Event-triggered control under time-varying rates and channel blackouts*

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Abstract

This paper studies event-triggered stabilization of linear time-invariant systems over time-varying rate-limited communication channels. We explicitly account for the possibility of channel blackouts, i.e., intervals of time when the communication channel is unavailable for feedback. We assume prior knowledge of the channel evolution and we introduce the notion of *bit capacity* as the maximum total number of bits that could be communicated over a given time interval. We then provide an efficient real-time algorithm to lower bound the bit capacity for a deterministic channel whose characteristics are piecewise constant in time. Building on these results, we design an event-triggering strategy that guarantees Zeno-free, exponential stabilization at a desired convergence rate even in the presence of intermittent channel blackouts. The contributions are the notion of channel blackouts, the effective event-triggered control despite their occurrence, and the analysis and quantification of the bit capacity for a class of time-varying continuous-time channels. Various simulations illustrate the results.

1 Introduction

Control under communication constraints has key theoretical and practical importance given the increasing ubiquity of networked cyber-physical systems in nearly every aspect of modern life. This has motivated a vast

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amount of research to address the challenges posed by communication channels with limited, time-varying, and unreliable bit rates. This paper is a contribution to the growing body of results that employ either information-theoretic or opportunistic triggered control to address the problem of stabilization under constrained resources. Specifically, we seek to combine both approaches to deal with the control of linear time-invariant systems under time-varying channels, including for the possibility of blackouts, i.e., intervals of time during which the channel is completely unavailable for control.

Potential applications: We assume that the channel evolution, including the occurrence of blackouts, is known a priori. This is an arguably simplifying assumption convenient for our design and analysis, that nevertheless is still applicable in a variety of scenarios. These include communication in contested environments with a fixed scheduling of shared communication resources, where there is prior knowledge of the channel evolution and the controller can plan its transmission schedule ahead of time or scenarios involving multi-agent systems where each individual is informed ahead of time when it will have access to a shared resource and the specific conditions of such access. Examples include spacecraft telemetry, tracking & command (TT&C) [4, 21, 34, 3], wherein due to the essentially deterministic trajectories of the spacecraft and the earth, communication windows are known a priori. Similarly, it is well known that radio frequency communication under water is highly impractical [13]. Thus, a potential method to remotely control underwater vehicles is to have fixed deterministic surfacing schedule for the vehicles, which can then communicate using radio frequency. Further, our setup is also applicable to time-varying channels that can be modeled deterministically (e.g., stochastic channels whose evolution can be predicted with sufficiently high accuracy using training data or in scenarios where a worst-case analysis is carried out by reasoning with guaranteed lower bounds on channel capacity). Note that, in the scenarios described above, the a priori knowledge of the availability of the shared resource does not violate causality.

Literature review: The literature of control under communication constraints focuses on identifying necessary and sufficient conditions on the bit rates that guarantee stabilization under various assumptions on the (often stochastically modeled) communication channels. Comprehensive overviews may be found in [30, 11]. Early data rate results [28, 29, 41] provided tight necessary and sufficient conditions on the data rate of the encoded feedback for asymptotic stabilization in the discrete-time setting. Since then, the problem has been studied under increasingly complex assumptions on the communication channels, see e.g., [23, 25, 24]. In the continuous-time setting, the problem has been studied under either periodic sampling or aperiodic sampling with known upper and lower bounds on the sampling period. The works [15, 16] deal with single-input systems, [32] deals with nonlinear feedforward systems, and [20] deals with switched linear systems and characterizes the convergence rate of the finite data-rate stabilization scheme. The recent work [31] explores the stabilization problem under a state-based aperiodic transmission policy, with the inter-transmission intervals being integral multiples of a fixed stepsize. In general, this literature has not explored the potential advantages of tuning the sampling period in the periodic case or if state-based aperiodic sampling can provide any gains in efficiency and performance. On the other hand, the event-triggered approach, see e.g. [36, 42, 14] and references therein, exploits the tolerance to measurement errors to design goal-driven, opportunistic state-based aperiodic sampling. The literature on event-triggered control mainly focuses on guaranteeing control performance while minimizing the number of transmissions but largely ignores quantization, bit capacity, and other important aspects of communication. Some of the few exceptions include [37, 12], which utilize static logarithmic quantization and [17, 18, 35] (see also references therein) which use dynamic quantization. All these works guarantee a positive lower bound on the inter-transmission times, while [17, 18, 35] also provide a uniform bound on the communication bit rate (i.e., the number of bits per transmission). However, these references do not address the inverse problem of triggering and quantization given a limit on the communication bit rate. Moreover, the channel is assumed to always be available to the control system and hence event-triggered designs typically do not take into account the possibility of channel blackouts. An important exception is [2], which uses the deadlines generated by a self-triggered controller to perform a kind of instantaneous or short-term scheduling. However, if the communication latency is time-varying either because of a time-varying channel or because of time-varying packet sizes it is difficult to guarantee long-term future schedulability and system performance.

Another approach in the literature to event-triggered control over shared channels is to consider independent identically distributed packet drops [7, 5, 22, 40] or to assume a model of a channel that guarantees an upper bound on the number of consecutive packet drops [8]. On the other hand, [6, 26, 27, 1] incorporate a centralized real-time arbitration mechanism for several processes to use a shared channel. A closely related problem is that of control under denial-of-service attacks. [10] assumes there is a lower bound on the time intervals during which there is no attack and proposes an event-triggered control strategies for cases with and without knowledge of such a lower bound. [33, 9] instead assume knowledge of average frequency and maximum duration of the attacks and propose an event-triggered control strategy for stabilization. However, these works assume infinite precision feedback and no time delays.

Compared to these works, this paper pursues the view that modeling the long term data requirements for a process, as in control under communication constraints, is helpful in either controlling a single system over a channel with blackouts or in online scheduling (as opposed to arbitration) of feedback communication of multiple systems over a shared channel. While a "denial-of-service" model of a shared channel as in [10, 33, 9] may require milder assumptions on the knowledge of the channel evolution, it may still lead to an inefficient usage of a shared communication resource essentially because there is no coordination between the users. On the other hand, an evaluation of the data requirements of each process sharing a communication channel may help coordinate its usage and lead to greater efficiency. In this context, our recent work [38] combines the information-theoretic and event-triggered control approaches to address the problem of event-triggered stabilization of continuous-time linear time-invariant systems under bounded bit rates. The event-triggered formulation allows us to guarantee, in the absence of channel blackouts, a specified rate of convergence under non-instantaneous communication.

Statement of contributions: We address the stabilization problem for linear time-invariant systems over time-varying rate-limited communication channels that may be subject to sporadic but known blackouts. Our starting point is a description of the communication channel through two time-varying channel functions representing, respectively, the minimum instantaneous communication-rate and the maximum packet size that can be successfully transmitted. Our model explicitly accounts for the possibility of channel blackouts, which are intervals of time during which no packet can be successfully transmitted. The proposed design critically relies on three elements: a performance-trigger function that measures how close the system state is to violating the control objective, a channel-trigger function that keeps track of the number of bits required at any moment to guarantee performance at least for a certain period of time in the future, and lower bounds on bit capacity provided by our real-time algorithm. The first two elements are extensions of our design in [38] to the case of time-varying channels. On the other hand, the third element is a distinctive part of the present treatment, and its design requires a number of contributions that we detail next.

Our first contribution is the definition of the concept of data capacity, i.e., the maximum number of bits that may be communicated over possibly multiple transmissions during an arbitrary time interval under complete knowledge of the channel evolution. This concept plays a key role in effectively controlling the system despite the occurrence of blackouts. The computation of bit capacity for general time-varying channels is challenging. We show that, for the class of piecewise-constant channel functions, the computation of bit capacity can be formulated as an allocation problem involving the number of bits to be transmitted over each interval where the channel functions are constant. This equivalence sets the basis for our second contribution, which is the design of an algorithm to lower bound in real time the bit capacity over an arbitrary time interval. Our third and final contribution is the synthesis of event-triggered control schemes that, using prior knowledge of the channel information, plans the transmissions in order to guarantee the exponential stabilization of the system at a desired convergence rate, even in the presence of intermittent channel blackouts. Our notion of scheduled channel blackouts and stabilization despite their occurrence is a key contribution in the context of event-triggered control, which typically assumes the channel is available for feedback on demand. Simulations illustrate our results.

Notation: We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{>0}$, and $\mathbb{Z}_{\geq 0}$ denote the set of real, nonnegative real, positive integer, and nonnegative integer numbers, resp. We let $|S|$ denote the cardinality of S. We denote by $\|.\|_2$ and $\|.\|_\infty$ the

Euclidean and infinity norm of a vector, resp., or the corresponding induced norm of a matrix. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we let $\lambda_m(A)$ and $\lambda_M(A)$ denote its smallest and largest eigenvalues, resp. For any matrix norm $\|.\|$, note that $\|e^{A\tau}\| \le e^{\|A\|\tau}$. For a number $a \in \mathbb{R}$, we let $[a]_+ \triangleq \max\{0, a\}$. For a function $f : \mathbb{R} \mapsto \mathbb{R}^n$ and any $t \in \mathbb{R}$, we let $f(t^-)$ and $f(t^+)$ denote the limit from the left, $\lim_{s\uparrow t} f(s)$ and the limit from the right, $\lim_{s\downarrow t} f(s)$, resp.

2 Problem statement

We describe here the system dynamics, the model for the communication channel, and the control objective.

2.1 System description

We consider a linear time-invariant control system,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state and $u \in \mathbb{R}^m$ the control input, while $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrices. We assume that the pair (A, B) is stabilizable. Thus, we can select a control gain $K \in \mathbb{R}^{m\times n}$ such that $\bar{A} = A + BK$ is Hurwitz, which mean the continuous-time feedback $u(t) = Kx(t)$ renders the origin of (1) globally exponentially stable.

The plant is equipped with a sensor (the *encoder*) and an controller (the *decoder*) that are not co-located. The sensor can measure the state exactly and the controller can exert the input to the plant with infinite precision. However, the sensor may transmit state information to the controller at the controller only at discrete time instants *of its choice*, using only a finite number of bits. We let $\{t_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\ge 0}$ be the sequence of *transmission times* at which the sensor transmits an encoded packet of data, $\{r_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\ge 0}$ the sequence of *reception times* at which the decoder receives a complete packet of data, and $\{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}} \subset \mathbb{R}_{\ge 0}$ the sequence of *update times* at which the encoder and the decoder update their copies of the controller state (this is possible because we assume a reliable acknowledgment from the decoder to the encoder). At a transmission time t_k, the sensor sends b_k bits, which encode the plant state. Due to causality, $\tilde{r}_k \ge r_k \ge t_k$, and we denote the k^{th} *communication time* and k^{th} *time-to-update*, respectively, by $\Delta_k \triangleq r_k - t_k$, $\tilde{\Delta}_k \triangleq \tilde{r}_k - t_k$. Figure 1 illustrates the major components of the overall networked control system.

