Event-Triggered Stabilization of Nonlinear Systems with Time-Varying Sensing and Actuation Delay

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Abstract

This paper studies the problem of stabilization of a nonlinear system with time-varying delays in both sensing and actuation using event-triggered control. Our proposed strategy seeks to opportunistically minimize the number of control updates while guaranteeing stabilization and builds on predictor feedback to compensate for arbitrarily large known time-varying delays. We establish, using a Lyapunov approach, the global asymptotic stability of the closed-loop system as long as the open-loop system is globally input-to-state stabilizable in the absence of time delays and event-triggering. We further prove that the proposed event-triggered law has inter-event times that are uniformly lower bounded and hence does not exhibit Zeno behavior. For the particular case of a stabilizable linear system, we show global exponential stability of the closed-loop system and analyze the trade-off between the rate of exponential convergence and average sampling frequency. We illustrate these results in simulation and also examine the properties of the proposed event-triggered strategy beyond the class of systems for which stabilization can be guaranteed.

1 Introduction

Event-triggered and self-triggered approaches have recently gained popularity for controlling cyberphysical systems. The basic premise is that of abandoning the assumption of continuous or periodic updating of the control signal and instead adopt an opportunistic perspective that leads to deliberate, aperiodic updates. The challenge resides in determining precisely when control signals should be updated to improve efficiency while still guaranteeing convergence. This paper expands the state-of-the-art in opportunistic state-triggered control by designing predictor-based event-triggered control strategies that stabilize nonlinear systems with *known* delays in both sensing and actuation that can be *arbitrarily large* and *time-varying*.

Literature review: There exists a vast literature on both event-triggered control and the control of time-delay systems. Here, we review the works most closely related to our treatment. Originating from event-based and discrete-event systems [Cassandras and Lafortune, 2007, Zou et al., 2017], the concept of event-triggered control (i.e., updating the control signal in an opportunistic fashion) was proposed in [Kopetz, 1991, Åström and Bernhardsson., 2002] and has found its way into the efficient use of sensing, computing, actuation, and communication resources in networked control systems, see e.g., [Tabuada, 2007, Wang and Lemmon, 2011, Heemels et al., 2012, Abdelrahim et al., 2017] and references therein. On the other hand, the notion of predictor feedback is a powerful method in dealing with controlled systems subject to time delay [Smith, 1959, Mayne, 1968, Manitius and Olbrot, 1979, Nihtila, 1991, Krstic, 2009]. In essence, a predictor feedback controller anticipates the future evolution of the plant using its forward model and sends the control signal early enough to compensate for the delay. Here, we pursue a Lyapunov-based analysis of predictor feedback following [Bekiaris-Liberis and Krstic, 2013]. Given that the numerical implementations of predictor feedback controllers are particularly challenging [Mirkin, 2004, Zhong, 2004], we further discuss several methods for the numerical implementation of our proposed controller and show that a carefully designed "closed-loop" method is numerically stable and robust to errors in delay compensation.

The joint treatment of time delay and event-triggering is particularly challenging. By its opportunistic nature, an event-triggered controller keeps the control value unchanged until the plant is close to instability and then updates the control value according to the current state. Now, if time delays exist, the controller only has access to some past state of the plant (delayed sensing) and it takes some time for an updated control action to reach the plant (delayed actuation), jointly increasing the possibility of the updated control value being already obsolete when it is implemented in the plant, resulting in instability. Therefore, the controller needs to be sufficiently proactive and update the control value sufficiently ahead of time to maintain closed-loop stability. This makes the design problem challenging. Delays in actuation and sensing may be due to communication delays between controller-actuator and controller-sensing

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pairs, and in that sense, previous work on the event-triggered control literature that specifically considers delays in the communication channel deals with a similar problem setup as the one considered here. Several event-triggered designs consider scenarios where the system dynamics are linear, see, e.g. [Zhang et al., 2017, Chen et al., 2017, Selivanov and Fridman, 2016, Ge and Han, 2015, Garcia and Antsaklis, 2013]. The inclusion of nonlinearity, however, makes the problem more challenging. When digital controllers are used and the delay is smaller than the sampling time, [Hetel et al., 2006, Wu et al., 2015] design event-triggered controllers for the resulting delay-free discretized system. Robust event-triggered stabilizing controllers are also designed for nonlinear systems with sensing delays in [Li et al., 2012] and with both sensing and actuation delays in [Dolk et al., 2017]. In all these works, however, a key assumption is that the (maximum) delay is smaller than the (minimum) inter-transmission time. This assumption (also called the small-delay case) allows for the *treatment of delay as a disturbance* and, by construction, can tolerate unknown delays. In reality, however, (minimum) inter-transmission times can be very small, making this assumption restrictive. We take a different perspective here and consider arbitrarily large delays, with the expected tradeoff in our treatment that the delay can no longer be unknown. The technical approach is based on using predictors that capture the effect of the delay on the system to compensate for it. We rigorously analyze the case when the delay is accurately known and show in simulation that our design is indeed robust to small variations when the delay is only approximately known.

Statement of contributions: Our contributions are three-fold. First, we design an event-triggered controller for stabilization of nonlinear systems with arbitrarily large sensing and actuation delays. We employ the method of predictor feedback to compensate for the delay in both and then co-design the control law and triggering strategy to guarantee the monotonic decay of a Lyapunov-Krasovskii functional. Our second contribution involves the closed-loop analysis of the event-triggered law, proving that the closed-loop system is globally asymptotically stable and the inter-event times are uniformly lower bounded (and thus no Zeno behavior may exist). Due to the importance of linear systems in numerous applications, we briefly discuss the simplifications of the design and analysis in this case. Our final contribution pertains to the trade-off between convergence rate and sampling. Our analysis in this part is limited to linear systems, where closed-form solutions are derivable for (exponential) convergence rate and minimum inter-event times. We provide a quantitative account of the well-known trade-off between sampling and convergence in event-triggered designs and show how this trade-off can be biased in either direction by tuning a design parameter. Finally, we present simulations to illustrate the effectiveness of our design and address its numerical implementation.

2 Preliminaries

We introduce notational conventions and briefly review notions on input-to-state stability. We denote by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ the sets of reals and nonnegative reals, respectively. Given a vector or matrix, we use $|\cdot|$ to denote the Euclidean norm. We denote by $\mathcal{K}$ the set of strictly increasing continuous functions $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$. α belongs to $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$. We denote by $\mathcal{KL}$ the set of functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that, for each $s \in [0,\infty)$, $r \mapsto \beta(r,s)$ is non-decreasing and continuous and $\beta(0,s) = 0$ and, for each $r \in [0,\infty)$, $s \mapsto \beta(r,s)$ is monotonically decreasing with $\beta(r,s) \to 0$ as $s \to \infty$. We use the notation $\mathcal{L}_f S = \nabla S \cdot f$ for the Lie derivative of a function $S : \mathbb{R}^n \to \mathbb{R}$ along the trajectories of a vector field f taking values in $\mathbb{R}^n$.

