on the continuous-time algorithm developed here sets the basis for the discrete-time implementations developed in [11].

Statement of contributions

We consider a power network consisting of an ISO and a group of competitive generators. The goal of the ISO is to solve the economic dispatch (ED) problem and regulate the frequency to the nominal value. However the generators are strategic and do not share information about their cost functions, making it unable for the ISO to solve the ED problem. Instead, the ISO allows the generators to make bids in the form of prices at which they are willing to produce and determines the generation setpoint based on them. We define the notions of Nash equilibrium and efficient Nash equilibrium of the underlying Bertrand game among the generators, where the latter corresponds to an optimal power generation allocation.

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Our first contribution is the characterization of efficient Nash equilibria in terms of the primal-dual optimizer of the ED problem, leading us to establishing its existence and identifying a condition for its uniqueness. Our second contribution is the design of a continuous-time update scheme for the ISO and the generators. Under this scheme, generators seek to maximize their profit by adjusting their bid based on the difference between their desired power generation and the production level that the ISO requests from them. On the other hand, the ISO seeks to minimize the total cost of matching generation and demand, while incorporating the local frequency error as a feedback signal for the negotiation process. We establish the local convergence of the combined ISO-generator-frequency dynamics to an optimizer of the ED problem, an efficient Nash equilibrium and the nominal network frequency. Moreover, we show that the nonnegativity constraints on the bids and power generation quantities are satisfied along the execution of the algorithm. Our technical approach relies on Lyapunov stability and the LaSalle Invariance Principle to obtain the performance guarantees. The proofs are omitted and will appear elsewhere.

Organization

The remainder of the paper is organized as follows. Section II introduces the notation and the dynamic model of the power network. Section III presents the problem statement and Section IV discusses the existence and uniqueness of efficient Nash equilibria. Section V introduces the ISO-generator bidding scheme and shows the local convergence to the efficient equilibria of the interconnection with the swing equations. Simulations illustrate the results in Section VI. Finally, we gather our conclusions and ideas for future work in Section VII.

Notation: Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}, \mathbb{N}$ be the set of real, nonnegative real, positive real, and positive integer numbers, respectively. We write the set $\{1, \ldots, n\}$ compactly as $[n]$. The notation $\mathbb{1} \in \mathbb{R}^n$ is used for the vector whose elements are equal to 1. Given a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, its gradient and its Hessian evaluated at x is denoted by $\nabla f(x)$ and $\nabla^2 f(x)$, respectively. For $v \in \mathbb{R}^n$, we let $\operatorname{diag}(v) \in \mathbb{R}^{n \times n}$ denote the diagonal matrix with entries $v_1, \ldots, v_n$ on the diagonal. A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* on $S \subset \mathbb{R}^n$ if it is convex and, for some $\mu > 0$, its Hessian satisfies $\nabla^2 f(x) > \mu I$ for all $x \in S$.

II. Power network model and dynamics

Here we present basic concepts on the dynamics of an electrical power network. Consider a power network consisting of n buses. The network is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes $\mathcal{V} = [n]$ represent buses and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ are the transmission lines connecting the buses. Let m denote the number of edges, which are arbitrarily labeled with a unique identifier in $[m]$. The ends of each edge are arbitrary labeled with '+' and '-', so that the incidence matrix $D \in \mathbb{R}^{n \times m}$ of the resulting directed graph is

$$D_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of edge } k, \\ -1 & \text{if } i \text{ is the negative end of edge } k, \\ 0 & \text{otherwise.} \end{cases}$$

Each bus represents a control area and is assumed to have one generator and one load. Following [12], the dynamics at the buses is given by (1); often referred to as the *swing equations*.

$$\begin{aligned} \dot{\delta} &= \omega \\ M\dot{\omega} &= -D\Gamma \sin(D^T\delta) - A\omega + P_g - P_d \end{aligned} \tag{1}$$

Here $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_m)$, where $\gamma_k = B_{ij}V_iV_j = B_{ji}V_iV_j$ and $k \in [m]$ corresponds to the edge between nodes i and j. Table I presents a list of symbols used in the model (1).