2.2 Communication channel

Our model for the time-varying communication channel is fully described by the map $R : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, where $R_a(t) = nR(t)$ is the *minimum instantaneous communication rate* at a given time t, and the piecewise-constant map $\bar{p} : \mathbb{R}_{\ge 0} \to \mathbb{Z}_{\ge 0}$, where $\bar{b}(t) = n\bar{p}(t)$ is the *maximum packet size* that can be successfully transmitted at a given time t. The k^{th} communication time and the k^{th} time-to-update satisfy

$$\tilde{\Delta}_k \ge \Delta_k \ge 0, \tag{2a}$$

$$\Delta_k \le \Delta(t_k, p_k) \triangleq \frac{p_k}{R(t_k)} = \frac{b_k}{R_a(t_k)}, \tag{2b}$$

where $b_k = np_k$ is the size of the packet (number of bits) that are actually transmitted at t_k, the k^{th} transmission time. The condition (2a) is that of causal communication while (2b) is an upper bound on the communication time. Note that the actual instantaneous communication rate at t_k is b_k/Δ_k and we can rewrite (2b) as

$$\frac{b_k}{\Delta_k} = \frac{np_k}{\Delta_k} \ge \frac{np_k}{\Delta(t_k, p_k)} = R_a(t_k).$$

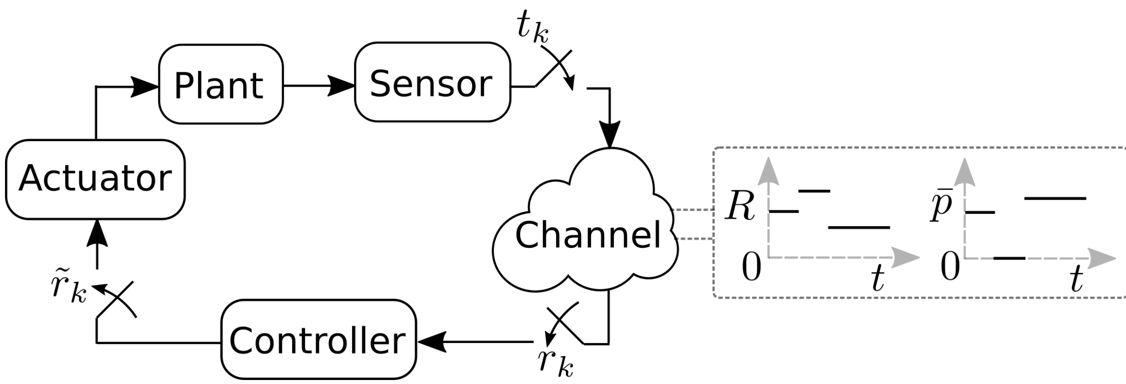


Figure 1: Major components of the networked control system. The sensor determines the time instants t_k at which to transmit np_k bits of data. The channel, if available at t_k, communicates this data at r_k to the controller. On receiving a packet at r_k, the controller chooses to update the control at $\tilde{r}_k$. We model the channel with two functions of time - minimum instantaneous communication rate $nR(.)$ and maximum packet size $n\bar{p}(.)$. During time intervals when $\bar{p}(t) = 0$, the channel is unavailable and we refer to such intervals as channel blackouts.

Thus, $R_a(t_k)$ is a lower bound on the number of bits communicated per unit time of all the bits transmitted at time t_k. Thus, for example, if $R_a(t_k) = \infty$, then the packet sent at t_k is received instantaneously.

Also note that the transmission policy has to ensure that $b_k \leq \bar{b}(t_k)$ (maximum packet size), or equivalently

$$p_k \leq \bar{p}(t_k), \quad p_k \in \mathbb{Z}_{\geq 0} \tag{3a}$$

for all $k \in \mathbb{Z}_{\geq 0}$. We refer to an interval of time during which $\bar{p} = \bar{b} = 0$ as a *(channel) blackout*. We assume that the encoder knows the functions $t \mapsto R(t)$ and $t \mapsto \bar{p}(t)$ a priori or sufficiently in advance, which we make clear in the sequel.

Since the channel has bounded bit capacity and in order to maintain synchronization between the encoder and the decoder, the transmission policy has to make sure that the encoder does not transmit a packet before a previous packet is received by the decoder and the controller updated, i.e.,

$$t_{k+1} \geq \tilde{r}_k, \tag{3b}$$

for all $k \in \mathbb{Z}_{\geq 0}$. We say the *channel is busy* at time t if $t \in [t_k, r_k)$, for some $k \in \mathbb{Z}_{>0}$. Finally, we refer to the sequences of transmission times $\{t_k\} \subset \mathbb{R}_{\geq 0}$, packet sizes $\{b_k\} \subset \mathbb{Z}_{\geq 0}$, and update times $\{\tilde{r}_k\} \subset \mathbb{R}_{\geq 0}$ as *feasible* if (2) and (3) are satisfied for every $k \in \mathbb{Z}_{>0}$.

2.3 Encoding and decoding

We use dynamic quantization for finite-bit transmissions from the encoder to the decoder. In dynamic quantization, there are two distinct phases: the zoom-out stage, e.g., [19], during which no control is applied while the quantization domain is expanded until it captures the system state at time $r_0 = t_0 \in \mathbb{R}_{\geq 0}$; and the zoom-in stage, during which the encoded feedback is used to asymptotically stabilize the system. We focus exclusively on the latter, i.e., for $t \geq t_0$. We assume both the encoder and the decoder have perfect knowledge of the plant system matrices, have synchronized clocks, and synchronously update their states at update times $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$. For simplicity, we assume that at transmission t_k the sensor (encoder) encodes each dimension of the plant state using p_k bits so that the total number of bits transmitted is $b_k = np_k$.

The state of the encoder/decoder is composed of the controller state $\hat{x} \in \mathbb{R}^n$ and an upper bound $d_e \in \mathbb{R}_{\geq 0}$ on $\|x_e\|_\infty$, where $x_e \triangleq x - \hat{x}$ is the encoding error. Thus, the actual input to the plant is $u(t) = K\hat{x}(t)$. During inter-update times, the state of the dynamic controller evolves as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) = \bar{A}\hat{x}(t), \quad t \in [\tilde{r}_k, \tilde{r}_{k+1}). \tag{4a}$$

Let the encoding and decoding functions at the k^{th} iteration be represented by $q_{E,k} : \mathbb{R}^n \times \mathbb{R}^n \mapsto G_k$ and $q_{D,k} : G_k \times \mathbb{R}^n \mapsto \mathbb{R}^n$, respectively, where G_k is a finite set of 2^{b_k} symbols. At t_k, the encoder encodes the plant state as $z_{E,k} \triangleq q_{E,k}(x(t_k), \hat{x}(t_k^-))$, where $\hat{x}(t_k^-)$ is the controller state just prior to the encoding time t_k, and sends it to the controller. The decoder can decode this signal as $z_{D,k} \triangleq q_{D,k}(z_{E,k}, \hat{x}(t_k^-))$ at any time during $[r_k, \tilde{r}_k]$. At the update time $\tilde{r}_k$, the sensor and the controller also update $\hat{x}$ using the jump map,

$$\begin{aligned}\hat{x}(\tilde{r}_k) &= e^{\bar{A}\tilde{\Delta}_k}\hat{x}(t_k^-) + e^{A\tilde{\Delta}_k}(z_{D,k} - \hat{x}(t_k^-)) \\ &\triangleq q_k(x(t_k), \hat{x}(t_k^-)),\end{aligned} \tag{4b}$$

where $q_k : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ represents the quantization that occurs as a result of the finite-bit coding. We allow the quantization domain, the number of bits and the resulting quantizer, q_k, for each transmission $k \in \mathbb{Z}_{>0}$ to be variable. The evolution of the plant state x and the encoding error x_e on $[\tilde{r}_k, \tilde{r}_{k+1})$ can be written as

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t), \tag{5a}$$

$$\dot{x}_e(t) = Ax_e(t). \tag{5b}$$

While the encoder knows x_e precisely, the decoder can only compute a bound $d_e(t)$ on $\|x_e(t)\|_\infty$ as follows

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty \delta_k, \;\; t \in [\tilde{r}_k, \tilde{r}_{k+1}), \;\; k \in \mathbb{Z}_{\geq 0}, \tag{6a}$$

$$\delta_{k+1} = d_e(t_{k+1})/2^{p_{k+1}}. \tag{6b}$$

The encoder and the decoder can use Algorithms 1 and 2, as described in Appendix A, to implement (4b). These algorithms are adapted from [38] with minor modifications. Further, Algorithms 1 and 2 maintain consistent $\hat{x}(t)$ and $d_e(t)$ signals and also ensure that $\|x_e(t)\|_\infty \leq d_e(t)$ for all $t \geq t_0$ if $\|x_e(t_0)\|_\infty \leq d_e(t_0)$, cf. Lemma A.1. For simplicity of exposition, we do not consider here the inclusion of additional bits in the transmitted packets dedicated to routing, error detection, or error correction. Note that the encoder and decoder algorithms can easily be augmented as in [38] to handle disturbances in the dynamics.

2.4 Control objective

We quantify the performance of the closed-loop system through a Lyapunov function. Given an arbitrary symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$, let P be the unique symmetric positive definite matrix that satisfies the Lyapunov equation $P\bar{A} + \bar{A}^T P = -Q$. Define $x \mapsto V(x) = x^T Px$ and let the desired *control performance* be

$$V_d(t) = V_d(t_0)e^{-\beta(t-t_0)}, \tag{7}$$

with $\beta > 0$. We assume that

$$W \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - a\beta > 0, \tag{8}$$

with $a > 1$ an arbitrary constant. Note that, given P, Q, this imposes a bound on the achievable convergence rates. Assumption (8) is sufficient to guarantee a convergence rate faster than β for (1) under the continuous-time and unquantized feedback control $u(t) = Kx(t)$.

Given the system and the communication channel model above, our objective is to design an event-triggered communication and control strategy that ensures exponential stability of the origin. Formally, we seek to synthesize an event-triggered control strategy that recursively determines the sequences of transmission times $\{t_k\}_{k\in\mathbb{Z}_{>0}}$ and update times $\{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}}$, along with a coding scheme for messages and a rule to determine the number of bits $\{b_k\}_{k\in\mathbb{Z}_{>0}}$ to be transmitted, so that

$$V(x(t)) \leq V_d(t),$$

holds for all $t \geq t_0$. This objective is especially challenging given the time-varying nature of the communication channel and the possibility of intermittent blackouts.

2.5 Proposed design solution

The main component of our solution is the event-triggering condition, which determines when the finite bit transmissions occur. The core principle of event-triggered control is to design an online condition that determines when the control goal is about to be violated based on the state and the dynamics of the system. We use the same principle in this paper, albeit with channel state and its evolution conceptually incorporated into the system state and dynamics.

Our event-triggering design has three main elements: (i) The first element is oblivious to bit capacity restrictions, or even to the finite precision feedback, and simply seeks to ensure the control performance $V(x(t)) \leq V_d(t)$ for all $t \geq t_0$ given the bounds on communication delays; (ii) The second element checks if the channel functions at the current time are sufficient to ensure that, if the encoder transmitted a packet, the encoding error is still manageable by the time the packet is received. A transmission is triggered when either of the conditions in (i) or (ii) are anticipated to be violated. However, both of these elements are oblivious of the time-varying nature of the channel and in particular of blackouts; Thus, (iii) the third element computes an upper bound on the amount of data required to ensure that the control objective is satisfied until the end of the next blackout. It also computes the maximum amount of data that could be transmitted from the current time until the beginning of the next blackout. The comparison between these two quantities leads to trigger a transmission when we anticipate the data requirement exceeding the available data capacity.

Comparison with [38]

Elements (i) and (ii) above come from our previous work [38], with some modifications that we outline in Section 3. Instead, the design of element (iii) is the major contribution of this paper. This task first requires us to clearly define bit capacity and to design an algorithm to compute it in real time, which we discuss in Section 4. In general, a transmission policy that maximizes the data throughput in a time interval may not be able to ensure the satisfaction of the control objective and vice versa. Hence, achieving the control objective at the current time while making sure that enough data can be transmitted in the future if needed to keep satisfying it requires some planning. Section 5 describes the way we incorporate this planning into the event-triggering rule and the integration of its various elements. In Section 5.1, we extend the results of [38] to the case of time-varying channels with explicit modeling of the instantaneous communication rate, albeit without channel blackouts. In this case, the requirements on the fore-knowledge of the channel evolution are also milder. In Section 5.2 we present our design and results for the case with channel blackouts. For reference, Table 1 lists the main elements of our design.

3 Performance- and channel-trigger functions

To achieve the control objective of Section 2.4 with opportunistic transmissions, we need a performance-trigger function that informs about how close the system state is to violating the convergence requirement. Bounded precision quantization further requires us to keep track (through a channel-trigger function) of the number of bits required at any moment to guarantee performance at least for a certain period of time. Threshold crossings of these two functions form the basis of our event-triggering mechanism. Further, to take care of

Performance and channel-trigger functions
Definition and properties: Section 3
Use: Control in absence of blackouts: Section 5.1
Control in presence of blackouts: Section 5.2
Bit capacity
Definition: Section 4.1
Under piece-wise constant channel functions: Section 4.2
Computation
Efficient sub-optimal solutions: Appendix B
Approximation via real-time algorithm: Section 4.3
Use: Control in presence of blackouts: Section 5.2

Table 1: Elements of the proposed event-triggered controller. Performance and channel-trigger functions are borrowed from [38] with minor modifications. Bit capacity is the primary contribution of this paper along with the design of its use in event-triggered control.

communication delays, the triggering mechanism instead uses guaranteed upper bounds on the performance and channel-trigger functions up to the maximum possible communication delay for the current channel state. In this section, we describe each of these components.