We follow [Sontag and Wang, 1995] to review the definition of input-to-state stability of nonlinear systems and its Lyapunov characterization. Consider a nonlinear system of the form

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable and satisfies $f(0,0) = 0$. For simplicity, we assume that this system has a unique solution which does not exhibit finite escape time. System (1) is (globally) input-to-state stable (ISS) if there exist $\alpha \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for any measurable locally essentially bounded input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and any initial condition $x(0) \in \mathbb{R}^n$, its solution satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \alpha\big(\sup_{t\geq 0} |u(t)|\big),$$

for all $t \geq 0$. For this system, a continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function if there exist $\alpha_1, \alpha_2, \gamma, \rho \in \mathcal{K}_\infty$ such that

$$\forall x \in \mathbb{R}^n \qquad \alpha_1(|x|) \leq S(x) \leq \alpha_2(|x|), \tag{2a}$$

$$\forall (x,u) \in \mathbb{R}^{n+m} \quad \mathcal{L}_f S(x,u) \leq -\gamma(|x|) + \rho(|u|). \tag{2b}$$

According to [Sontag and Wang, 1995, Theorem 1], the system (1) is ISS if and only if it admits an ISS-Lyapunov function.

3 Problem Statement

Consider the nonlinear system ("plant") with dynamics

$$\dot{x}(t) = f(x(t), u_p(t)), \qquad t \geq 0, \tag{3}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable and $f(0,0) = 0$. We assume the plant (3) does not exhibit finite escape time for any initial condition and any bounded input [1]. Our goal is to provide a state-feedback controller ensuring global asymptotic stability under the following challenges:

(i) **Actuation delay:** Let $u(t)$ be the control signal generated by the controller. Actuation delay is modeled as

$$u_p(t) = u(\phi(t)), \quad t \geq 0,$$

where $t - \phi(t) > 0$ is the amount of time that it takes for a control action generated at time $\phi(t)$ to reach the

[1] This is a technical assumption and is satisfied by any real system due to bounded vector fields.

plant/actuator [2]. This delay may be due to the controller computation time, signal transmission time from controller to actuator (e.g, when they are not co-located), actuation lag due internal actuator dynamics, or a combination thereof. For simplicity, we assume ϕ is continuously differentiable. We also assume ϕ is monotonically increasing ($\dot{\phi}(t) > 0$), so the time argument of the control does not go back in time We further assume the delay and its derivative are bounded, i.e., there exist $M_0, M_1, m_2 > 0$ such that

$$t - \phi(t) \le M_0 \quad \text{and} \quad m_2 \le \dot{\phi}(t) \le M_1, \quad t \ge 0. \quad (4)$$

In the case of a constant actuation delay D, we have $\phi(t) = t - D$, trivially satisfying (4) with $M_0 = D$ and $M_1 = m_2 = 1$.
(ii) **Sensing delay:** We assume the sensor information (state values) take $t-\psi(t)$ seconds to reach the controller, where the delay function ψ is also monotonically increasing but need *not* be known a priori. Again, this delay may be due to processing times at the sensor, the transmission time from sensor to controller (e.g, if they are not co-located), or a combination thereof. We rely on a posteriori knowledge of the values of ψ only at the times when state information is available.
(iii) **Actuation event-triggering:** We aim to design opportunistic event-triggered controllers that do not require continuous updating of the control signals. This is motivated by practical concerns about the implementability of the controller in real time, and also by considerations about the efficient use of the available resources (to prevent, for instance, wear-and-tear of the actuator, or to accommodate bandwidth limitations when sensor, controller, and actuator are not co-located). We seek to design a controller that updates $u(t)$ only at a sequence of discrete times $\{t_k\}_{k=0}^{\infty}$,

$$u(t) = u(t_k), \quad t \in [t_k, t_{k+1}), \quad k \ge 0. \quad (5)$$

(iv) **Sensing event-triggering:** We assume the sensor can only send an event-triggered sequence of states $\{x(\tau_\ell)\}_{\ell=0}^{\infty}$ to the controller. However, $\{\tau_\ell\}_{\ell=0}^{\infty}$ is fully determined by the sensor independently of our design. The sensor ensures that $\lim_{\ell\to\infty} \tau_\ell = \infty$. For simplicity, we let $\tau_0 = 0$ and $t_0 = \psi^{-1}(0)$ ($u(t)$ is arbitrarily set in $[0, t_0)$ as the controller has not received any state information yet).

Since our focus is on the challenges imposed by time delays and event-triggered control, we assume the origin of (3) is robustly globally asymptotically stabilizable in the absence of delays and with continuous sensing and actuation. Formally, we assume that there exists a globally Lipschitz feedback law $K : \mathbb{R}^n \to \mathbb{R}^m$, $K(0) = 0$, that makes

$$\dot{x}(t) = f(x(t), K(x(t)) + w(t)), \quad (6)$$

ISS with respect to the additive input disturbance w. The availability of this feedback law is the premise to tackle the challenges posed by the network implementation.

[2] The initial control $\{u(t)\}_{t=\phi(0)}^{0}$ is given and continuously differentiable.

Problem 1 *(Event-Triggered Stabilization under Sensing and Actuation Delay): Design the sequence of actuation triggering times*[3] $\{t_k\}_{k=1}^{\infty}$ *and the corresponding control values* $\{u(t_k)\}_{k=0}^{\infty}$ *such that* $\lim_{k\to\infty} t_k = \infty$ *and the closed-loop system* (3) *is globally asymptotically stable using the piecewise constant control* (5) *and the delayed information* $\{x(\tau_\ell)\}_{\ell=0}^{\infty}$ *received, resp., at* $\{\psi^{-1}(\tau_\ell)\}_{\ell=0}^{\infty}$. [4] •

The requirement $\lim_{k\to\infty} t_k = \infty$ ensures the resulting design is implementable by avoiding finite accumulation points.

4 Event-Triggered Design and Analysis

In this section, we propose an event-triggered control policy to solve Problem 1. We start our analysis with the simpler case where the controller receives state feedback continuously (i.e., $\{x(t)\}_{t=0}^{\infty}$ instead of $\{x(\tau_\ell)\}_{\ell=0}^{\infty}$) without delays (i.e., $\psi(t) = t$), and later extend it to the general case.