$\delta \in \mathbb{R}^n$	(vector of) voltage phase angles
$\omega \in \mathbb{R}^n$	frequency deviation w.r.t. the nominal frequency
$V_i \in \mathbb{R}_{>0}$	voltage magnitude at bus i
$P_d \in \mathbb{R}^n_{\geq 0}$	power load
$P_g \in \mathbb{R}^n_{\geq 0}$	power generation
$M \in \mathbb{R}^{n\times n}_{\geq 0}$	diagonal matrix of moments of inertia
$A \in \mathbb{R}^{n\times n}_{\geq 0}$	diagonal matrix of asynchronous damping constants
$B_{ij} \in \mathbb{R}_{>0}$	negative of the susceptance of transmission line (i, j)

Table I
PARAMETERS AND STATE VARIABLES OF SWING EQUATIONS (1).

To avoid rotational symmetry issues in (1), see e.g., [13], we define the voltage phase angle differences $\varphi = D_t^T\delta \in \mathbb{R}^{n-1}$. Here $D_t \in \mathbb{R}^{(n-1)\times n}$ is the incidence matrix of a tree graph on the set of buses $[n]$ (this could be a spanning tree of the physical network, for example). Furthermore, let $U(\varphi) = -\mathbb{1}^T\Gamma\cos(D^TD_t^{\dagger T}\varphi)$ where $D_t^\dagger = (D_t^TD_t)^{-1}D_t^T$ denotes the Moore-Penrose pseudo-inverse of D_t. By observing that $D_tD_t^\dagger D = (I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)D = D$, the physical system (1) in the (φ, ω)-coordinates takes the form

$$\begin{aligned} \dot{\varphi} &= D_t^T\omega \\ M\dot{\omega} &= -D_t\nabla U(\varphi) - A\omega + P_g - P_d. \end{aligned} \tag{2}$$

In the sequel, we assume that there exists an equilibrium $(\bar{\varphi}, \bar{\omega})$ of (2) for the power generation $\bar{P}_g$ that satisfies $D^TD_t^{\dagger T}\bar{\varphi} \in (-\pi/2, \pi/2)^m$. The latter assumption is standard and often referred to as the *security constraint* [12].

III. Problem statement

In this section we discuss the problem setting, introduce the necessary game-theoretic tools and then explain the paper objectives. We start from the electrical power network model described in Section II and describe the elements of the ISO-generator coordination problem following the exposition of [9], [10]. We denote by $C_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the cost incurred by generator $i \in [n]$ in producing P_{gi} units of power and we assume that C_i is strongly convex on the domain $\mathbb{R}_{\geq 0}$ and satisfies $\nabla C_i(0) \geq 0$. Given the total network cost

$$C(P_g) := \sum_{i\in[n]} C_i(P_{gi}) \tag{3}$$

and the power loads P_d the ISO seeks to solve the *economic dispatch (ED) problem* (4), and, at the same time, regulate the frequency of the physical power network.

$$\begin{aligned} &\underset{P_g}{\text{minimize}} && C(P_g), && \text{(4a)}\\ &\text{subject to} && \mathbb{1}^T P_g = \mathbb{1}^T P_d, && \text{(4b)}\\ & && P_g \geq 0. && \text{(4c)}\end{aligned}$$

We assume the total load to be positive, i.e., $\mathbb{1}^T P_d > 0$. Since the function C is strongly convex, and the constraints (4b), (4c) are affine, Slater's condition holds and there exists a unique primal-dual optimizer $(P_g^*, \lambda^*, \mu^*)$ of (4) with zero duality gap [14]. However, the ISO is unable to find the optimizer of the ED problem (4) because generators are strategic and they do not reveal their cost functions to anyone. Instead, the ISO operates a market where each generator $i \in [n]$ submits a bid $b_i \in \mathbb{R}_{\geq 0}$ in the form of a price at which it is willing to provide power. Based on these bids, the ISO aims to find the power allocation that meets the load and that minimizes the total payment to the generators. Thus instead of solving the ED problem (4) directly, the ISO considers, given a bid $b \in \mathbb{R}^n_{\geq 0}$, the convex optimization problem (5).

$$\begin{aligned} &\underset{P_g}{\text{minimize}} && b^T P_g && \text{(5a)}\\ &\text{subject to} && \mathbb{1}^T P_g = \mathbb{1}^T P_d && \text{(5b)}\\ & && P_g \geq 0 && \text{(5c)}\end{aligned}$$