3.1 Performance-trigger function

We define the *performance-trigger* function as the ratio $h_{\text{pf}}(t) \triangleq V(x(t))/V_d(t)$. Thus, the control objective is to maintain $h_{\text{pf}}(t) \leq 1$ at all times. This is why, in general, it is of interest to characterize the open-loop evolution of the performance-trigger function. The next result provides an upper bound on the value of h_{pf} in the future as a function of the information available now.

Lemma 3.1. *(Upper bound on open-loop evolution of performance-trigger function [38]). Given $t_k \in \mathbb{R}_{>0}$ such that $h_{\text{pf}}(t_k) \leq 1$, then*

$$h_{\text{pf}}(\tau + t_k) \leq \bar{h}_{\text{pf}}(\tau, h_{\text{pf}}(t_k), \epsilon(t_k)),$$

for $\tau \geq 0$, where

$$\epsilon(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}}, \quad \bar{h}_{\text{pf}}(\tau, h_0, \epsilon_0) \triangleq \frac{f_1(\tau, h_0, \epsilon_0)}{f_2(\tau)},$$
$$f_1(\tau, h_0, \epsilon_0) \triangleq h_0 + \frac{W\epsilon_0}{w+\mu}(e^{(w+\mu)\tau} - 1), \;\; f_2(\tau) \triangleq e^{w\tau},$$
$$c \triangleq \frac{W\sqrt{\lambda_m(P)}}{2\sqrt{n}\|PBK\|_2}, \;\; w \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - \beta > 0, \;\; \mu \triangleq \|A\|_2 + \frac{\beta}{2}. \tag{9}$$

As we will see in the sequel, $\epsilon(t)$ in Lemma 3.1 is an important signal. In particular, it compares the bound on the estimation error $d_e(t)$ against the control performance function $V_d(t)$. Loosely, if $V_d(t)$ is large then the control algorithm can tolerate larger bounds $d_e(t)$ on the estimation error. Thus, in the sequel, we seek to maintain $\epsilon(t)$ within a certain bound that can guarantee the control performance. The result in Lemma 3.1 motivates the definition of the function

$$\Gamma_1(h_0, \epsilon_0) \triangleq \min\{\tau \geq 0 : \bar{h}_{\text{pf}}(\tau, h_0, \epsilon_0) = 1, \; \frac{\mathrm{d}\bar{h}_{\text{pf}}}{\mathrm{d}\tau} \geq 0\},$$

as a lower bound on the time it takes h_{pf} to evolve to 1 starting from $h_{\text{pf}}(t_k) = h_0$ with $\epsilon(t_k) = \epsilon_0$. Some useful properties of Γ_1 are listed in the following result.

Lemma 3.2. *(Properties of the function Γ_1 [38]). The following holds true,*

(i) $\Gamma_1(1,1) > 0$.

(ii) If $h_1 \geq h_0$ and $\epsilon_1 \geq \epsilon_0$, then $\Gamma_1(h_0, \epsilon_0) \geq \Gamma_1(h_1, \epsilon_1)$. In particular, if $h_0 \in [0,1]$, then $\Gamma_1(h_0, \epsilon_0) \geq \Gamma_1(1, \epsilon_0)$.

(iii) For $T > 0$, if $h_0 \in [0,1]$ and

$$\epsilon_0 \leq \rho_T(h_0) \triangleq \frac{(w+\mu)(1-h_0)}{W(e^{(w+\mu)T}-1)} + 1, \tag{10}$$

then $\Gamma_1(h_0, \epsilon_0) \geq \min\{\Gamma_1(1,1), T\}$.

(iv) For $T > 0$ and $h_0 \in [0,1]$,

$$\Gamma_1(h_0, \epsilon_0) \geq T \iff \bar{h}_{\mathrm{pf}}(T, h_0, \epsilon_0) \leq 1.$$

The statement with strict inequalities is also true.

Note that the function Γ_1 may not be amenable to evaluation in real time. However, we can provide a simple rule to check in real time, if $\Gamma_1(h_0, \epsilon_0)$ is larger or smaller than a given threshold. We present this method in the next result.

Lemma 3.3. *(Algebraic Condition to Check if $h_{\mathrm{pf}} < 1$ for the next T° units of time [38]). Let $T^\circ > 0$. For any $h_0 \in [0,1]$, $\Gamma_1(h_0, \epsilon_0) > T^\circ$ if and only if $\bar{h}_{\mathrm{pf}}(T^\circ, h_0, \epsilon_0) < 1$. Further, the corresponding statement with the inequalities replaced by equalities is true.*

3.2 Channel-trigger function

We define the *channel-trigger* function as the ratio $h_{\mathrm{ch}}(t) \triangleq \epsilon(t)/\rho_T(h_{\mathrm{pf}}(t))$, where $T > 0$ is a fixed design parameter and the function $\rho_T(.)$ is defined in (10). The channel-trigger function h_{ch} depends on the bound on the encoding error d_e through ϵ. Since d_e evolves as (6), the channel-trigger function h_{ch} also jumps at the update times $\tilde{r}_k$. Lemma 3.2(iii) implies that for any time $s_0 \geq t_0$, if $h_{\mathrm{ch}}(s_0) \leq 1$, then $h_{\mathrm{pf}}(t) \leq 1$ for at least $t \in [s_0, s_0 + \min\{T, \Gamma_1(1,1)\})$ even without any transmissions or receptions. Thus, assuming that the communication delays are smaller than $\min\{T, \Gamma_1(1,1)\}$, a transmission strategy ($\{t_k\}_{k\in\mathbb{Z}_{>0}}$ and $\{b_k\}_{k\in\mathbb{Z}_{>0}}$ such that $b_k = np_k$) is to ensure that, for each k, $h_{\mathrm{ch}}(\tilde{r}_k) \leq 1$ so that $\Gamma_1(h_{\mathrm{pf}}(\tilde{r}_k), \epsilon(\tilde{r}_k)) \geq \min\{T, \Gamma_1(1,1)\}$. Thus, we now require an upper bound on the open-loop evolution of h_{ch}, which is provided in the following result. Its proof follows from the definitions of ϵ and ρ_T in (9) and (10), respectively, and the evolution of d_e described in (6).

Lemma 3.4. *(Upper bound on the channel-trigger function at the update times $\tilde{r}_k$). If $t_k \in \mathbb{R}_{>0}$ is such that $h_{\mathrm{pf}}(t_k) \in [0,1]$, then*

$$h_{\mathrm{ch}}(\tilde{r}_k) \leq \bar{h}_{\mathrm{ch}}(\tilde{r}_k - t_k, h_{\mathrm{pf}}(t_k), \epsilon(t_k), p_k), \tag{11}$$

where $b_k = np_k$ bits are transmitted at t_k and

$$\bar{h}_{\mathrm{ch}}(\tau, h_0, \epsilon_0, p) \triangleq \frac{\|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}\epsilon_0}{\rho_T(\bar{h}_{\mathrm{pf}}(\tau, h_0, \epsilon_0))} \cdot \frac{1}{2^p}. \tag{12}$$

Note that for $t, t+\tau \in [\tilde{r}_k, t_{k+1})$, for any $k \in \mathbb{Z}_{\geq 0}$, we have $h_{\mathrm{ch}}(t+\tau) \leq \bar{h}_{\mathrm{ch}}(\tau, h_{\mathrm{pf}}(t), \epsilon(t), 0)$.

Now, analogous to Γ_1, we define

$$\Gamma_2(b_0, \epsilon_0, p) \triangleq \min\{\tau \geq 0 : \bar{h}_{\mathrm{ch}}(\tau, b_0, \epsilon_0, p) = 1\},$$

which essentially is an lower bound on the communication delay $\tilde{r}_k - t_k$, for which we can still guarantee $h_{\mathrm{ch}}(\tilde{r}_k) \leq 1$. Given the interpretation of Γ_2, one of the conditions in our event-triggering rule would be to check if Γ_2 is less than a maximum communication delay. Lemma 3.5 provides a way to check this in real time.

Lemma 3.5. *(Algebraic condition to compare value of Γ_2 against a given constant [38]). Let $T^\circ > 0$. For any $h_0 \in [0,1]$ and $\epsilon_0 \in [0, \rho_T(h_0)]$, $\Gamma_2(h_0, \epsilon_0, p) > T^\circ$ if and only if $\bar{h}_{ch}(T^\circ, h_0, \epsilon_0, p) < 1$. Further, the statement with equalities is also true.*

The following result provides a lower bound for Γ_2 uniform in its first two arguments. This bound will be useful in our event-triggered design later.

Lemma 3.6. *(Lower bound on Γ_2). If $\epsilon_0 \in [0, \rho_T(h_0)]$ then $\Gamma_2(h_0, \epsilon_0, p) \geq T^*(p)$ with*

$$T^*(p) \triangleq \min\{\tau \geq 0 : g(\tau, p) = 1\},$$
$$g(\tau, p) \triangleq \frac{\|e^{A\tau}\|_\infty e^{\frac{\beta}{2}\tau}}{2^p} \cdot \frac{e^{(w+\mu)T} - 1}{e^{(w+\mu)T} - e^{(w+\mu)\tau}}.$$

Proof. From the definition of $h_{pf}$and (10), we have

$$\begin{aligned}
&\rho_T(\bar{h}_{pf}(\tau, h_0, \epsilon_0)) \\
&= \frac{(w+\mu)(1 - e^{-w\tau}(h_0 + \frac{W\epsilon_0}{w+\mu}(e^{(w+\mu)\tau} - 1)))}{W(e^{(w+\mu)T} - 1)} + 1 \\
&= \rho_T(e^{-w\tau}h_0) - \frac{e^{(w+\mu)\tau} - 1}{e^{(w+\mu)T} - 1} e^{-w\tau}\epsilon_0 \\
&\geq \rho_T(e^{-w\tau}h_0) \frac{e^{(w+\mu)T} - e^{(w+\mu)\tau}}{e^{(w+\mu)T} - 1},
\end{aligned}$$

where the inequality follows from the assumption that $\epsilon_0 \leq h_0$. Now, substituting this lower bound in (12) and noting the fact that $\rho_T(e^{-w\tau}h_0) \geq \rho_T(h_0)$ gives

$$\bar{h}_{ch}(\tau, h_0, \epsilon_0, p) \leq g(\tau, p).$$

The claim now follows from the definition of Γ_2. □

4 Characterization of the bit capacity

Our study of bit capacity here is motivated by the need of the encoder to know how much data can be transmitted successfully before a channel blackout. We structure our discussion into three parts. In Section 4.1, we define the notion of bit capacity and motivate the assumption of a piecewise-constant evolution of the channel communication rate R. Under this assumption, we formulate in Section 4.2 the computation of the bit capacity problem as an optimal allocation problem which is combinatorial in nature. Thus, in Section 4.3, we present a real-time algorithm to compute a sub-optimal solution to the bit capacity problem, in particular from the current time until the next blackout. We then use this in our event-triggered policy.