4.1 Predictor Feedback Control for Time-Delay Systems

Here we review the continuous-time stabilization of the dynamics (3) by means of a predictor-based feedback control [Bekiaris-Liberis and Krstic, 2013]. For convenience, we denote the inverse of ϕ by $\sigma(t) = \phi^{-1}(t)$, for all $t \ge 0$. The inverse exists since ϕ is strictly monotonically increasing. From (4), for all $t \ge \phi(0)$,

$$\frac{1}{\sigma(t) - t} \ge m_0 \quad \text{and} \quad m_1 \le \dot{\sigma}(t) \le M_2,$$

for $m_0 = \frac{1}{M_0}$, $m_1 = \frac{1}{M_1}$, and $M_2 = \frac{1}{m_2}$. To compensate for the delay, at any time $t \ge \phi(0)$, the controller makes the following prediction of the future state of the plant,

$$p(t) = x(\sigma(t)) = x(t^+) + \int_{\phi(t^+)}^{t} \dot{\sigma}(s) f(p(s), u(s)) ds, \quad (7)$$

where $t^+ = \max\{t, 0\}$. This integral is computable by the controller since it only requires knowledge of the initial or current state of the plant (gathered from the sensors) and the history of $u(t)$ and $p(t)$, both of which are available to the controller. Nevertheless, for general nonlinear vector fields f, (7) may not have a closed-form solution and it has to be computed using numerical integration methods, cf. Remark 6.1 below. The controller applies the control law K on the prediction p in order to compensate for the delay, i.e.,

$$u(t) = K(p(t)), \qquad t \ge 0. \quad (8)$$

[3] Recall that $t_0 = \psi^{-1}(0)$ is fixed.
[4] We require that the control law is causal, i.e., t_k and $u(t_k)$ depend only on the states $\{x(\tau_\ell)\}$ that have reached the controller by the time t_k. While sampling may be modeled as a specific type of delay, we capture it with the prediction error $e(t)$ (defined later). The values $\phi(t)$ and $\psi(t)$ only capture the delays in actuation and sensing, resp.

The next result shows convergence for the closed-loop system.

Proposition 4.1 *(Asymptotic Stabilization by Predictor Feedback [Bekiaris-Liberis and Krstic, 2013]): Under the aforementioned assumptions, the closed-loop system* (3) *under the controller* (8) *is globally asymptotically stable, i.e., there exists* $\beta \in \mathcal{KL}$ *such that for any* $x(0) \in \mathbb{R}^n$ *and bounded* $\{u(t)\}_{t=\phi(0)}^{0}$*, for all* $t \geq 0$,

$$|x(t)| + \sup_{\phi(t)\leq\tau\leq t} |u(\tau)| \leq \beta\Big(|x(0)| + \sup_{\phi(0)\leq\tau\leq 0} |u(\tau)|, t\Big).$$

4.2 Design of Event-triggered Control Law

Following Section 4.1, we let the controller make the prediction $p(t)$ according to (7) for all $t \geq \phi(0)$. Since the controller can only update $u(t)$ at discrete times $\{t_k\}_{k=0}^{\infty}$, it uses the piecewise-constant control (5) and assigns the control

$$u(t_k) = K(p(t_k)), \tag{9}$$

for all $k \geq 0$. In order to design the triggering times $\{t_k\}_{k=1}^{\infty}$, we use Lyapunov stability tools to determine when the controller has to update $u(t)$ to prevent instability. We define the triggering error for all $t \geq \phi(0)$ as

$$e(t) = \begin{cases} p(t_k) - p(t) & \text{if } t \in [t_k, t_{k+1}) \text{ for } k \geq 0, \\ 0 & \text{if } t \in [\phi(0), t_0), \end{cases} \tag{10}$$

so that $u(t) = K(p(t) + e(t))$, for $t \geq t_0$. Let

$$w(t) = u(t) - K(p(t) + e(t)), \qquad t \geq \phi(0), \tag{11}$$

where $w(t) = 0$ for $t \geq t_0$ but $w(t)$ is in general nonzero for $t \in [\phi(0), t_0)$. Computing $u(\phi(t))$ from (11) and substituing it in (3), the closed-loop system can be written as

$$\dot{x}(t) = f\big(x(t), K\big(x(t) + e(\phi(t))\big) + w(\phi(t))\big), \tag{12}$$

for all $t \geq 0$. Notice that (12) simplifies to [Tabuada, 2007, Eq. (3)] in the absence of delay ($\phi(t) = t$). Let $g(x, w) = f(x, K(x) + w)$ for all x, w. By the assumption that $\dot{x} = g(x, w)$ is ISS with respect to w, there exists a continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}$ and $\alpha_1, \alpha_2, \gamma, \rho \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x(t)|) \leq S(x(t)) \leq \alpha_2(|x(t)|), \tag{13}$$

and $(\mathcal{L}_g S)(x, w) \leq \gamma(|x|) + \rho(|w|)$. Therefore, we have

$$\begin{aligned} &(\mathcal{L}_f S)\big(x(t), K\big(x(t) + e(\phi(t))\big) + w(\phi(t))\big) \qquad (14) \\ &= (\mathcal{L}_g S)\big(x(t), K\big(x(t)+e(\phi(t))\big)+w(\phi(t))-K(x(t))\big) \\ &\leq -\gamma(|x(t)|) + \rho\big(\big|K\big(x(t)+e(\phi(t))\big)+w(\phi(t))-K(x(t))\big|\big). \end{aligned}$$

We assume that ρ is such that $\int_0^1 \frac{\rho(r)}{r} < \infty$. This assumption is not restrictive and is satisfied by most well-known class $\mathcal{K}$ functions. Then, define

$$V(t) = S(x(t)) + \frac{2}{b}\int_0^{2L(t)} \frac{\rho(r)}{r} dr, \tag{15a}$$

$$L(t) = \sup_{t\leq\tau\leq\sigma(t)} |e^{b(\tau-t)} w(\phi(\tau))|, \tag{15b}$$

and $b > 0$ is a design parameter. The next result establishes an upper bound on the time derivative of V.

Proposition 4.2 *(Upper-bounding $\dot{V}(t)$): For the system* (3) *under the control defined by* (5) *and* (9) *and the predictor* (7)*, we have*

$$\dot{V}(t) \leq -\gamma(|x(t)|) - \rho(2L(t)) + \rho(2L_K|e(\phi(t))|), \tag{16}$$

for all $t \neq \bar{t}$ *and* $V(\bar{t}^-) \geq V(\bar{t}^+)$*, where* L_K *is the Lipschitz constant of* K *and* $\bar{t} \in [0, \sigma(0)]$ *is the greatest time such that* $w(t) = 0$ *for all* $t > \bar{t}$.

Proof. Using (14), we have

$$\begin{aligned} &\mathcal{L}_f S(x(t)) \\ &\leq -\gamma(|x(t)|) + \rho\big(|w(\phi(t))| + |K(x(t)+e(\phi(t))) - K(x(t))|\big) \\ &\leq -\gamma(|x(t)|) + \rho\big(|w(\phi(t))| + L_K|e(\phi(t))|\big) \\ &\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|). \qquad (17) \end{aligned}$$

Since $e^{-b(t-\tau)}w(\phi(\tau))$ is bounded for $\tau \in [t, \sigma(t)]$ and any $t \geq 0$ and $[t, \sigma(t)]$ has finite measure, the sup-norm in (15b) equals the limit of the corresponding p-norm as $p \to \infty$, i.e.,

$$L(t) = \lim_{n\to\infty} \left[\int_t^{\sigma(t)} e^{2nb(\tau-t)} w(\phi(\tau))^{2n} d\tau\right]^{\frac{1}{2n}} \triangleq \lim_{n\to\infty} L_n(t).$$