A fundamental difference between (4) and (5) is that the latter optimization is linear and may in general have multiple solutions. Let $P_g^{\text{opt}}(b)$ be the optimizer of (5) the ISO selects given bids b and note that this might not be unique. Knowing the ISO's strategy, each generator i bids a quantity $b_i \geq 0$ to maximize its payoff

$$\Pi_i(b_i, P_{gi}^{\text{opt}}(b)) := P_{gi}^{\text{opt}}(b) b_i - C_i(P_{gi}^{\text{opt}}(b)), \tag{6}$$

where $P_{gi}^{\text{opt}}(b)$ is the i-th component of the optimizer $P_g^{\text{opt}}(b)$. Since each generator is strategic, we analyze the market clearing, and hence the dispatch process explained above using tools from game theory [15], [16]. We define the *inelastic market game* as

- Players: the set of generators $[n]$.
- Action: for each player i, the bid $b_i \in \mathbb{R}_{\geq 0}$.
- Payoff: for each player i, the payoff Π_i in (6).

In the sequel we interchangeably use the notation $b \in \mathbb{R}^n_{\geq 0}$ and $(b_i, b_{-i}) \in \mathbb{R}^n_{\geq 0}$ for the bid vector where b_{-i} represents the bids of all players except i. We note that the payoff of generator i not only depends on the bids of the other players but also on the optimizer $P_g^{\text{opt}}(b)$ the ISO selects. We therefore use a slightly different notion of the Nash equilibrium than the usual one, which has also been defined in [10].

Definition III.1 (Nash equilibrium [10])**.** We define the *Nash equilibrium* of the inelastic electricity market game as the bid profile $b^* \in \mathbb{R}^n_{\geq 0}$ for which there exists an optimizer $P_g^{\text{opt}}(b^*)$ of (5) such that for each $i \in [n]$,

$$\Pi_i(b_i, P_{gi}^{\text{opt}}(b_i, b^*_{-i})) \leq \Pi_i(b_i^*, P_{gi}^{\text{opt}}(b^*))$$

for all $b_i \in \mathbb{R}_{\geq 0}$ with $b_i \neq b_i^*$ and all optimizers $P_{gi}^{\text{opt}}(b_i, b^*_{-i})$ of (5) given bids (b_i, b^*_{-i}).

We are particularly interested in bid profiles for which the optimizer of (4) is also a solution to (5). This is captured in the following definition.

Definition III.2 (Efficient bid and efficient Nash equilibrium)**.** An *efficient bid* of the inelastic electricity market is a bid $b^* \in \mathbb{R}^n_{\geq 0}$ for which the optimizer P_g^* of (4) is also an optimizer of (5) given bids $b = b^*$ and

$$P_{gi}^* = \underset{P_{gi}\geq 0}{\arg\max}\{P_{gi} b_i^* - C_i(P_{gi})\} \quad \text{for each } i \in [n]. \tag{7}$$

A bid $b^* \in \mathbb{R}^n_{\geq 0}$ is an *efficient Nash equilibrium* of the inelastic electricity market game if it is an efficient bid and a Nash equilibrium.

At the efficient Nash equilibrium, the optimizer of the ED problem coincides with the production levels that maximize the individual profits (6) of the generators, which justifies the study of efficient Nash equilibria. Our objective is to study the interconnection between the bidding process that takes place to carry out the ISO-generator coordination and the physical dynamics of the power network. In previous work [9], [10], we have focused only on the bidding process, without regard for the power network dynamics, while in [11] we have included frequency dynamics but disregarding the nonnegativity constraints on the bids and the production levels. In our forthcoming discussion, we introduce a continuous-time dynamics to perform the bidding process without violating the nonnegativity constraints throughout its execution, and establish the stability of its interconnection with the power system dynamics.

IV. Existence and uniqueness of Nash equilibria

Here we establish the existence of an efficient Nash equilibrium of the inelastic electricity market game described in the previous section. Thereafter, we provide a condition under which this equilibrium is unique.