4.1 Bit capacity

We denote the number of bits *communicated* (transmitted by the encoder and completely received by the decoder) during $[\tau_1, \tau_2]$ under the feasible sequences $\{t_k\}$, $\{p_k\}$, and $\{\tilde{\Delta}_k\}$ (satisfying (2) and (3)) as

$$D(\tau_1, \tau_2, \{t_k\}, \{\tilde{\Delta}_k\}, \{p_k\}) \triangleq n \sum_{k=\underline{k}_{\tau_1}}^{\bar{k}_{\tau_2}} p_k,$$

where $\underline{k}_{\tau_1} = \min\{k : t_k \geq \tau_1\}$ and $\overline{k}_{\tau_2} = \max\{k : t_k + \tilde{\Delta}_k \leq \tau_2\}$. Notice that we count only the bits that are communicated during $[\tau_1, \tau_2]$. We define the bit capacity during $[\tau_1, \tau_2]$ as the maximum data that can be communicated during the time interval under *all* possible communication delays, i.e.,

$$\mathcal{D}(\tau_1, \tau_2) \triangleq \max_{\substack{\{t_k\},\{p_k\} \\ \text{s.t. (3) holds} \\ \forall \Delta_k \leq \Delta(t_k, p_k)}} D(\tau_1, \tau_2, \{t_k\}, \{\Delta_k\}, \{p_k\}).$$

Notice that to maximize the data communicated, it must be that $\tilde{r}_k = r_k$ ($\tilde{\Delta}_k = \Delta_k$) for all $k \in \mathbb{Z}_{>0}$. This explains the fact that only the sequences $\{t_k\}$ and $\{p_k\}$ are the optimization variables. Next, notice that maximization under *all* possible communication delays ($\Delta_k \leq \Delta(t_k, p_k)$) is the same as maximization under maximum communication delays ($\Delta_k = \Delta(t_k, p_k)$). Thus,

$$\mathcal{D}(\tau_1, \tau_2) \triangleq \max_{\substack{\{t_k\},\{p_k\} \\ \text{s.t. (3) holds}}} D(\tau_1, \tau_2, \{t_k\}, \{\Delta(t_k, p_k)\}, \{p_k\}).$$

In general, the precise computation of $\mathcal{D}(\tau_1, \tau_2)$ involves solving an integer program with non-convex feasibility constraints. Therefore, we seek a class of channel functions R and $\bar{b}$ that are meaningful and yet simple enough to efficiently compute a lower bound for the bit capacity. To this end, we make the following observation.

Lemma 4.1. *(Bit capacity under constant communication rate). Suppose $\forall t \in [\tau_1, \tau_2]$ (i) $R(t) = R \geq 0$ and (ii) $\bar{p}(t) \geq 1$ (no blackouts). Then, $\mathcal{D}(\tau_1, \tau_2) = n\lfloor R(\tau_2 - \tau_1)\rfloor$.*

The proof of Lemma 4.1 follows directly by noting that an optimal solution can be constructed by choosing $p_k = 1$ and $t_{k+1} = \tilde{r}_k = r_k$ for all $k \in \mathbb{Z}_{\geq 0}$. Motivated by this result, we assume in the sequel that R is piecewise constant so that the problem of finding a reasonable lower bound on $\mathcal{D}(\tau_1, \tau_2)$ is tractable while also ensuring that the overall problem is meaningful. Note that any given R can be approximated to an arbitrary degree of accuracy by a piecewise-constant function. In addition, according to (2b), R is a lower bound on the instantaneous communication rate and it is reasonable to assume it is piecewise constant. Specifically, we assume that

$$R(t) = R_j, \quad \bar{p}(t) = \bar{\pi}_j, \quad \forall t \in (\theta_j, \theta_{j+1}] \tag{13}$$

where $\{\theta_j\}_{j=0}^{\infty}$ is a strictly increasing sequence of time instants and $\bar{\pi}_j \in \mathbb{Z}_{\geq 0}$ for each j. We denote $T_j \triangleq \theta_{j+1} - \theta_j$ as the length of the j^{th} time slot $I_j \triangleq (\theta_j, \theta_{j+1}]$. Again note that identical $\{\theta_j\}$ sequences for R and $\bar{p}$ is not a restriction because one can always refine the sequence $\{\theta_j\}$. In order to concisely express the constraints in the definition of $\mathcal{D}$, we assume, without loss of generality, that $\tau_1 = \theta_{j_0}$ and $\tau_2 = \theta_{j_f}$, for some $j_0, j_f \in \mathbb{Z}_{\geq 0}$. Finally, we choose left-open intervals in (13) as it provides a slight technical advantage in lowering the gap between the optimal and our sub-optimal solutions.

4.2 Formulation as an allocation problem

Here we show that, for piecewise-constant channel functions, we can think of the computation of $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ as an allocation problem: that of allocating the number of bits $\{n\phi_j\}$, with $\phi_j \in \mathbb{Z}_{\geq 0}$, to be transmitted in the time slots $\{I_j\}$ for $j \in \mathcal{N}_{j_0}^{j_f} \triangleq \{j_0, \ldots, j_f - 1\}$. For convenience, we let $\phi_{j_0}^{j_f} \triangleq (\phi_{j_0}, \ldots, \phi_{j_f - 1})$. Given $\phi_{j_0}^{j_f}$, the sequences $\{t_k\}$ and $\{p_k\}$ are determined so that transmissions start at the earliest possible time in I_j and the channel is not idle until all the allocated bits ϕ_j are received, i.e., $t_{k+1} = \tilde{r}_k = r_k = \Delta(t_k, p_k)$ during I_j and $\{p_k\}$ during I_j is any sequence that respects the channel upper bound $\bar{\pi}_j$ and adds up to ϕ_j. Given this correspondance, our forthcoming discussion focuses on expressing the constraints in the optimization problem in terms of the ϕ variables. In the sequel, a standing constraint is that $\phi_j \in \mathbb{Z}_{\geq 0}$ for each j, unless we mention otherwise.

4.2.1 Constraints of the allocation problem

Maximum bits that may be transmitted: First, we present the constraint that describes the maximum bound on the number of bits that may be transmitted in each slot I_j. Note that according to Lemma 4.1, in the time slot I_j, $n\lfloor R_j T_j \rfloor$ bits could be transmitted and received within $\lfloor R_j T_j \rfloor / R_j \leq T_j$ units of time. In addition, $n\bar{\pi}_j$ more bits could be transmitted during the closed interval $[\ \lfloor R_j T_j \rfloor, \theta_{j+1}]$, though these bits are received only in subsequent time slots. Thus, we have for each $j \in \mathcal{N}_{j_0}^{j_f}$

$$n\phi_j \leq \begin{cases} nR_j T_j + n\bar{\pi}_j, & \text{if } \bar{\pi}_j > 0 \\ 0, & \text{if } \bar{\pi}_j = 0 \end{cases} \tag{14}$$

where in the first case we have used the fact that $\phi_j \in \mathbb{Z}_{\geq 0}$ to avoid the use of the floor function.

Reduced channel availability in a time slot due to prior transmissions: As noted above, if $\phi_j > \lfloor R_j T_j \rfloor$, then these bits take up some of the time in I_{j+1} and possibly even subsequent slots. Thus, effectively the time available in I_{j+1} and consequently the upper bound on ϕ_{j+1} is reduced. Moreover, in general, the number of bits transmitted in I_j has an effect on the number that could be transmitted in all subsequent intervals either directly or indirectly. Thus, for each $j_1, j \in \mathcal{N}_{j_0}^{j_f}$, we introduce

$$\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \triangleq \Big(T_j - \sum_{j=j_1}^{j-1} \big(\frac{\phi_j}{R_j} - T_j \big) \Big) = \theta_{j+1} - \theta_{j_1} - \sum_{j=j_1}^{j-1} \frac{\phi_j}{R_j}$$

The next result shows that these functions determine the available time in slot I_j given $\phi_{j_0}^{j_f}$. Its proof appears in Appendix C.

Lemma 4.2. *(Available time in slot I_j). Let $\bar{T}_j(\phi_{j_0}^{j_f})$ be the time available for transmissions in the slot I_j given the bit allocation $\phi_{j_0}^{j_f}$. Then,*

$$\bar{T}_j(\phi_{j_0}^{j_f}) = \Big[\min_{j_1 \in \mathcal{N}_{j_0}^{j_f}} \{ \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}), T_j \} \Big]_+ .$$

As a consequence of Lemma 4.2, for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1]$, consider the constraints

$$n\phi_j \leq \begin{cases} nR_j \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) + n\bar{\pi}_j, & \text{if } \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{15a}$$

which we obtain using the same reasoning as in (14) with T_j replaced by $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f})$. Note that if $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \geq T_j$, then the constraint (15a) is weaker than (14) and hence inactive. For $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \in (0, T_j)$, the constraint reflects the reduced available time in the time slot I_j and if $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \leq 0$, for some $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1]$, then it corresponds to the case when the channel is busy for the whole of the time slot I_j ($\bar{T}_j(\phi_{j_0}^{j_f}) = 0$). Thus (15a) accurately reflects the effect of possibly reduced available time during the slot I_j due to prior transmissions.

Counting only the bits transmitted and received during $[\theta_{j_0}, \theta_{j_f}]$: Finally, since in the computation of $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$, we are interested in the maximum number of bits that can be communicated (transmitted and received) during the time interval, we also require that any bits transmitted during the slot I_j are received before θ_{j_f}, i.e.,

$$\frac{\phi_j}{R_j} \leq \begin{cases} \bar{T}_j(\phi_{j_0}^{j_f}) + \theta_{j_f} - \theta_{j+1}, & \text{if } \bar{T}_j(\phi_{j_0}^{j_f}) > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Using the definition of $\bar{T}_j(\phi_{j_0}^{j_f})$, this can be rewritten giving the following constraints for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j]$

$$\frac{\phi_j}{R_j} \leq \begin{cases} \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) + \theta_{j_f} - \theta_{j+1}, & \text{if } \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{15b}$$

Then, the bit capacity is given as

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) = \max_{\substack{\phi_j \in \mathbb{Z}_{\geq 0}, \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (14), (15) hold}}} n \sum_{j=j_0}^{j_f - 1} \phi_j. \tag{16}$$

Ignoring the fact that this is an integer program, the constraints (15) still make the problem combinatorial. Often, we may be satisfied with an efficiently computable sub-optimal solution $\{\phi_j^s\}_{j \in \mathbb{Z}_{\geq 0}}$ to give a lower bound on the bit capacity of the form

$$\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f - 1} \phi_j^s, \tag{17}$$

Appendix B presents efficient methods to achieve this.

4.3 Computing bit capacity in real time

As mentioned earlier, we want the encoder to compute a lower bound for the bit capacity up to the end of the next blackout period. However, the computation of $\mathcal{D}(\tau_1, \tau_2)$ or even $\mathcal{D}_s(\tau_1, \tau_2)$ as presented in Appendix B may not be suitable for real-time computation. Thus, given $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ (or $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f})$), we propose a simpler procedure to compute a lower bound on $\mathcal{D}(t, \theta_{j_f})$ (or $\mathcal{D}_s(t, \theta_{j_f})$) for any $t \in [\theta_{j_0}, \theta_{j_0+1})$. We present the procedure in the following result, whose proof appears in Appendix C.

Theorem 4.3. *(Real-time computation of bit capacity). Let ϕ^* (or ϕ^s) be any optimizing solution to $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ (or $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f})$). Let*

$$\begin{aligned} \hat{\mathcal{D}}(t, \theta_{j_f}) &\triangleq \left[n \left\lfloor \phi_{j_0}^* - R_{j_0}(t - \theta_{j_0}) \right\rfloor \right]_+ + n \sum_{j=j_0+1}^{j_f - 1} \phi_j^* \\ \hat{\mathcal{D}}_s(t, \theta_{j_f}) &\triangleq \left[n \left\lfloor \phi_{j_0}^s - R_{j_0}(t - \theta_{j_0}) \right\rfloor \right]_+ + n \sum_{j=j_0+1}^{j_f - 1} \phi_j^s, \end{aligned} \tag{18}$$

for any $t \in [\theta_{j_0}, \theta_{j_0+1})$. Then, $0 \leq \mathcal{D}(t, \theta_{j_f}) - \hat{\mathcal{D}}(t, \theta_{j_f}) \leq n$ and $0 \leq \mathcal{D}_s(t, \theta_{j_f}) - \hat{\mathcal{D}}_s(t, \theta_{j_f}) \leq n$.