In fact, it can be shown that this convergence is uniform over $[0, t_1]$ for any $t_1 < \bar{t}$. Therefore, since $\dot{L}_n(t) = -bL_n(t) - \frac{L_n}{2n}\left(\frac{w(\phi(t))}{L_n}\right)^{2n}$, $\frac{w(\phi(t))}{L_n} < 1$ for $t \in [0, t_1]$ and sufficiently large n and b, and $t_1 \in [0, \bar{t})$ is arbitrary, it follows from [Rudin, 1976, Thm 7.17] that $\dot{L}(t) = -bL(t)$ for $t \in (0, \infty) \setminus \{\bar{t}\}$. Combining this and (17), we get

$$\begin{aligned} \dot{V}(t) &\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|) \\ &\quad + \frac{2}{b} 2\dot{L}(t)\frac{\rho(2L(t))}{2L(t)} \\ &\leq -\gamma(|x(t)|) + \rho(2|w(\phi(t))|) + \rho(2L_K|e(\phi(t))|) \\ &\quad - 2\rho(2L(t)). \end{aligned}$$

for $t \in (0, \infty) \setminus \{\bar{t}\}$. Equation (16) thus follows since $|w(\phi(t))| \leq L(t)$ (c.f. (15b)) and the fact that ρ is strictly increasing. Finally, since $S(x(t))$ is continuous, $L(\bar{t}^-) \geq 0$, and $L(\bar{t}^+) = 0$, we get $V(\bar{t}^-) \geq V(\bar{t}^+)$. ■

Proposition 4.2 is the basis for our event-trigger design. Formally, we select $\theta \in (0, 1)$ and require

$$\rho(2L_K|e(\phi(t))|) \leq \theta\gamma(|x(t)|), \qquad t \geq 0,$$

which can be equivalently written as

$$|e(t)| \le \frac{\rho^{-1}(\theta\gamma(|p(t)|))}{2L_K}, \qquad t \ge \phi(0). \qquad (18)$$

Notice from (10) and the fact $t = 0$ that (18) holds on $[\phi(0), t_0]$. Equation (18) fully specifies the sequence of times $\{t_k\}_{k=1}^{\infty}$ and its dependence on the actuation delay. For each k, after each time t_k, the controller keeps evaluating (18) until it reaches equality. At this time, labeled t_{k+1}, the controller triggers the next event that sets $e(t_{k+1}) = 0$ and maintains (18). Notice that "larger" γ and "smaller" ρ (corresponding to "stronger" input-to-state stability in (2)) are then more desirable, as they allow the controller to update u less often. Our ensuing analysis shows global asymptotic stability of the closed-loop system and the existence of a uniform lower bound on the inter-event times.

4.3 Convergence Analysis under Event-triggered Law

In this section we show that our event triggered law (18) solves Problem 1 by showing, in the following result, that the inter-event times are uniformly lower bounded (so, in particular, there is no finite accumulation point in time) and the closed-loop system achieves global asymptotic stability.

Theorem 4.3 *(Uniform Lower Bound for the Inter-Event Times and Global Asymptotic Stability): Suppose that the class $\mathcal{K}_\infty$ function $\mathcal{G} : r \mapsto \gamma^{-1}(\rho(r)/\theta)$ is (locally) Lipschitz. For the system* (3) *under the control* (5)-(9) *and the triggering condition* (18)*, the following hold:*

(i) there exists $\delta > 0$ such that $t_{k+1} - t_k \ge \delta$ for all $k \ge 1$,
(ii) there exists $\beta \in \mathcal{KL}$ such that for any $x(0) \in \mathbb{R}^n$ and bounded $\{u(t)\}_{t=\phi(0)}^{0}$, we have for all $t \ge 0$,

$$|x(t)| + \sup_{\phi(t)\le\tau\le t} |u(\tau)| \le \beta\Big(|x(0)| + \sup_{\phi(0)\le\tau\le 0} |u(\tau)|, t\Big). \quad (19)$$

Proof. Let $[0, t_{\max})$ be the maximal interval of existence of the solutions of the closed-loop system. The proof involves three steps. First, we prove that (ii) holds for $t < t_{\max}$. Then, we show that (i) holds until $t_{\max}$, and finally that $t_{\max} = \infty$.

Step 1: From Proposition 4.2 and (18), we have

$$\begin{aligned}\dot V(t) &\le -(1-\theta)\gamma(|x(t)|) - \rho(2L(t)) \\ &\le -\gamma_{\min}(|x(t)| + L(t)), \qquad t \in [0, t_{\max}) \setminus \{\bar t\},\end{aligned}$$

where $\gamma_{\min}(r) = \min\{(1-\theta)\gamma(r), \rho(2r)$ for all $r \ge 0$, so $\gamma_{\min} \in \mathcal{K}$. Also, note that

$$V(t) \le \alpha_2(|x(t)|) + \alpha_0(L(t)) \le 2\alpha_{\max}(|x(t)| + L(t)),$$

where $\alpha_{\max}(r) = \max\{\alpha_2(r), \alpha_0(r)\}$ and $\alpha_0(r) = \frac{2}{b}\int_0^{2r} \frac{\rho(s)}{s} ds$ for all $r \ge 0$. Since $\alpha_0, \alpha_2 \in \mathcal{K}_\infty$, we have $\alpha_{\max} \in \mathcal{K}_\infty$, so $\alpha_{\max}^{-1} \in \mathcal{K}$. Hence,

$$\dot V(t) \le -\alpha_{\min}(\alpha_{\max}^{-1}(V(t)/2)) \triangleq \overline{\alpha}(V(t)), \; t \in [0, t_{\max}) \setminus \{\bar t\},$$

where $\overline{\alpha} \in \mathcal{K}$. Therefore, using the Comparison Principle [Khalil, 2002, Lemma 3.4], [Khalil, 2002, Lemma 4.4], and $V(\bar t^-) \ge V(\bar t^+)$, there exists $\beta_1 \in \mathcal{KL}$ such that $V(t) \le \beta_1(V(0), t)$, $t < t_{\max}$. Therefore,

$$|x(t)| + L(t) \le \beta_2(|x(0)| + L(0), t), \qquad t < t_{\max},$$

where $\beta_2(r, s) = \alpha_{\min}^{-1}(\overline{\beta}(2\alpha_{\max}(r), s))$ for any $r, s \ge 0$. Note that $\beta_2 \in \mathcal{KL}$. Since we have

$$\sup_{\phi(t)\le\tau\le t} |w(\tau)| \le L(t) \le e^{bM_0} \sup_{\phi(t)\le\tau\le t} |w(\tau)|,$$

it then follows that

$$|x(t)| + \sup_{\phi(t)\le\tau\le t} |w(\tau)| \le \beta_3\Big(|x(0)| + \sup_{\phi(0)\le\tau\le 0} |w(\tau)|, t\Big),$$

for all $t < t_{\max}$, where $\beta_3(r, s) = \beta_2(e^{bM_0} r, s)$. This inequality leads to (19) using the same steps as in [Bekiaris-Liberis and Krstic, 2013, Lemmas 8.10, 8.11] (the only difference being the multiplicity of inputs).