Proposition IV.1. *(Existence of efficient Nash equilibria): Let $(P_g^*, \lambda^*, \mu^*)$ be a primal-dual optimizer of* (4)*, that is, $P_g^* \in \mathbb{R}^n, \lambda^* \in \mathbb{R}, \mu^* \in \mathbb{R}^n$ satisfy the Karush-Kuhn-Tucker (KKT) conditions*

$$\begin{aligned} \nabla C(P_g^*) = \mathbb{1}\lambda^* + \mu^*, \qquad \mathbb{1}^T P_g^* = \mathbb{1}^T P_d,\\ P_g^* \geq 0, \qquad \mu^* \geq 0, \qquad (P_g^*)^T \mu^* = 0. \end{aligned} \tag{8}$$

Then, any $b^* \in \mathbb{R}^n_{\geq 0}$ satisfying $\mathbb{1}\lambda^* \leq b^* \leq \nabla C(P_g^*)$ is an efficient Nash equilibrium of the inelastic electricity market game.

The proof of Proposition IV.1 shows that at a Nash equilibrium, either generator i's bid $\bar{b}_i$ is equal to the Lagrange multiplier $\bar{\lambda}$ associated to the power balance (4b), or the generation $\bar{P}_{gi}$ is zero. In case the latter holds, generator i's bid is larger than or equal to $\bar{\lambda}$. This represents the case that generator i's marginal costs at zero power production is larger than or equal to the market clearing price, and hence generator i is not willing to produce any electricity in that case. These observations lead us to identify a sufficient condition to guarantee the uniqueness of the efficient Nash equilibrium.

Corollary IV.2 (Uniqueness of the efficient Nash equilibrium)**.** *Let* $(P_g^*, \lambda^*, \mu^*)$ *be a primal-dual optimizer of* (4) *such that* $P_g^* > 0$*, then* $b^* = \nabla C(P_g^*) = \mathbb{1}\lambda^*$ *is the unique efficient Nash equilibrium of the inelastic electricity market game.*

Remark IV.3 (Any efficient Nash equilibrium is positive). We observe from the optimality conditions (8) that, since $\mathbb{1}^T P_d > 0$, and $P_g^* \geq 0$, we must have that $P_{gi}^* > 0, \mu_i^* = 0$ for some $i \in [n]$. As $\nabla C_i(P_{gi}^*) > 0$ by the strict convexity of C_i and the assumption $\nabla C_i(0) \geq 0$, this implies that $\lambda^* > 0$ and therefore also $b^* > 0$. •

V. Interconnection of continuous-time bid update scheme with power network dynamics

This section introduces a continuous-time ISO-generator coordination mechanism where generators update their bids based on the power generation setpoints received from the ISO, and the ISO updates the power generation setpoints depending on the generator bids and the frequency of the network. Each generator can only communicate with the ISO and is not aware of the number of other generators participating, their respective cost functions, or the load at its own bus. We study the stability of the interconnection of this bidding process with the physical dynamics of the system and establish local convergence to an efficient Nash equilibrium, an optimizer of the ED problem, and zero frequency deviation.

A. Continuous-time price-bidding mechanism

In our design, each generator $i \in [n]$ updates its bid $b_i \geq 0$ according to

$$\tau_{bi}\dot{b}_i = b_i(P_{gi} - \nabla C_i^*(b_i)), \tag{9a}$$

where $\tau_{bi} > 0$. The map $C_i^* : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes the *convex conjugate* of the cost function C_i and is defined by

$$C_i^*(b_i) := \max_{P_{gi} \geq 0} \{b_i P_{gi} - C_i(P_{gi})\}.$$

Using tools from convex analysis [17, Section I.6], one can show that C_i^* is convex and continuously differentiable on the domain $\mathbb{R}_{\geq 0}$, strictly convex on the domain $[\nabla C_i(0), \infty)$, and satisfies $\nabla C_i^*(b_i) = \arg\max_{P_{gi} \geq 0}\{P_{gi} b_i - C(P_{gi})\}$ for all $b_i \geq 0$.

The update law (9a) can be interpreted as follows: given a bid $b_i > 0$, generator i wants to produce the amount of power that results in maximal profit, i.e., $P_{gi}^{\text{des}} = \nabla C_i^*(b_i) = \arg\max_{P_{gi} \geq 0}\{C_i(P_{gi}) - b_i P_{gi}\}$. Therefore, if the ISO requests generator i to produce more power than its desired quantity, i.e., $P_{gi} > P_{gi}^{\text{des}}$, it increases its bid, and vice versa. We assume in (9a) that the initial bid is positive, i.e. $b_i(0) > 0$, which ensures that the bid remains nonnegative for all future time. Interestingly, (9a) can be seen as the continuous-time version of the discrete-time dynamics proposed in [9], [10] where instead of using a projection, a smooth vector field is used to deal with the nonnegativity constraint.