Theorem 4.3 guarantees that $\hat{\mathcal{D}}(t, \theta_{j_f})$ (resp. $\hat{\mathcal{D}}_s(t, \theta_{j_f})$) under-approximates $\mathcal{D}(t, \theta_{j_f})$ (resp. $\mathcal{D}_s(t, \theta_{j_f})$) by at most n bits, one per each dimension of the plant state. Thus, $\hat{\mathcal{D}}(t, \theta_{j_f})$ and $\hat{\mathcal{D}}_s(t, \theta_{j_f})$ are tight sub-optimal solutions to the bit capacity problem. Moreover, given any optimizing solution to $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$ (resp. $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f})$) $\hat{\mathcal{D}}(t, \theta_{j_f})$ (resp. $\hat{\mathcal{D}}_s(t, \theta_{j_f})$) can be found in real time. The implication is that, if one has the computational resources, then one may solve the full optimization problem $\mathcal{D}(\theta_{j_1}, \theta_{j_2})$ for $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and use the above result to find a tight sub-optimal solution $\hat{\mathcal{D}}(t, \theta_{j_2})$ for any $t \in [\theta_{j_1}, \theta_{j_1+1}]$.

5 Event-triggered stabilization

In this section, we address the problem of event-triggered control under a time-varying channel. Section 5.1 addresses the case with no channel blackouts. Section 5.2 builds on this design and analysis to deal with the presence of channel blackouts.

5.1 Control in the absence of channel blackouts

In the case of no channel blackouts, the encoder may choose to transmit at any time and, in addition, we assume the channel rate R is sufficiently high (the exact technical assumption is specified later) so that there is no need to resort to the computation of bit capacity. For this reason, we are able to consider arbitrary (i.e., not necessarily piecewise constant) functions $t \mapsto R(t)$. Note that, by its discrete nature, the function $t \mapsto \bar{p}(t)$ is always piecewise constant. For any $p \in \mathbb{Z}_{\geq 0}$, let

$$T_M(p) = \sigma \min\{\Gamma_1(1,1), T, T^*(p)\}, \tag{19}$$

where $\sigma \in (0,1)$ is a design parameter, T is the parameter chosen in (10) and T^* is as defined in Lemma 3.6. As we show in the sequel, if $T_M(p)$ is an upper bound on the communication delay when $b = np$ bits are transmitted, then it is sufficient to design an event-triggering rule that guarantees the control objective is met.

In the presence of communication delays, we need to make sure (i) that the control objective is not violated between a transmission and the resulting control update and (ii) that at the control update times, the encoding error is sufficiently small to ensure future performance. To this end, we define

$$\mathcal{L}_1(t) \triangleq \bar{h}_{\mathrm{pf}}\left(T_M(\bar{p}(t)), h_{\mathrm{pf}}(t), \epsilon(t)\right), \tag{20a}$$

$$\mathcal{L}_2(t) \triangleq \bar{h}_{\mathrm{ch}}\left(T_M(\bar{p}(t)), h_{\mathrm{pf}}(t), \epsilon(t), \bar{p}(t)\right), \tag{20b}$$

to take care of each of these requirements. From the discussion of Section 3, we know that ensuring that these two functions do not grow beyond 1 guarantees that both objectives (i) and (ii) are met. Specifically, if up to $\bar{b} = n\bar{p}$ bits are transmitted at time t, then $\mathcal{L}_1(t)$ provides an upper bound on the performance-trigger function h_{pf} at the reception time (which would be less than $t + T_M(\bar{p}(t))$), while $\mathcal{L}_2(t)$ provides an upper bound on the channel-trigger function h_{ch} if the control is updated as soon as the packet is received. Now, we present our first main result. Its proof appears in Appendix C.

Theorem 5.1. *(Event-triggered control in the absence of blackouts). Suppose $t \mapsto \bar{p}(t)$ is piecewise constant, as in (13), with a uniform lower bound 1 (i.e., no blackouts) and a uniform upper bound $p^{\max}$. Assume that*

$$R(t) \geq \frac{p}{T_M(p)}, \ \forall p \in \{1, \ldots, \bar{p}(t)\}, \ \forall t. \tag{21}$$

Consider the system (1) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$ determined recursively by

$$\begin{aligned} t_{k+1} = \min\{t \geq \tilde{r}_k : \mathcal{L}_1(t) \geq 1 \ \vee \ \mathcal{L}_1(t^+) \geq 1 \vee \\ \mathcal{L}_2(t) \geq 1 \ \vee \ \mathcal{L}_2(t^+) \geq 1\}. \end{aligned} \tag{22}$$

Let $\{r_k\}_{k \in \mathbb{Z}_{\geq 0}}$ and $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0 = t_0$ and $\tilde{r}_k = r_k \leq t_k + \Delta_k$ for $k \in \mathbb{Z}_{>0}$. Assume the encoding scheme is such that (6) is satisfied for all $t \geq t_0$. Further assume that $\mathcal{L}_1(t_0) \leq 1$, $\mathcal{L}_2(t_0) \leq 1$ and that (8) holds. Define

$$\underline{p_k} \triangleq \min\{p \in \mathbb{Z}_{>0} : \bar{h}_{\mathrm{ch}}\Big(\frac{p}{R(t_k)}, h_{\mathrm{pf}}(t_k), \epsilon(t_k), p\Big) \leq 1\}. \tag{23}$$

Then, the following hold:

(i) $\underline{p_1} \leq \bar{p}(t_1)$. Further for each $k \in \mathbb{Z}_{>0}$, if $p_k \in \mathbb{Z}_{>0} \cap [\underline{p_k}, \bar{p}(t_k)]$, then $\underline{p_{k+1}} \leq \bar{p}(t_{k+1})$.

(ii) the inter-transmission times $\{t_{k+1} - t_k\}_{k \in \mathbb{Z}_{>0}}$ and inter-update times $\{\tilde{r}_{k+1} - \tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$ have a uniform positive lower bound,

(iii) the origin is exponentially stable for the closed-loop system, with $V(x(t)) \leq V_d(t_0)e^{-\beta(t-t_0)}$ for $t \geq t_0$.

The interpretation of the three claims of the result is as follows. Claim (i) essentially states that if the number of bits transmitted in the past is according to the given recommendation, then in the future, the sufficient number of bits $\underline{b_k} = n\underline{p_k}$ to guarantee continued performance will respect the time-varying channel constraints. Claim (ii) is sufficient to guarantee non-Zeno behavior and claim (iii) states that indeed the control objective is met.

Remark 5.2. *(Requirements on the knowledge of channel information).* Note that in the scenario with no channel blackouts, the encoder needs to know the channel information given by R and $\bar{p}$ only over a time horizon of length δ_t. Further, if a uniform lower bound on $t \mapsto \bar{p}(t)$ greater than or equal to 1 is known, then it is sufficient for the encoder to know only the channel information at the current time and use this bound to schedule the transmissions (however, this might result in more frequent transmissions with smaller packet sizes). •

5.2 Control in the presence of channel blackouts

Here, we address the scenario of channel blackouts. The main difficulty comes from the fact that, in the presence of blackouts, the channel might be completely unavailable. Thus, the event-triggering condition not only needs to be based on $\mathcal{L}_1$ and $\mathcal{L}_2$ in (20) as in Section 5.1, but also on the available bit capacity up to the next blackout. Throughout the section, we assume both R and $\bar{p}$ are piecewise-constant functions, as in Section 4, and, without loss of generality, that blackout time slots are not consecutive. We let $B_k \triangleq (\theta_{j_k}, \theta_{j_k+1}]$ denote the k^{th} blackout slot, with $k \in \mathbb{Z}_{>0}$. Also, for any $t \geq t_0$, we let

$$\tau_l(t) \triangleq \min\{s \geq t : \bar{p}(s) = 0\},$$
$$\tau_u(t) \triangleq \min\{s \geq \tau_l(t) : \bar{p}(s) > 0\},$$

give, respectively, the beginning and the end times of the next channel blackout slot from the current time t. When there is no confusion, we simply use τ_l and τ_u, dropping the argument t. Hence, for $t \in [t_0, \theta_{j_1})$, we have $\tau_l(t) = \theta_{j_1}$ and $\tau_u(t) = \theta_{j_1+1}$. Similarly, for any $k \in \mathbb{Z}_{>0}$ and $t \in (\theta_{j_k}, \theta_{j_{k+1}}]$, we have $\tau_l(t) = \theta_{j_{k+1}}$ and $\tau_u(t) = \theta_{j_{k+1}+1}$. To build up to the main result of this section, we first investigate the question of amenability of the time-varying channel (and in particular of the blackouts) to the control objective.

5.2.1 Control feasibility under channel blackouts

At time t, the length of the next channel blackout slot, $T_b(t) \triangleq \tau_u(t) - \tau_l(t)$, determines a sufficient upper bound on the encoding error $d_e(\tau_l)$, or equivalently $\epsilon(\tau_l)$, for non-violation of the control objective during the blackout or immediately subsequent to it. We quantify this bound next (the proof appears in Appendix C).

Lemma 5.3. *(Upper bound on required ϵ before blackout). For $t \in [t_0, \infty)$, suppose*

$$\epsilon(\tau_l(t)) \leq \epsilon_r(t) \triangleq \min\left\{\frac{(e^{wT_b(t)} - 1)(w + \mu)}{W(e^{(w+\mu)T_b(t)} - 1)}, \frac{1}{e^{\bar{\mu}T_b(t)}}\right\},$$

where $\bar{\mu} \triangleq \|A\|_\infty + \frac{\beta}{2}$. If $h_{\text{pf}}(\tau_l(t)) \leq 1$, then $h_{\text{pf}}(s) \leq 1$ for all $s \in [\tau_l(t), \tau_u(t)]$ and $h_{\text{ch}}(\tau_u(t)) \leq 1$ (in particular $\epsilon(\tau_u(t)) \leq 1$).

The ability to ensure that $\epsilon(\tau_l)$ is sufficiently small is determined by the bit capacity $\mathcal{D}(t, \tau_l)$. To have a real-time implementation, we make use of the sub-optimal lower bound $\hat{\mathcal{D}}_s(t, \tau_l)$, cf. Theorem 4.3. However, notice that maximizing the data throughput and satisfying the primary control goal of exponential convergence may not be compatible in general - if maximizing data throughput is the only goal, then certain transmissions might be delayed and this might lead to the violation of the primary control objective. Conversely, if the control objective is the only goal, this might lead to an inefficient use of the channel that could be detrimental later. Thus, to use the building blocks of Section 5.1, we need to impose a time-varying artificial bound on the allowed packet size in place of $\bar{p}(t)$ that prevents the system from affecting the bit capacity until the next blackout. To this end, we store in the variable $\mathcal{P}_j$ the value of ϕ^s_j, where ϕ^s is defined in Appendix B for $\mathcal{D}_s(\theta_j, \tau_l(\theta_j))$. Then, we define

$$\Phi^{\tau_l}(t) \triangleq [\lfloor \mathcal{P}_j - R_j(t - \theta_j) \rfloor]_+, \ \ t \in (\theta_j, \theta_{j+1}]. \tag{24}$$

We notice from (18) that $n\Phi^{\tau_l}(t)$ is the optimal number of bits to be transmitted during $(t, \theta_{j+1}]$ to obtain the sub-optimal bit capacity $\hat{\mathcal{D}}_s(t, \tau_l(t))$. Note that some of $n\Phi^{\tau_l}(t)$ bits may be received after θ_{j+1}. Now, we let

$$\psi^{\tau_l}(t) \triangleq \min\{\bar{p}(t), \Phi^{\tau_l}(t)\} \tag{25}$$

be the artificial bound on the packet size for transmissions. Notice that $\Phi^{\tau_l}(t)$ may at times be zero, even when $\bar{p}(t) > 0$, which means letting $\psi^{\tau_l}(t)$ be the bound on packet size may itself introduce *artificial blackouts*. However, we can upper bound their length as follows (the proof appears in Appendix C).

Lemma 5.4. *(Upper bound on the length of artificial blackouts). Let $\tilde{B}_j \triangleq \{t \in I_j = (\theta_j, \theta_{j+1}] : \psi^{\tau_l}(t) = 0\}$. Then, for each $j \in \mathbb{Z}_{\geq 0}$, $\tilde{B}_j$ is an interval and if $\bar{\pi}_j > 0$, then the length of $\tilde{B}_j$ is less than $2/R_j = 2/R(\theta_{j+1})$.*

In the sequel, we use the upper bound of Lemma 5.4 on the length of the artificial blackouts in our event-triggering rule and the analysis of the resulting closed-loop system of Section 5.2.2.