Step 2: Equation (18) can be rewritten as

$$|p(t)| \ge \gamma^{-1}\Big(\frac{\rho(2L_K|e(t)|)}{\theta}\Big).$$

From step 1, the prediction $p(t) = x(\sigma(t))$ and its error $e(t) = p(t_k) - p(t)$ are bounded. Therefore, there exists $L_{\gamma^{-1}\rho/\theta} > 0$ such that for all $t \ge 0$,

$$\gamma^{-1}\Big(\frac{\rho(2L_K|e(t)|)}{\theta}\Big) \le 2L_{\gamma^{-1}\rho/\theta} L_K |e(t)|.$$

where $L_{\gamma^{-1}\rho/\theta}$ is the Lipschitz constant of $\mathcal{G}$ on the compact set that contains $\{e(t)\}_{t=0}^{t_{\max}}$. Hence, a sufficient (stronger) condition for (18) is

$$|p(t)| \ge 2L_{\gamma^{-1}\rho/\theta} L_K |e(t)|. \qquad (20)$$

Note that (20) is only for the purpose of analysis and is *not* executed in place of (18). Clearly, if the inter-event times of (20) are lower bounded, so are the inter-event times of (18). Let $r(t) = \frac{|e(t)|}{|p(t)|}$ for any $t \ge 0$ (with $r(t) = 0$ if $p(t) = 0$). For any $k \ge 0$, we have $r(t_k) = 0$ and $t_{k+1} - t_k$ is greater than or equal to the time that it takes for $r(t)$ to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$. Note that for any $t \ge 0$,

$$\begin{aligned}\dot r &= \frac{d}{dt}\frac{|e|}{|p|} = \frac{d}{dt}\frac{(e^T e)^{1/2}}{(p^T p)^{1/2}} \\ &= \frac{(e^T e)^{-1/2} e^T \dot e (p^T p)^{1/2} - (p^T p)^{-1/2} p^T \dot p (e^T e)^{1/2}}{p^T p} \\ &= -\frac{e^T \dot p}{|e||p|} - \frac{|e| p^T \dot p}{|p|^3} \le \frac{|\dot p|}{|p|} + \frac{|e||\dot p|}{|p|^2} = (1 + r)\frac{|\dot p|}{|p|},\end{aligned}$$

where the time arguments are dropped for better readability. To upper bound the ratio $|\dot p(t)|/|p(t)|$, we have from (7) that $\dot p(t) = \dot\sigma(t) f(p(t), u(t))$ for all $t \ge \phi(0)$. By continuous differentiability of f (which implies Lipschitz continuity on compacts) and global asymptotic stability of the closed loop system, there exists $L_f > 0$ such that

$$\begin{aligned}
|\dot{p}(t)| &= |\dot{\sigma}(t)f(p(t),u(t))| \leq M_2|f(p(t),K(p(t)+e(t)))| \\
&\leq M_2L_f|(p(t),K(p(t)+e(t)))| \\
&\leq M_2L_f(|p(t)|+|K(p(t)+e(t))|) \\
&\leq M_2L_f(|p(t)|+L_K|p(t)+e(t)|) \\
&\leq M_2L_f(1+L_K)|p(t)|+M_2L_fL_K|e(t)| \\
\Rightarrow \dot{r}(t) &\leq M_2(1+r(t))(L_f(1+L_K)+L_fL_K|r(t)|).
\end{aligned}$$

Thus, using the Comparison Principle [Khalil, 2002, Lemma 3.4], we have $t_{k+1}-t_k \geq \delta, k \geq 0$ where δ is the time that it takes for the solution of

$$\dot{r} = M_2(1+r)(L_f(1+L_K)+L_fL_Kr), \qquad (21)$$

to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$.

Step 3: Since all system trajectories are bounded and $t_k \xrightarrow{k\to\infty} \infty$, we have $t_{\max} = \infty$, completing the proof. ∎

A particular corollary of Theorem 4.3 is that the proposed event-triggered law does not suffer from Zeno behavior, i.e., t_k accumulating to a finite point $t_{\max}$. Also, note that the lower bound δ in general depends on the initial conditions $x(0)$ and $\{u(t)\}_{t=\phi(0)}^0$ through the Lipschitz constant $L_{\gamma^{-1}\rho/\theta}$.

4.4 Delayed and Event-Triggered Sensing

So far, we have not considered any delays in the availability of the sensing information about the plant state. To address the general scenario in Problem 1, let

$$\bar{\ell} = \bar{\ell}(t) = \max\{\ell \geq 0 \mid \tau_\ell \leq \psi(t)\},$$

be the index of the last plant state available at the controller at time t. Then, the best estimate of $x(\sigma(t))$ available to the controller, namely,

$$p(t) = x(\tau_{\bar{\ell}}) + \int_{\phi(\tau_{\bar{\ell}})}^t \dot{\sigma}(s)f(p(s),u(s))ds, \quad t \geq \psi^{-1}(0), \qquad (22)$$

is used as the prediction signal in place of (7).[5] Since $p(t)$ is not available before $\psi^{-1}(0)$, the control signal (5), (9) is updated as

$$u(t) = \begin{cases} K(p(t_k)) & \text{if} \quad t \in [t_k, t_{k+1}), \ k \geq 0, \\ 0 & \text{if} \quad t \in [0, t_0), \end{cases} \qquad (23)$$

where the first event time is now $t_0 = \psi^{-1}(0)$. We next provide the same guarantees as Theorem 4.3 for this scenario.

Theorem 4.4 *Consider the plant dynamics* (3) *driven by the predictor-based event-triggered controller* (23) *with the predictor* (22) *and triggering condition* (18). *Under the aforementioned assumptions, the closed-loop system is globally asymptotically stable, namely, there exists $\beta \in \mathcal{KL}$ such that* (19) *holds for all $x(0) \in \mathbb{R}^n$, continuously differentiable $\{u(t)\}_{t=\phi(0)}^0$, and $t \geq 0$. Furthermore, there exists $\delta > 0$ such that $t_{k+1} - t_k \geq \delta$ for all $k \geq 0$.*

Proof. For simplicity, let $U(t) = \sup_{\phi(t)\leq\tau\leq t}|u(t)|$. Since the open-loop system exhibits no finite escape time behavior, the state remains bounded during the initial period $[0, t_0]$. Hence, for any $x(0)$ and any $\{u(t)\}_{t=\phi(0)}^0$ there exists $\Xi > 0$ such that $|x(t)| \leq \Xi$ for $t \in [0, t_0]$. Without loss of generality, Ξ can be chosen to be a class $\mathcal{K}$ function of $|x(0)| + U(0)$. Thus,

$$\begin{aligned}
|x(t)|+U(t) &\leq \Xi(|x(0)|+U(0))+U(0) \qquad (24)\\
&\leq \big[\Xi(|x(0)|+U(0))+U(0)\big]e^{-(t-t_0)}, \quad t \in [0,t_0].
\end{aligned}$$

As soon as the controller receives $x(0)$ at t_0, it can estimate the state $x(t)$ by simulating the dynamics (3), i.e.,

$$x(t) = x(0) + \int_0^t f(x(s), u(\phi(s)))ds. \qquad (25)$$

This estimation is updated whenever a new state $x(\tau_\ell)$ arrives and used to compute the predictor (7), which combined with (25) takes the form (22). Since the controller now has access to the same prediction signal $p(t)$ as before, the same Lyapunov analysis as above holds for $[t_0, \infty)$. Therefore, let $\hat{\beta} \in \mathcal{KL}$ be such that (19) holds for $t \geq t_0$. By (24),

$$|x(t)| + U(t) \leq \hat{\beta}\big(\Xi(|x(0)|+U(0))+U(0), t-t_0\big) \quad t \geq t_0.$$

Therefore, (19) holds by choosing $\beta(r,t) = \max\big\{\hat{\beta}\big(\Xi(r)+r, t-t_0\big), \big[\Xi(r)+r\big]e^{-(t-t_0)}\big\}$. Finally, since the triggering condition (18) has not changed, $t_{k+1}-t_k \geq \delta, k \geq 0$ for the same $\delta > 0$ as in Theorem 4.3. ∎

While the controller can theoretically discard the received states $\{x(\tau_\ell)\}_{\ell=1}^\infty$ and rely on $x(0)$ for estimating the state at all future times, closing the loop using the most recent state value $x(\tau_{\bar{\ell}})$ has the advantage of preventing the estimator (25) from drifting due to noise and un-modeled dynamics.