We now provide an update law for the ISO depending on the bid $b \in \mathbb{R}^n_{\geq 0}$ and the network frequency. Based on a smooth primal-dual dynamics [18] associated to the problem (5), the ISO updates its actions according to

$$\begin{aligned} \tau_g \dot{P}_g &= \text{diag}\,(P_g)(\mathbb{1}\lambda - b - \rho\mathbb{1}\mathbb{1}^T(P_g - P_d) - \sigma^2\omega) \\ \tau_\lambda \dot{\lambda} &= \mathbb{1}^T(P_d - P_g) \end{aligned} \tag{9b}$$

with parameters $\rho, \sigma, \tau_\lambda \in \mathbb{R}_{>0}$ and diagonal positive definite matrix $\tau_g \in \mathbb{R}^{n \times n}$. Also here we assume that $P_g(0) > 0$ and from (9b) we observe that $P_g(t) \geq 0$ for all $t \geq 0$. The dynamics (9b) can be interpreted as follows. If generator i bids higher than the Lagrange multiplier λ associated with the power balance constraint (5b), then the power generation (setpoint) of node i is decreased, and vice versa. We include the term $\rho\mathbb{1}\mathbb{1}^T(P_g - P_d)$ to add more damping in (9b) and we add the feedback signal $-\sigma^2\omega$ to compensate for the frequency deviations in the physical system. In our forthcoming discussion, we analyze the equilibria and the stability of the interconnection of the physical power network dynamics (2) with the bidding process (9).

B. Equilibrium analysis of the interconnected system

The evolution of the coupled system between the bidding scheme introduced in Section V-A and the physical dynamics of the power network is given by (10) where $C^*(b) := \sum_{i \in [n]} C_i^*(b_i)$, and $\tau_b = \text{diag}(\tau_{b1}, \ldots, \tau_{bn}) \in \mathbb{R}^{n \times n}$.

$$\dot{\varphi} = D_t^T \omega \tag{10a}$$

$$M\dot{\omega} = -D_t \nabla U(\varphi) - A\omega + P_g - P_d \tag{10b}$$

$$\tau_b \dot{b} = \text{diag}\,(b)(P_g - \nabla C^*(b)) \tag{10c}$$

$$\tau_g \dot{P}_g = \text{diag}\,(P_g)(\mathbb{1}\lambda - b - \rho\mathbb{1}\mathbb{1}^T(P_g - P_d) - \sigma^2\omega) \tag{10d}$$

$$\tau_\lambda \dot{\lambda} = \mathbb{1}^T(P_d - P_g) \tag{10e}$$

Note that in general there exist many equilibria of (10). However, we are particularly interested in the equilibria that correspond simultaneously to an efficient Nash equilibrium, economic dispatch and zero frequency deviation, as specified next.

Definition V.1 (Efficient equilibrium)**.** An equilibrium $\bar{x} = \text{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{\lambda})$ of (10) is called *efficient* if $\bar{b}$ is an efficient Nash equilibrium, $\bar{P}_g$ is a primal optimizer of (4), and $\bar{\omega} = 0$.

The next result shows necessary and sufficient conditions for an equilibrium of (10) to be efficient.

Lemma V.2. *(Characteristics of efficient equilibria): The vector $\bar{x} = \mathrm{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{\lambda})$ is an efficient equilibrium of* (10) *if and only if it satisfies*

$$-D_t \nabla U(\bar{\varphi}) + \bar{P}_g - P_d = 0, \tag{11a}$$

$$\bar{\omega} = 0, \tag{11b}$$

$$\bar{P}_g = \nabla C^*(\bar{b}), \tag{11c}$$

$$\bar{P}_g^T(\mathbb{1}\bar{\lambda} - \bar{b}) = 0, \tag{11d}$$

$$\mathbb{1}\bar{\lambda} \leq \bar{b}, \tag{11e}$$

$$\mathbb{1}^T \bar{P}_g = \mathbb{1}^T P_d. \tag{11f}$$

In the next section we investigate the stability of the dynamics (10) and prove local asymptotic convergence to the set of efficient equilibria.