Before proceeding to our event-triggered design, there is a final point of contention that we need to address: clearly, we cannot satisfactorily control the system for arbitrary channel characteristics with arbitrary channel blackout slots. To make this precise, we define the function $\mathcal{L}_3$ that captures the effect of bit capacity,

$$\mathcal{L}_3(t, \epsilon) \triangleq n \log_2 \left(\frac{e^{\bar{\mu}(\tau_l(t)-t)}\epsilon}{\epsilon_r(t)} \right) - \sigma_1 \hat{\mathcal{D}}_s(t, \tau_l(t)), \tag{26a}$$

where $\sigma_1 \in (0, 1)$ is a design parameter. Recall that the notation $\tau_l(t)$ denotes the time of the beginning of the next blackout given t. From Lemma 5.3, recall that $\epsilon_r(t)$ is the upper-bound on $\epsilon(\tau_l(t))$ that would ensure non-violation of the control objective during the blackout. Thus, the first term in the function $\mathcal{L}_3$ quantifies the "least amount of data required to successfully get through the next blackout" while the second term quantifies the "maximum amount of data that can be communicated before the next blackout". Given this interpretation, the objective is then to keep the value of this function below 0 at all times. Using this function, the next result presents a sufficient condition on the length of the blackout slots and the available bit capacity to ensure the control objective is met during each blackout slot. The proof can be found in Appendix C.

Lemma 5.5. *(Control feasibility in the presence of blackouts). Suppose $t \mapsto R(t)$ and $t \mapsto \bar{p}(t)$ are piecewise-constant functions as in (13). Let $\{(\theta_{j_k}, \theta_{j_k+1}]\}_{k \in \mathbb{Z}_{>0}}$ be a sequence of channel blackout slots. Assume $\bar{p}(t_0) > 0$, $\mathcal{L}_3(t_0, \epsilon(t_0)) \leq 0$ and, for each $k \in \mathbb{Z}_{>0}$, $\mathcal{L}_3(\theta_{j_k+1}, 1) \leq 0$. Then, there exists a transmission policy ensuring the control objective is met $h_{\mathrm{pf}} \leq 1$ during each blackout.*

There are three conditions in Lemma 5.5 related to the channel. The first is that $\bar{p}(t_0) > 0$, i.e., initially the channel does not start in a blackout state. We make this assumption purely for convenience (it could easily be replaced by a more general condition in case the channel starts in a blackout state). The second condition $\mathcal{L}_3(t_0, \epsilon(t_0)) \leq 0$ merely states that the initial condition of the system and the channel is such that it is possible to successfully overcome the first blackout. In the third condition, we consider the maximum value of ϵ (or equivalently the maximum bound on the encoding error) at the end of each blackout such that $h_{\mathrm{ch}} \leq 1$ is guaranteed for all possible values of $h_{\mathrm{pf}} \in [0, 1]$. This maximum value is $\epsilon = 1$. Thus, $\mathcal{L}_3(\theta_{j_k+1}, 1) \leq 0$ implies that even in such a worst-case scenario the bit capacity until the next blackout is sufficient.

5.2.2 Event-triggered control under channel blackouts

With the study of control feasibility in place, we are ready to describe our event-triggered design. In addition to $\mathcal{L}_3$ in (26a) to capture the effect of bit capacity, our design involves functions analogous to $\mathcal{L}_1$ and $\mathcal{L}_2$ to, resp., monitor the compliance with the control objective and ensure the encoding error is sufficiently small at the control update times to ensure future performance,

$$\tilde{\mathcal{L}}_1(t) \triangleq \bar{h}_{\mathrm{pf}}\left(\mathcal{T}(t), h_{\mathrm{pf}}(t), \epsilon(t)\right), \tag{26b}$$

$$\tilde{\mathcal{L}}_2(t) \triangleq \bar{h}_{\mathrm{ch}}\left(\mathcal{T}(t), h_{\mathrm{pf}}(t), \epsilon(t), \psi^{\tau_l}(t)\right), \tag{26c}$$

where

$$\mathcal{T}(t) \triangleq \begin{cases} T_M(\psi^{\tau_l}(t)), & \text{if } \psi^{\tau_l}(t) \geq 1 \\ \frac{2}{R(t)}, & \text{if } \psi^{\tau_l}(t) = 0. \end{cases}$$

We are ready to present our next main result. Its proof appears in Appendix C.

Theorem 5.6. *(Event-triggered control in the presence of blackouts). Suppose $t \mapsto R(t)$ and $t \mapsto \bar{p}(t)$ satisfy the assumptions of Lemma 5.5. In addition, assume that $\bar{p}$ is uniformly upper bounded by $p^{\max} \in \mathbb{Z}_{>0}$. Also, assume*

$$R(t) \geq \frac{(p+2)}{T_M(p)}, \ \forall p \in \{1, \dots, p^{\max}\}, \ \forall t. \tag{27}$$

Consider the system (1) under the feedback law $u = K\hat{x}$, with $t \mapsto \hat{x}(t)$ evolving according to (4) and the sequence $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$ determined recursively by

$$\begin{aligned} t_{k+1} = \min\big\{t \geq \ \tilde{r}_k : \ \psi^{\tau_l}(t) \geq 1 \ \wedge \\ \big(\max\{\tilde{\mathcal{L}}_1(t), \tilde{\mathcal{L}}_1(t^+), \tilde{\mathcal{L}}_2(t), \tilde{\mathcal{L}}_2(t^+)\} \geq 1 \\ \max\{\tilde{\mathcal{L}}_3(t), \tilde{\mathcal{L}}_3(t^+)\} \geq 0\big)\big\}, \end{aligned} \tag{28}$$

where $\tilde{\mathcal{L}}_3(t) \triangleq \mathcal{L}_3(t, \epsilon(t))$. Let $\{r_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0 = t_0$ and $r_k \leq t_k + \Delta_k$ for $k \in \mathbb{Z}_{>0}$. Let the update times $\{\tilde{r}_k\}_{k \in \mathbb{Z}_{\geq 0}}$ be given as $\tilde{r}_0 = r_0$ and for $k \in \mathbb{Z}_{>0}$

$$\tilde{r}_k = \min\{t \geq r_k : \psi^{\tau_l}(t) \geq 1 \ \vee \ \bar{p}(t) = 0\}. \tag{29}$$

Assume the encoding scheme is such that (6) is satisfied for all $t \geq t_0$. Further assume that $\tilde{\mathcal{L}}_1(t_0) \leq 1$, $\tilde{\mathcal{L}}_2(t_0) \leq 1$ and that (8) holds. Define

$$\underline{p_k} \triangleq \min\{p \in \mathbb{Z}_{>0} : \bar{h}_{\mathrm{ch}}\left(T_M(p), h_{\mathrm{pf}}(t_k), \epsilon(t_k), p\right) \leq 1\}. \tag{30}$$

Then, the following hold:

(i) $\underline{p_1} \leq \psi^{\tau_l}(t_1)$. Further for each $k \in \mathbb{Z}_{>0}$, if $p_k \in \mathbb{Z}_{>0} \cap [\underline{p_k}, \psi^{\tau_l}(t_k)]$, then $\underline{p_{k+1}} \leq \psi^{\tau_l}(t_{k+1})$.

(ii) the inter-transmission times $\{t_{k+1} - t_k\}_{k \in \mathbb{Z}_{>0}}$ and inter-update times $\{\tilde{r}_{k+1} - \tilde{r}_k\}_{k \in \mathbb{Z}_{>0}}$ have a uniform positive lower bound,

(iii) the origin is exponentially stable for the closed-loop system, with $V(x(t)) \leq V_d(t_0)e^{-\beta(t-t_0)}$ for $t \geq t_0$.

Claim (i) in the result may be interpreted as the satisfaction of the constraints imposed by the channel. The use of ψ^{τ_l} in (28) and (29) also ensures that the bit capacity is not lowered at any time in the future due to past transmissions. The interpretation of claims (ii) and (iii) is the same as in Theorem 5.1.

Remark 5.7. *(Requirements on the knowledge of channel information).* In the scenario with channel blackouts, the encoder needs to know, at $t \in [t_0, \infty)$, the time at which the next blackout will occur $\tau_l(t)$ and its duration $T_b(t)$, from which $\epsilon_r(t)$ may be computed. The encoder also needs to know the channel functions $s \mapsto R(s)$ and $s \mapsto \bar{p}(s)$ for all $s \in [t, \tau_l(t)]$. Using this information, the encoder can compute the lower bound on the remaining bit capacity by computing $\hat{\mathcal{D}}_s(t, \tau_l(t))$. •

Remark 5.8. *(Application of time-varying channels and channel blackouts in shared communication channels).* The idea of time-varying channels and channels with blackouts can be utilized in scenarios where multiple processes share communication resources. In such scenarios, the shared communication resource may be 'split' among all the processes and each process can be assigned its own channel functions R and $\bar{p}$. Figure 2 shows an example of a channel that is shared by two processes, red and blue. When the maximum packet size function $\bar{p}$ for a process is zero, then that process cannot utilize the channel. A process may be given exclusive access to the channel or the channel resources may be 'split' non-exclusively, akin to time-division and frequency-division multiplexing, respectively. Thus, for each control system that shares the communication channel, it is sufficient to know the 'portion' of the channel that is allotted to it. This effectively decouples the transmission decisions of the individual processes while also coordinating effective usage of the limited communication resources. The question of how to exactly 'split' the channel among multiple processes is a design problem in itself and control feasibility results such as Lemma 5.5 can aid such a design. •

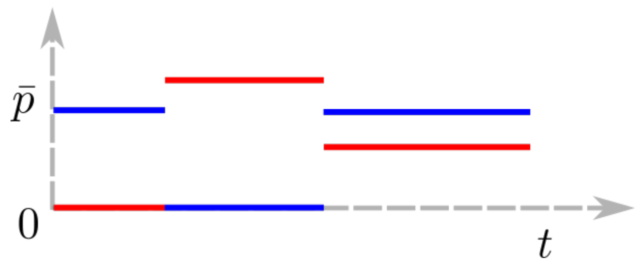


Figure 2: Channel is 'split' among two processes - blue and red. During each slot, except the last one, one of the processes gets exclusive access to the channel. In the last slot, both processes share the channel.

6 Simulation results

In this section we illustrate the execution of our event-triggered design of Section 5. The simulation results we present correspond to the strategy described in Theorem 5.6 on the system given by (1) with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} 2 & -8 \end{bmatrix}.$$

The plant matrix A has eigenvalues at 2 and 3, while the control gain matrix K places the eigenvalues of the matrix $\bar{A} = A + BK$ at -1 and -2. We select the matrix $Q = \mathrm{I}_2$, with solution

$$P = \begin{bmatrix} 2.2500 & -0.9167 \\ -0.9167 & 0.5833 \end{bmatrix}$$

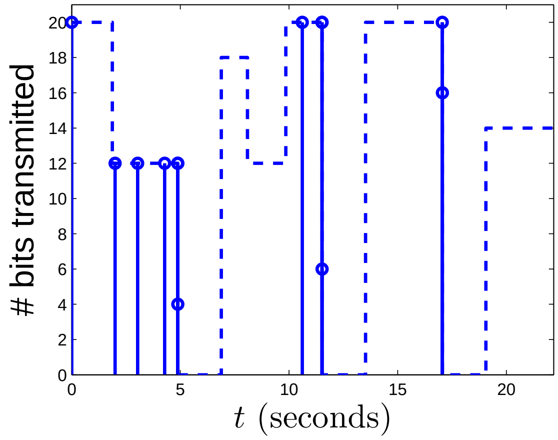
to the Lyapunov equation. The desired control performance is specified by

$$V_d(t_0) = 1.2V(x(t_0)), \quad \beta = 0.8\frac{\lambda_m(Q)}{\lambda_M(P)}.$$

We set $a = 1.2$ in (8), so that $W > 0$, and assume, without loss of generality, $t_0 = 0$. The initial condition is $x(t_0) = (6, -4)$, and the encoder and decoder use the information

$$\hat{x}(t_0) = (0, 0), \quad d_e(t_0) = 1.5\|x(t_0) - \hat{x}(t_0)\|_\infty.$$

In (26), we chose $\sigma_1 = 0.8$. For these parameters, $\Gamma_1(1, 1) = 0.5699$. We select $T = 0.1 \times \Gamma(1, 1)$ and $T_M(p) = 0.06 \times \min\{\Gamma(1, 1), T, T^*(p)\}$. The time-varying channel functions $n\bar{p}$ and R are plotted in Figures 3(a) and 3(b) respectively with dashed lines. Figure 3(a) also shows the times of transmission and the number of bits transmitted on each one. Note that, in this simulation, the maximum possible number of bits are transmitted on each transmission. Figure 4(a) shows the evolution of V and V_d and it is clear that the control goal is satisfied. Notice that, just before a blackout, V decreases to a low value in anticipation to ensure that the control goal is