5 The Linear Case

In this section, we show how the general treatment of Section 4 is specialized and simplified if the dynamics (3) is linear, i.e, when we have

$$\dot{x}(t) = Ax(t) + Bu(\phi(t)), \qquad t \geq 0, \qquad (26)$$

subject to initial conditions $x(0) \in \mathbb{R}^n$ and bounded $\{u(t)\}_{t=\phi(0)}^0$. For simplicity, we restrict our attention to the perfect sensing case, as the generalization to sensing channels with time delay does not change the controller or stability guarantees (cf. Theorem 4.4). Assuming that the pair (A, B) is stabilizable, we can use pole placement to find a linear feedback law $K : \mathbb{R}^n \to \mathbb{R}$ that makes (6) ISS. Moreover, $p(t)$ can be explicitly solved from (7) to

[5] This only requires the controller to know $\psi(\tau_\ell)$ for every received state (not the full function ψ), which is realized by having a time-stamp for $x(\tau_\ell)$.

obtain

$$p(t) = e^{A(\sigma(t)-t^+)}x(t^+) + \int_{\phi(t^+)}^{t} \dot{\sigma}(s)e^{A(\sigma(t)-\sigma(s))}Bu(s)ds, \tag{27}$$

for all $t \geq \phi(0)$ and the closed-loop system takes the form

$$\dot{x}(t) = (A + BK)x(t) + Bw(\phi(t)) + BKe(\phi(t)).$$

Furthermore, given an arbitrary $Q = Q^T > 0$, the continuously differentiable function $S : \mathbb{R}^n \to \mathbb{R}$ is $S(x) = x^T Px$, where $P = P^T > 0$ is the unique solution to the Lyapunov equation $(A + BK)^T P + P(A + BK) = -Q$. Clearly, (13) holds with $\alpha_1(r) = \lambda_{\min}(P)r^2$ and $\alpha_2(r) = \lambda_{\max}(P)r^2$. To show (14), notice that using Young's inequality [Young, 1912],

$$\begin{aligned}\mathcal{L}_f S(x(t)) = &- x(t)^T Qx(t) \\ &+ 2x(t)^T PB(w(\phi(t)) + Ke(\phi(t))),\end{aligned}$$

so (14) holds with $\gamma(r) = \frac{1}{2}\lambda_{\min}(Q)r^2$ and $\rho(r) = \frac{2|PB|^2}{\lambda_{\min}(Q)}r^2$. In this case, the trigger (18) takes the simpler form

$$|e(t)| \leq \frac{\lambda_{\min}(Q)\sqrt{\theta}}{4|PB||K|}|p(t)|. \tag{28}$$

In addition to the simplifications, we show next that the closed-loop system is globally exponentially stable in the linear case.

5.1 Exponential Stabilization under Event-triggered Control

We next show that, in the linear case, we obtain the stronger feature of global exponential stability, though this requires a slightly different Lyapunov-Krasovskii functional.

Theorem 5.1 *(Exponential Stability of the Linear Case): The system* (26) *subject to the piecewise-constant closed-loop control* $u(t) = Kp(t_k)$, $t \in [t_k, t_{k+1})$, *with* $p(t)$ *given in* (27) *and* $\{t_k\}_{k=1}^{\infty}$ *determined according to* (28) *satisfies*

$$|x(t)|^2 + \int_{\phi(t)}^{t} u(\tau)^2 d\tau \leq Ce^{-\mu t}\Big(|x(0)|^2 + \int_{\phi(0)}^{0} u(\tau)^2 d\tau\Big),$$

for some $C > 0$, $\mu = \frac{(2-\theta)\lambda_{min}(Q)}{4\lambda_{max}(P)}$, *and all* $t \geq 0$.

Proof. For $t \geq 0$, let $L(t) = \int_t^{\sigma(t)} e^{b(\tau-t)}w(\phi(\tau))^2 d\tau$. One can see that $\dot{L}(t) = -w(\phi(t))^2 - bL(t)$, $t \geq 0$. Define $V(t) = x(t)^T Px(t) + \frac{4|PB|^2}{\lambda_{\min}(Q)}L(t)$. Therefore, using (28),

$$\begin{aligned}\dot{V}(t) = &-x(t)^T Qx(t) + 2x(t)^T PBw(\phi(t)) - \frac{4|PB|^2 b}{\lambda_{\min}(Q)}L(t) \\ &+ 2x(t)^T PBKe(\phi(t)) - \frac{4|PB|^2}{\lambda_{\min}(Q)}w(\phi(t))^2 \\ &\leq -\frac{2-\theta}{4}\lambda_{\min}(Q)|x(t)|^2 - \frac{4|PB|^2 b}{\lambda_{\min}(Q)}L(t) \leq -\mu V(t),\end{aligned}$$

where $\mu = \min\left\{\frac{(2-\theta)\lambda_{\min}(Q)}{4\lambda_{\max}(P)}, b\right\} = \frac{(2-\theta)\lambda_{\min}(Q)}{4\lambda_{\max}(P)}$ if b is chosen sufficiently large. Hence, by the Comparison Principle [Khalil, 2002, Lemma 3.4], we have $V(t) \leq e^{-\mu t}V(0)$, $t \geq 0$. Let $W(t) = |x(t)|^2 + \int_{\phi(t)}^{t} u(\tau)^2 d\tau$. From [Bekiaris-Liberis and Krstic, 2013, Eq. (6-99)-(6-100)], $c_1 W(t) \leq V(t) \leq c_2 W(t)$, for some $c_1, c_2 > 0$ and all $t \geq 0$. Hence, the result follows with $C = c_2/c_1$. ■

From Theorem 5.1, the convergence rate μ depends both on the ratio $\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$ and the parameter θ. The former can be increased by placing the eigenvalues of $A + BK$ at larger negative values, though large eigenvalues result in noise amplification. Decreasing θ, however, comes at the cost of faster control updates, a trade-off we study in detail next.