C. Convergence analysis of the interconnected system

In the previous section we restricted our attention to equilibria of (10) that are efficient. We show next that under the security constraint mentioned in Section II, local convergence of the set of efficient equilibria is guaranteed.

Theorem V.3. *(Local convergence of closed-loop system): Consider the system* (10) *and suppose the set*

$$\mathcal{X} := \{\bar{x} \mid \bar{x} = \mathrm{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{\lambda}) \text{ satisfies } (11) \text{ and } D^T D_t^{\dagger T} \bar{\varphi} \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})^m\} \tag{12}$$

is non-empty. Then $\mathcal{X}$ is locally asymptotically stable under (10). *Moreover, convergence is to a point.*

VI. Simulations

In this section we illustrate the convergence properties of the interconnected system (10). We consider the power network depicted in Figure 1, where each node has one generator and one load. We assume quadratic generator cost functions

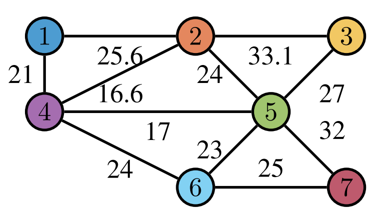

Figure 1. A 7-node power network. Each node contains one generator and one load. The edges correspond to transmission lines and the numbers indicate the values of the line susceptances.

of the form $C_i(P_{gi}) = \frac{1}{2} q_i P_{gi}^2 + c_i P_{gi}$. Table II shows the parameter values appearing in the dynamics (2)-(9a). The parameter values for the ISO dynamics (9b) are $\tau_g = 0.5I$, $\tau_\lambda = 0.2$, $\rho = 3$, $\sigma = 2$.

The initial load is $P_d = \mathrm{col}(3, 1, 2, 1, 2, 3, 2)$. For this load, the unique efficient Nash equilibrium is $b_i^* = 3.64$ for all $i \in [7]$ and the optimizer of (4) amounts to

$$P_g^* = \mathrm{col}(2.93, 2.19, 0.28, 2.40, 1.26, 3.30, 1.64), \tag{13}$$

Parameter	Node 1	Node 2	Node 3	Node 4	Node 5	Node 6	Node 7
A_i	3.46	2.62	2.97	3.05	3.14	2.73	2.82
M_i	5.22	3.98	4.49	4.22	5.01	4.03	4.24
τ_{bi}	.554	.741	.611	.622	.534	.713	.602
V_i	1.0	1.2	1.1	1.1	1.0	1.1	1.3
q_i	.90	.75	4.0	1.1	1.3	.80	1.0
c_i	1.0	2.0	2.5	1.0	2.0	1.0	2.0

Table II

Parameter values of the benchmark system.

which we calculate using the Matlab function `quadprog`. The the closed-loop system (10) is initialized at this equilibrium.

At $t = 1$ s, the power load at node 7 is increased from 2 to 3 units, and the new unique efficient Nash equilibrium becomes $b_i^* = 3.79, i \in [7]$ and the primal optimizer of (4) changes to $P_g^* = \mathrm{col}(3.102.390.322.541.383.491.79)$.

At time $t = 15$ s, we change the cost function of generator 3 to $C_3(P_{g3}) = 2P_{g3}^2 + 4P_{g3}$. In this case, the nonnegativity constraint of generator 3 becomes active and we calculate the primal optimizer of (4) as $P_g^* = \mathrm{col}(3.16, 2.45, 0, 2.58, 1.42, 3.55, 1.84)$. The Lagrange multiplier of the power balance constraint becomes $\lambda^* = 3.84$. By Proposition IV.1, any bid b^* satisfying $3.84 \leq b_3^* \leq 4 = \nabla C_3(0)$, and $b_i^* = 3.84$ for the remaining generators is an efficient Nash equilibrium.