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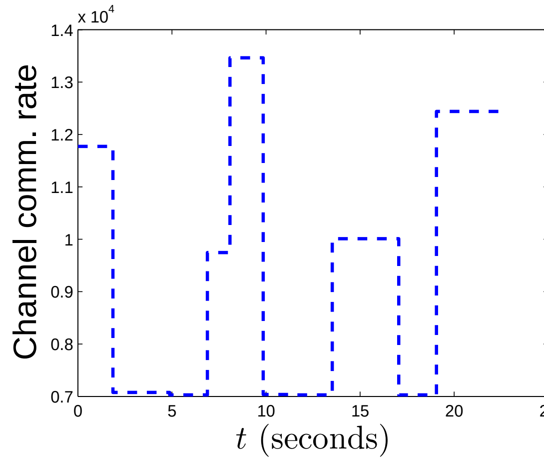
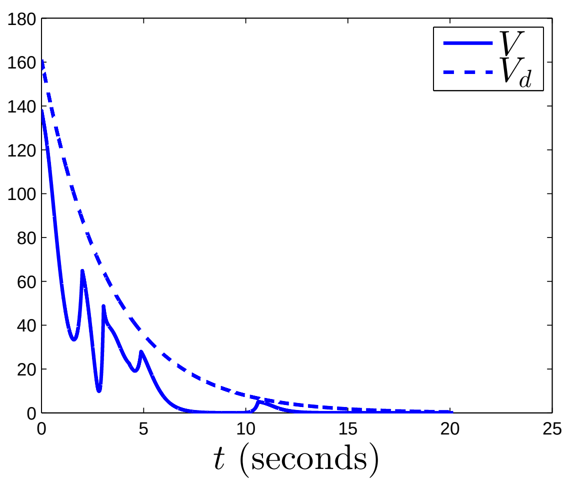
(a)

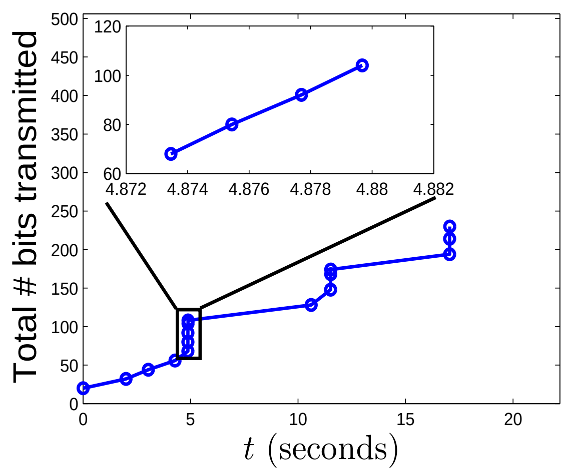


(b)

Figure 3: (a) shows the transmission times, the number of bits transmitted on each transmission and the time-varying function $n\bar{p}$ (dashed line). The three intervals, $(4.88, 6.88]$, $(11.52, 13.52]$ and $(17.05, 19.05]$, with $\bar{p} = 0$ are the blackouts. (b) shows the time-varying function R.



(a)



(b)

Figure 4: Evolution of (a) V and V_d and (b) total number of bits transmitted, the inset shows that the transmission times are separated.

not violated during the blackout. Figure 4(b) shows the (interpolated) cumulative number of bits transmitted as a function of time. We see that there is a rush of transmissions just prior to 4.88 units of time, which we see from Figure 3(a) is the beginning of the first blackout. The number of transmissions in the 20 units of time in the simulation are 16, with the average inter-transmission interval as 1.26 and the minimum as 0.002. From Figure 4(b), we also see that on an average 11.5 bits are transmitted per unit time.

7 Conclusions

We have addressed the problem of event-triggered control of linear time-invariant systems under time-varying rate-limited communication channels. The class of channels we consider is broad enough to include intermittent occurrence of channel blackouts, which are intervals of time when the communication channel is unavailable for feedback. We have designed an event-triggered control scheme that, using prior knowledge about the channel, guarantees the exponential stabilization of the system at a desired convergence rate, even in the presence of intermittent channel blackouts. Key enablers of our design are the definition and analysis of the bit capacity, which measures the maximum number of bits that can be communicated over a given time interval through one or more transmissions. We have also provided an efficient real-time algorithm to lower bound the bit capacity for a time-slotted model of channel evolution. An important assumption we make is that the encoder has knowledge of the channel evolution sufficiently ahead of time so that it can plan its transmission schedule. In many practical scenarios, the channel will have to be estimated, and only uncertain knowledge of its future evolution may be available. Nevertheless, our results show that the problem of estimating the bit capacity, which is needed to design a meaningful mechanism to guarantee exponential stability, is challenging even assuming full channel information. Future work will explore the reduction of the conservatism of the proposed design, the determination of design parameters that optimize performance and communication cost, the comparison with alternative design solutions that combine model predictive control and periodic event-triggered control, scenarios with bounded disturbances, a stochastic model of channel evolution. Other promising future directions of research include developing and quantifying notions of data capacity under partial knowledge of the channel evolution, such as knowledge of only frequency and duration of blackouts, which could also be extended to the cases with blackouts caused by an adversary, and the trade-off between the available information pattern at the encoder and the ability to perform event-triggered control.

Acknowledgments

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A Encoder and decoder schemes

Here, we detail for completeness, the schemes (cf. Algorithms 1 and 2) executed by the encoder and decoder during the system evolution.

The next result is a slight extension of [38, Lemma V.1] to accommodate the distinction between reception and update times and guarantees that the output of the encoder's and decoder's algorithms are consistent. The proof is analogous to that of [38, Lemma V.1] and we omit it here for brevity.

Lemma A.1. *(Consistency of Algorithms 1 and 2). If initially the encoder and the decoder share identical values for $\hat{x}(t_0)$ and $d_e(t_0)$, with $\|\hat{x}(t_0)\|_\infty \leq d_e(t_0)$, then Algorithms 1 and 2 result in consistent $\hat{x}(t)$ and $d_e(t)$ signals for all $t \geq t_0$. Further, $t \mapsto \hat{x}(t)$ evolves according to (4) and $\|x_e(t)\|_\infty \leq d_e(t)$ for all $t \in [t_0, \infty)$ with d_e defined according to (6).*

Algorithm 1: Update of encoder variables

At $t = t_0 = r_0$, the encoder initializes

1: $\delta_0 \leftarrow d_e(t_0)$ {store initial bound on encoding error}

At $t \in \{t_k\}_{k\in\mathbb{Z}_{>0}}$, the encoder sets

2: $z_k \leftarrow \hat{x}(t_k^-)$ {store encoder variable}

3: $z_{E,k} \leftarrow q_{E,k}(x(t_k), z_k)$ {encode plant state with p_k bits}

4: $\delta_k \leftarrow d_e(t_k^-)/2^{p_k}$ {compute bound on encoding error}

At $t \in \{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}}$, the encoder sets

5: $z_{D,k} \leftarrow q_{D,k}(z_{E,k}, z_k)$ {decode plant state at t_k}

6: $\hat{x}(\tilde{r}_k) \leftarrow e^{\bar{A}\tilde{\Delta}_k} z_k + e^{A\tilde{\Delta}_k}(z_{D,k} - z_k)$ {update controller state}

7: $d_e(\tilde{r}_k) \leftarrow \|e^{A\tilde{\Delta}_k}\|_\infty \delta_k$ {update bound on encoding error}

Algorithm 2: Update of decoder variables

At $t = t_0 = r_0$, the decoder initializes

1: $\delta_0 \leftarrow d_e(t_0)$ {store initial bound on encoding error}

At $t \in \{\tilde{r}_k\}_{k\in\mathbb{Z}_{>0}}$, the decoder sets

2: $z_k \leftarrow e^{-\bar{A}\tilde{\Delta}_k}\hat{x}(\tilde{r}_k^-)$ {compute encoder state at t_k}

3: $z_{E,k}$ {received from the encoder}

4: $\delta_k \leftarrow \frac{1}{2^{p_k}}(\|e^{A(t_k^- - t_{k-1})}\|_\infty \delta_{k-1}$ {compute bound on encoding error at t_k}

5: $z_{D,k} \leftarrow q_{D,k}(z_{E,k}, z_k)$ {decode plant state at t_k}

6: $\hat{x}(\tilde{r}_k) \leftarrow e^{\bar{A}\tilde{\Delta}_k} z_k + e^{A\tilde{\Delta}_k}(z_{D,k} - z_k)$ {update controller state}

7: $d_e(\tilde{r}_k) \leftarrow \|e^{A\tilde{\Delta}_k}\|_\infty \delta_k$ {update bound on encoding error}

B Efficient approximation of bit capacity

Here, we propose methods to efficiently approximate bit capacity. The next result is the basis for the construction of a sub-optimal solution to the problem (16).

Lemma B.1. *(Bound on "channel variation"). If there exists $J \in \mathbb{Z}_{\geq 0}$ such that*

$$\frac{\bar{\pi}_j}{R_j} < \sum_{i=j+1}^{i=j+1+J} T_i, \quad \forall j \in \mathcal{N}_{j_0}^{j_f}, \tag{31}$$

then, for any $j \in \mathcal{N}_{j_0}^{j_f}$, any bits transmitted in time slot I_j would be received strictly before the end of the slot I_{j+1+J}.

Proof. The term $\bar{\pi}_j/R_j$ is the time it takes a packet of size up to $n\bar{\pi}_j$ bits transmitted during I_j to reach the decoder. Thus, the claim follows by noting that any bits transmitted during I_j would be received before $t = \theta_{j+1} + (\bar{\pi}_j/R_j)$. ☐

Lemma B.1 relates the three sequences of parameters, $\{R_j\}$, $\{\bar{\pi}_j\}$ and $\{T_j\}$, that define the channel state at any given time. The result may be interpreted as the imposition of a bound on how often there is a change in the channel state as measured by the time slot lengths T_j. The parameter J may be interpreted as a uniform upper bound on the number of consecutive time slots that may be fully occupied due to a prior transmission.

B.1 Guaranteed channel availability in each time slot

The case of $J = 0$ is of special interest and will be addressed next. This case is interesting because the constraints (15) reduce to a simpler form, as presented in the following result, and using which we can compute a good sub-optimal solution subsequently.

Lemma B.2. *(bit capacity in the case of $J = 0$). Suppose the channel is such that $J = 0$ for all $j \in \mathcal{N}_{j_0}^{j_f}$. Then, the constraints (15a) reduce to*

$$n\phi_j + nR_j \sum_{i=j_1}^{j-1} \frac{\phi_i}{R_i} \le nR_j(\theta_{j+1} - \theta_{j_1}) + n\bar{\pi}_j, \tag{32a}$$

for each $j \in \mathcal{N}_{j_0}^{j_f}$ and $j_1 \in \mathbb{Z}_{\ge 0} \cap [j_0, j-1]$ while the constraints (15b) reduce to

$$\sum_{i=j_1}^{j_f-1} \frac{\phi_i}{R_i} \le \theta_{j_f} - \theta_{j_1}, \tag{32b}$$

for each $j_1 \in \mathbb{Z}_{\ge 0} \cap [j_0, j_f - 1]$. The bit capacity is

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) = \max_{\substack{\phi_j \in \mathbb{Z}_{\ge 0}, \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (14), (32) hold}}} n \sum_{j=j_0}^{j_f-1} \phi_j. \tag{33}$$

Proof. Indeed, if $J = 0$ then for each j and $j_1 \in \mathcal{N}_{j_0}^{j_f}$, $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) > 0$ and hence $\bar{T}_j > 0$ also. Thus, the constraints (15a) reduce to $n\phi_j \le nR_j \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) + n\bar{\pi}_j$, which after using the definition of $\bar{T}$ gives us (32a). Note that Lemma B.1, with $J = 0$, guarantees that the constraints (15b) are satisfied for all $j \in \{j_0, \dots, j_f - 2\}$, while for $j_f - 1$ (15b) reduce to

$$\frac{\phi_{j_f-1}}{R_{j_f-1}} \le \bar{T}_{j_1,j}(\phi_{j_0}^{j_f}),$$

which by expanding and rearranging the terms, we get the constraints (32b). Bit capacity (33) follows from (16) and the equivalence of (15) and (32). ☐

Note that for $J = 0$ all the constraints, (14) and (32) are linear, though ϕ_j are still restricted to be integers. This brings us to the next result.