5.2 Optimizing the Sampling-Convergence Trade-off

In this section, we analyze the trade-off between sampling and convergence speed in our proposed event-triggered scheme. In general, it is clear from the Lyapunov analysis of Section 4 that more updates (smaller θ) hasten the decay of $V(t)$ and help the convergence. This trade-off becomes clearer in the linear case since explicit expressions are derivable for convergence rate and minimum inter-event times. To this end, we define two objective functions and formulate the trade-off as a multi-objective optimization. Let δ be the time that it takes for the solution of (21) to go from 0 to $\frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K}$. As shown in Section 4.3, the inter-event times are lower bounded by δ, so it can be used to roughly measure the cost of implementing the control scheme. Let

$$a = M_2 L_f L_K, \quad c = M_2 L_f(1 + L_K), \quad R = \frac{1}{2L_{\gamma^{-1}\rho/\theta}L_K},$$

where $L_f = \sqrt{2}(|A| + |B|)$, $L_K = |K|$, and $L_{\gamma^{-1}\rho/\theta} = \frac{2|PB|}{\lambda_{\min}(Q)\sqrt{\theta}}$. Then, the solution of (21) with initial condition $r(0) = 0$ is given by $r(t) = \frac{ce^{at}-ce^{ct}}{ae^{ct}-ce^{at}}$. Solving $r(\delta) = R$ for δ gives $\delta = \frac{\ln\frac{c+Ra}{c+Rc}}{a-c}$. The objective is to maximize δ and μ by tuning the optimization variables θ and Q. For simplicity, let $\theta = \nu^2$ and $Q = qI_n$ where $\nu, q > 0$. Then,

$$\delta(\nu) = \frac{1}{a-c}\ln\frac{c + \frac{\nu}{|P_1B||K|}a}{c + \frac{\nu}{|P_1B||K|}c}, \quad \mu(\nu) = \frac{2-\nu^2}{4\lambda_{\max}(P_1)},$$

where $P_1 = q^{-1}P$ is the solution of the Lyapunov equation $(A + BK)^T P_1 + P_1(A + BK) = -I_n$. Figure 1(a) depicts δ and μ as functions of ν and illustrates the sampling-convergence trade-off.

To balance these two objectives, we define the aggregate objective function as a convex combination of δ and μ, i.e.,

$$J(\nu) = \lambda\delta(\nu) + (1-\lambda)\mu(\nu),$$

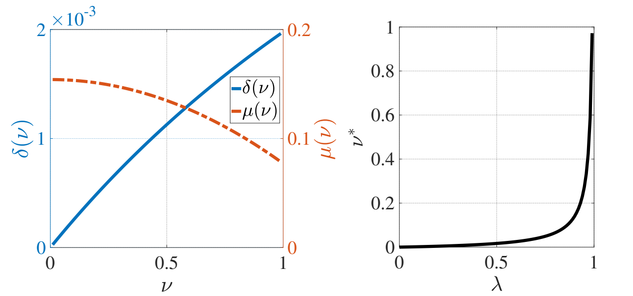

Fig. 1. Sampling-convergence trade-off for event-triggered control of linear systems. On the left, values of the lower bound of the inter-event times (δ) and exponential rate of convergence (μ) for different values of the optimization parameter ν for a third-order unstable linear system with $M_2 = 1$. On the right, the unique maximizer ν^* of the aggregate objective function $J(\nu)$ for different values of the weighting factor λ. As λ goes from 0 to 1, more weight is given to the maximization of δ, which increases ν^*.

where $\lambda \in [0, 1]$ determines the (subjective) relative importance of convergence rate and sampling. Notice that due to the difference between the (physical) units of δ and μ, one might multiply either one by a unifying constant, but we are not doing this as it leads to an equivalent optimization problem with a different λ. It is straightforward to verify that J is strongly convex and its unique maximizer is given by the positive real solution of $c_3\nu^3 + c_2\nu^2 + c_1\nu + c_0 = 0$ where $c_3 = a(1-\lambda)$, $c_2 = (a + c)|P_1B||K|(1-\lambda)$, $c_1 = c|P_1B|^2|K|^2(1-\lambda)$, and $c_0 = -2\lambda_{\max}(P_1)|P_1B||K|\lambda$. Figure 1(b) illustrates the optimizer of the aggregate objective function $J(\nu)$ for different values of the weighting factor λ.

6 Simulations

Here we illustrate the performance of our event-triggered predictor-based design. Example 6.2 is a two-dimensional nonlinear system that satisfies all the hypotheses required to ensure global asymptotic convergence of the closed-loop system. Example 6.3 is a different two-dimensional nonlinear system which instead does not, but for which we observe convergence in simulation. We start by discussing some numerical challenges that arise because of the particular hybrid nature of our design, along with our approach to tackle them.

Remark 6.1 *(Numerical implementation of event-triggered control law):* The main challenge in the numerical simulation of the proposed event-trigger law is the computation of the prediction signal $p(t) = x(\sigma(t))$. To this end, at least three methods can be used, as follows:
(i) *Open-loop*: One can solve $\dot{p}(t) = \dot{\sigma}(t)f(p(t), u(t))$ directly starting from $p(\phi(0)) = x(0)$. The closed-loop system takes the form of a time-delay hybrid system [Goebel et al., 2012] with flow map

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(\phi(t))), && t \geq 0, && \text{(29a)}\\
\dot{p}(t) &= \dot{\sigma}(t)f(p(t), u(t)), && t \geq \phi(0), && \text{(29b)}\\
\dot{p}_{tk}(t) &= 0, && t \geq t_0, && \text{(29c)}\\
u(t) &= K(p_{tk}(t)), && t \geq t_0, && \text{(29d)}
\end{aligned}
$$

jump map $p_{tk}(t_k^+) = p(t_k^+)$, jump set $D = \big\{(x, p, p_{tk}) \mid |p_{tk} - p| = \frac{\rho^{-1}(\theta\gamma(|p|))}{2L_K}\big\}$, and flow set $C = \mathbb{R}^{3n} \setminus D$. This formulation is computationally efficient but, if the original system is unstable, it is prone to numerical instabilities. The reason, suggesting the name "open-loop", is that the (p, p_{tk})-subsystem is completely decoupled from the x-subsystem. Therefore, if any mismatch occurs between $x(t)$ and $p(\phi(t))$ due to numerical errors, the x-subsystem tends to become unstable, and this is not "seen" by the (p, p_{tk})-subsystem.
(ii) *Semi-closed-loop*: One can add a feedback path from the x-subsystem to the (p, p_{tk}) subsystem by computing p directly from (7) at every integration time step of x. This requires a numerical integration of $f(p(s), u(s))$ over the "history" of (p, u) from $\phi(\tau_{\bar{\ell}})$ to t. This method is more computationally intensive but improves the numerical robustness. However, since we are still integrating over the history of p, any mismatch in the prediction takes more time to die out, which may not be tolerable for an unstable system.
(iii) *Closed-loop*: To further increase robustness, one can solve (29b) at every step of the integration of (29a) from $\phi(\tau_{\bar{\ell}})$ to t with "initial" condition $p(\phi(\tau_{\bar{\ell}})) = x(\tau_{\bar{\ell}})$. This method is the most computationally intensive of the three, but does not propagate prediction mismatch and is quite robust to numerical errors. We use this method in Examples 6.2 and 6.3. •