At $t = 25$ s, the power load at each node is increased by 0.5 units to $P_d = \mathrm{col}(3.5, 1.5, 2.5, 1.5, 2.5, 3.5, 3.5)$. As a result, the nonnegativity constraint of generator 3 becomes inactive again. Specifically, the primal optimizer of (4) becomes

$$P_g^* = \mathrm{col}(3.75, 3.17, 0.09, 3.07, 1.83, 4.22, 2.38)$$

and the efficient Nash equilibrium is $b_i^* = 4.37, i \in [7]$. Figure 2 shows the evolution of the closed-loop continuous-time (10) system. As predicted by Theorem V.3, the frequency is regulated and the dynamics converges to an equilibrium corresponding to an efficient Nash equilibrium, economic dispatch, and zero frequency deviation.

VII. Conclusions

We have studied the stability of a dynamic competitive market model coupled with the frequency dynamics of the power network. On the market side, we have formulated a Bertrand game among strategic generators that seek to respond to the ISO's demand for power generation. We have established the existence of an efficient Nash equilibrium and designed a dynamic bidding interaction scheme between the ISO and the generators. We have shown that the interconnection of this scheme with the physical dynamics of the network leads to a stable closed-loop system. In particular, we have proven that trajectories locally converge an efficient Nash equilibrium and economic dispatch, and the frequency approaches the nominal frequency. In future research, we aim to include power flow constraints in the economic dispatch problem and look at a more realistic implementation of the proposed bidding process in which the ISO and the generators iteratively update their actions while respecting the constraints.

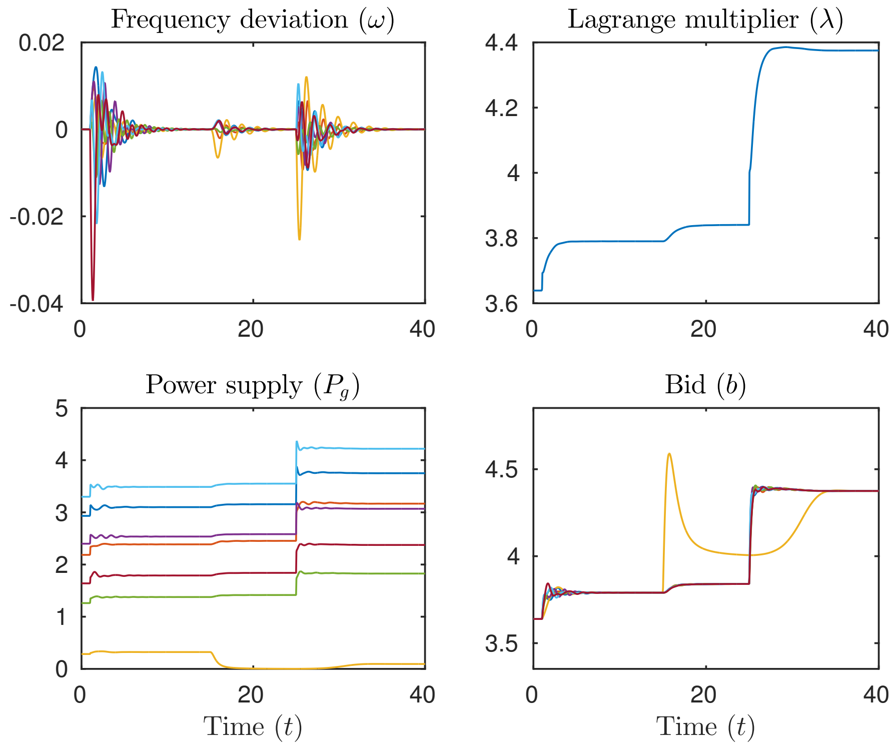

Figure 2. Simulation of the continuous-time system (10) for the power network in Figure 1 with the parameter values in Table II. The system is initialized at steady state corresponding to (13). At $t = 1\,\mathrm{s}$, the load at node 7 is increased to 3 units. At $t = 15\,\mathrm{s}$ the cost function of generator 3 is changed to $C_3(P_{g3}) = 2P_{g3}^2 + 4P_{g3}$ such that its nonnegativity constraint becomes active at economic dispatch, resulting in a non-unique efficient Nash equilibrium. At $t = 25\,\mathrm{s}$ each load in increased by 0.5 units making the nonnegativity constraint of generator 3 inactive. After each change, the bids of the generators converge to a new efficient Nash equilibrium and the power production levels converge to the associated optimizer of (4). In addition, the frequency is regulated.

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