Example 6.2 *(Compliant Nonlinear System):* Consider the 2-dimensional system given by

$$
\begin{aligned}
f(x, u) &= \begin{bmatrix} x_1 + x_2 \\ \tanh(x_1) + x_2 + u \end{bmatrix}, & \phi(t) &= t - \frac{(t-5)^2 + 2}{2(t-5)^2 + 2},\\
\tau_\ell &= \ell\Delta_\tau, \quad \ell \geq 0, & \psi(t) &= t - D_\psi,
\end{aligned}
$$

where Δ_τ and D_ψ are constants. This system satisfies all the aforementioned assumptions with the feedback law $K(x) = -6x_1 - 5x_2 - \tanh(x_1)$ and

$$
\begin{aligned}
&L_f = 2\sqrt{3}, \quad L_K = 7\sqrt{2}, \quad M_0 = 1, \quad (M_1, m_2) = 1 \pm \frac{3\sqrt{3}}{16},\\
&S(x) = x^T P x, \quad \gamma(r) = \frac{\lambda_{\min}(Q)}{2} r^2, \quad \rho(r) = \frac{2|PB|^2}{\lambda_{\min}(Q)} r^2,
\end{aligned}
$$

where $P = P^T > 0$ is the solution of $(A+Bk)^T P + P(A+Bk) = -Q$ for $A = [1\ 1;\ 0\ 1]$, $B = [0;\ 1]$, $k = [-6\ \ -5]$, and arbitrary $Q = Q^T > 0$. A sample simulation result of this system is depicted in Figure 2(a). It is to be noted that for this example, (18) simplifies to $|e(t)| \leq \overline{\rho}|p(t)|$ with $\overline{\rho} = 0.015$, but the closed-loop system remains stable when increasing $\overline{\rho}$ until 0.7. Further, in order to study the effect of limitations in sensing on closed-loop stability, we varied Δ_τ and D_ψ and computed $|x(25)|$ as a measure of asymptotic stability. The average result is depicted in Figure 2(b) for 10 random initial conditions, showing that unlike our theoretical expectation, large Δ_τ and/or D_ψ result in instability even in the absence of noise because of the numerical error that degrades the estimation (25) over time (c.f. Remark 6.1). •

Example 6.3 *(Non-compliant Nonlinear System):* Here, we consider an example that violates several of our assumptions and study the performance of the proposed

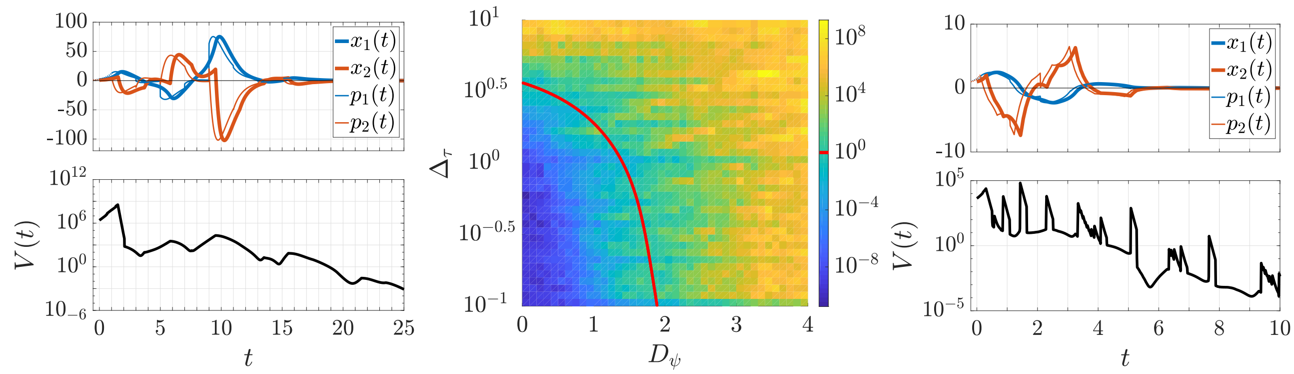

Fig. 2. Left plots, simulation of the compliant system in Example 6.2 with $x(0) = (1, 1)$, $\theta = 0.5$, $b = 10$, $\Delta_\tau = 2$, and $D_\psi = 1$. The non-monotonicity of V (bottom plot) is due to the numerical mismatch between $p(t)$ and $x(\sigma(t))$ (top plot), cf. Remark 6.1. The dotted portion of $p(t)$ corresponds to the times $[\phi(0), \psi^{-1}(0))$ and is plotted only for illustration purposes (not used by the controller). Center plot, heat map of the average of $|x(25)|$ over 10 random initial conditions drawn from standard normal distribution for the compliant system of Example 6.2. The red line shows an approximate border of stability. Right plots, simulation of the non-compliant system in Example 6.3 with $x(0) = (1, 1)$, $\theta = 0.5$, $b = 10$, $a = 0.01$, $D = 0.2$, $\Delta_\tau = 1$, $\mu_\psi = 0.1$, $\sigma_\psi = 0.02$, and triggering condition $|e(t)| \leq 0.5|p(t)|$. All simulations use an Euler discretization of the continuous-time dynamics with stepsize 10^{-2}.

algorithm. Let

$$f(x, u) = \begin{bmatrix} x_1 + x_2 \\ x_1^3 + x_2 + u \end{bmatrix}, \quad t - \phi(t) = D + a\sin(t),$$

$$\tau_\ell = \ell\Delta_\tau, \quad \ell \geq 0, \qquad \psi(t) = t - D_\psi,\ D_\psi \sim \mathcal{N}(\mu_\psi, \sigma_\psi^2).$$

where the nominal delay $D = 0.5$ is known but its perturbation magnitude $a = 0.05$ is not (the controller *assumes* $\phi(t) = t - D$) and D_ψ is generated independently at every τ_ℓ. Further, the control law $K(x) = -6x_1 - 5x_2 - x_1^3$ makes the closed-loop system ISS but is not globally Lipschitz, and the zero-input system exhibits finite escape time. The simulation results of this example are illustrated in Figure 2(c). It can be seen that although V is significantly non-monotonic, the event-triggered controller is able to stabilize the system, showing that the proposed scheme is applicable to a wider class of systems than those satisfying the assumptions. •

7 Conclusions and Future Work

We have proposed a prediction-based event-triggered control scheme for the stabilization of nonlinear systems with sensing and actuation delays. Under the assumptions of known time delay, globally-Lipschitz input-to-state stabilizability, and state feedback, we have shown that the closed-loop system is globally asymptotically stable and the inter-event times are uniformly lower bounded. We have particularized our results for the case of linear systems, providing explicit expressions for our design and analysis steps, and further studied the critical sampling-convergence trade-off characteristic of event-triggered strategies. Finally, we have addressed the numerical challenges that arise in the computation of predictor feedback and demonstrated the effectiveness of our proposed approach in simulation. Regarding future work, we highlight the extension of our results to systems with disturbances, unknown input delays, or output feedback, the characterization of the robustness properties resulting from incorporating the most recently available state information, the relaxation of the global Lipschitz requirement on the input-to-state stabilizer.

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