Proposition B.3. *(A sub-optimal solution and quantification of sub-optimality in the case of $J = 0$). Suppose the channel is such that $J = 0$ for all $j \in \mathcal{J} = \{j_0, \dots, j_f\}$. Let $\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f-1} \phi_j^s$ where*

$$\begin{aligned} \phi^s &\triangleq \lfloor \phi^r \rfloor \triangleq (\lfloor \phi_{j_0}^r \rfloor, \dots, \lfloor \phi_{j_f-1}^r \rfloor), \\ \phi^r &= \operatorname*{argmax}_{\substack{\phi_j \in \mathbb{R}_{\ge 0},\ \forall j \in \mathcal{N}_{j_0}^{j_f} \\ \text{s.t. (14), (32) hold}}} \sum_{j=j_0}^{j_f-1} \phi_j. \end{aligned} \tag{34}$$

Then ϕ^s is a sub-optimal solution to (33), and

$$\begin{aligned} &\mathcal{D}(\theta_{j_0}, \theta_{j_f}) - \mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \\ &\qquad \le n |\{j \in \mathbb{Z}_{\ge 0} \cap [j_0, j_{f-1}] : \bar{\pi}_j > 0\}|. \end{aligned}$$

Proof. Clearly, ϕ^s satisfies the constraints (14) and (32) since ϕ^r does and for each j, $\phi_j^s \le \phi_j^r$ and $\phi^s \in \mathbb{Z}_{\ge 0}$. Thus, ϕ^s is a sub-optimal solution to (33). The sub-optimality bound follows from the fact that for any $a \in \mathbb{R}$, $(a - \lfloor a \rfloor) \in [0, 1)$. ☐

B.2 No guaranteed channel availability

If $J > 0$, we forgo optimality in favor of an easily computable lower bound of the bit capacity. With a slight abuse of notation, we let

$$\phi_j^s = \lfloor R_j(\theta_{j+1} - \theta_j) \rfloor, \quad j \in \mathbb{Z}_{\geq 0},$$

which is the number of bits that can be communicated during the time slot $I_j = [\theta_j, \theta_{j+1})$. Hence, $\{\phi_j^s\}_{j \in \mathbb{Z}_{\geq 0}}$ is a feasible solution and, again with an abuse of notation, we denote

$$\mathcal{D}_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f-1} \phi_j^s,$$

which is a sub-optimal lower bound of the bit capacity.

C Proofs of the main results

In this appendix, we present the proofs of the main results and of the intermediate lemmas.

Proof of Lemma 4.2. Observe that for any $j_1, j \in \mathcal{N}_{j_0}^{j_f}$, $\theta_{j+1} - \theta_{j_1}$ is the total time in the slots j_1 to j, while $\sum_{i=j_1}^{j-1} \frac{\phi_i}{R_i}$ is the total time taken by the bits transmitted in slots j_1 to $j-1$. Thus, $\left[\bar{T}_{j_1,j}(\phi_{j_0}^{j_f})\right]_+$ is an upper bound on the time available for transmission in the slot I_j. Now, let

$$j_2 = \max\{i \in \mathbb{Z}_{\geq 0} \cap [j_0, j-1] : \bar{T}_i(\phi_{j_0}^{j_f}) = T_i\}$$

Then clearly, $\{\phi_i\}_{i=j_2}^{j-1}$ is sufficient to determine $\bar{T}_j(\phi_{j_0}^{j_f})$. Next, for the allocation $\phi_{j_0}^{j_f}$, the bits transmitted during the time slots I_i for $i \in \{j_2, j-1\}$ are received by $\theta_{j_2} + \sum_{j=j_2}^{j-1} \frac{\phi_j}{R_j}$ and thus in deed $\bar{T}_j(\phi_{j_0}^{j_f}) = \left[\min\{\bar{T}_{j_2,j}(\phi_{j_0}^{j_f}), T_j\}\right]_+$. Finally, for each $j_1 \in \mathbb{Z}_{\geq 0} \cap [j_0, j_2 - 1]$, $\bar{T}_{j_1,j}(\phi_{j_0}^{j_f}) \geq \bar{T}_{j_2,j}(\phi_{j_0}^{j_f})$, which proves the result. □

Proof of Theorem 4.3. Here we prove only the statements about $\mathcal{D}(t, \theta_{j_f})$ as the proof of the statements for $\mathcal{D}_s(t, \theta_{j_f})$ are exactly analogous to those of $\mathcal{D}(t, \theta_{j_f})$. First of all notice that for any $\tau_1 < \tau_2 < \tau_3$

$$\mathcal{D}(\tau_1, \tau_3) \geq \mathcal{D}(\tau_1, \tau_2) + \mathcal{D}(\tau_2, \tau_3). \tag{35}$$

Now, let $\mathcal{T}_0 = \theta_{j_0} + \frac{\phi_{j_0}^*}{R_{j_0}}$. Clearly, from the optimality of $\mathcal{D}(\theta_{j_0}, \theta_{j_f})$, it follows that

$$\mathcal{D}(\theta_{j_0}, \mathcal{T}_0) = n\phi_{j_0}^*, \quad \mathcal{D}(\mathcal{T}_0, \theta_{j_f}) = n \sum_{j=j_0+1}^{j_f-1} \phi_j^*. \tag{36}$$

Thus, for the special choice of $\mathcal{T}_0$, we have the stronger relation $\mathcal{D}(\theta_{j_0}, \theta_{j_f}) = \mathcal{D}(\theta_{j_0}, \mathcal{T}_0) + \mathcal{D}(\mathcal{T}_0, \theta_{j_f})$. Now, using (35) twice we get

$$\begin{aligned}\mathcal{D}(\theta_{j_0}, \theta_{j_f}) &\geq \mathcal{D}(\theta_{j_0}, t) + \mathcal{D}(t, \theta_{j_f}) \\ &\geq \mathcal{D}(\theta_{j_0}, t) + \mathcal{D}(t, \mathcal{T}_0) + \mathcal{D}(\mathcal{T}_0, \theta_{j_f}),\end{aligned}$$

which implies

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) - \mathcal{D}(\theta_{j_0}, t) \geq \mathcal{D}(t, \theta_{j_f}) \geq \mathcal{D}(t, \mathcal{T}_0) + \mathcal{D}(\mathcal{T}_0, \theta_{j_f}).$$

Notice that $\mathcal{D}(t, \mathcal{T}_0) + \mathcal{D}(\mathcal{T}_0, \theta_{j_f}) = \hat{\mathcal{D}}(t, \theta_{j_f})$. Now, we compute the difference between the upper and lower bounds on $\mathcal{D}(t, \theta_{j_f})$

$$\mathcal{D}(\theta_{j_0}, \theta_{j_f}) - \mathcal{D}(\theta_{j_0}, t) - \hat{\mathcal{D}}(t, \theta_{j_f})$$

$$
\begin{aligned}
&= \mathcal{D}(\theta_{j_0}, \mathcal{T}_0) + \mathcal{D}(\mathcal{T}_0, \theta_{j_f}) - \mathcal{D}(\theta_{j_0}, t) - \hat{\mathcal{D}}(t, \theta_{j_f}) \\
&= n\left[R_{j_0}(\mathcal{T}_0 - \theta_{j_0}) - \lfloor R_{j_0}(t - \theta_{j_0})\rfloor - \lfloor R_{j_0}(\mathcal{T}_0 - t)\rfloor\right] \\
&= n\left[-\lfloor R_{j_0}(t - \theta_{j_0})\rfloor - \lfloor -R_{j_0}(t - \theta_{j_0})\rfloor\right] \le n,
\end{aligned}
$$

where, in arriving at the second last relation, we have used $\lfloor R_{j_0}(\mathcal{T}_0 - t)\rfloor = \lfloor R_{j_0}(\mathcal{T}_0 - \theta_{j_0}) - R_{j_0}(t - \theta_{j_0})\rfloor$ and the fact that $R_{j_0}(\mathcal{T}_0 - \theta_{j_0}) = \phi^*_{j_0}$ is an integer. The statement now follows. □

Proof of Theorem 5.1. We start by establishing two claims that we later invoke to establish the result.

Claim (a): First, we show that for any $t \ge t_0$, if $h_{\mathrm{pf}}(t) \le 1$ and $h_{\mathrm{ch}}(t) \le 1$ then $\mathcal{L}_1(s) < 1$ and $\mathcal{L}_2(s) < 1$, with $s = t$ and $s = t^+$. Indeed, if $h_{\mathrm{pf}}(t) \le 1$ and $h_{\mathrm{ch}}(t) \le 1$, then Lemma 3.2 says $\Gamma_1(h_{\mathrm{pf}}(t), \epsilon(t)) \ge \min\{\Gamma_1(1,1), T\}$. Then, from (19), (20a) and from Lemma 3.2(iv), we see that the claim is true for $\mathcal{L}_1$. Again, the conditions $h_{\mathrm{pf}}(t) \le 1$ and $h_{\mathrm{ch}}(t) \le 1$ along with Lemma 3.6 guarantee that for any $p \in \mathbb{Z}_{\ge 0}$, $\Gamma_2(h_{\mathrm{pf}}(t), \epsilon(t), p) \ge T^*(p)$. Thus, (19), (20b) and Lemma 3.5 imply that the claim is true for $\mathcal{L}_2$.

Claim (b): Next, we claim that for any $k \in \mathbb{Z}_{\ge 0}$, if $h_{\mathrm{pf}}(\tilde{r}_k) \le 1$ and $h_{\mathrm{ch}}(\tilde{r}_k) \le 1$, then $\mathcal{L}_i(t_{k+1}) \le 1$, for $i \in \{1, 2\}$. If the signal $\bar{p}$ is constant during $[\tilde{r}_k, t_{k+1}]$, the claim immediately follows from Claim (a) and (22). Now, let us suppose there exists $\theta \in [\tilde{r}_k, t_{k+1})$ at which time $\bar{p}$ is discontinuous, i.e., $\theta \in \{\theta_j\}_{j \in \mathbb{Z}_{>0}}$ as defined by (13). Then, from (22), it is clear that, for $i \in \{1, 2\}$, $\mathcal{L}_i(\theta) < 1$ and $\mathcal{L}_i(\theta^+) < 1$. This implies that there exists an interval $\mathcal{I}_\theta = [\theta, \theta + \epsilon)$ such that $\mathcal{L}_i(s) < 1$ for each $s \in \mathcal{I}_\theta$ and $i \in \{1, 2\}$. Then, by continuity of $\mathcal{L}_i$ on each interval $(\theta_j, \theta_{j+1}]$ and by invoking induction over the discontinuity times of $\bar{p}$, we conclude the claim is true.

Now, we show that (i) holds. The facts $\mathcal{L}_1(t_0) \le 1$ and $\mathcal{L}_2(t_0) \le 1$ together with the arguments used above ensure that $\mathcal{L}_1(t_1) \le 1$ and $\mathcal{L}_2(t_1) \le 1$. Then, (23) ensures that $\underline{p_1} \le \bar{p}(t_1)$. Now, for each $k \in \mathbb{Z}_{>0}$, if $\mathcal{L}_1(t_k) \le 1$ and $\mathcal{L}_2(t_k) \le 1$ and $p_k \in \mathbb{Z}_{>0} \cap [\underline{p_k}, \bar{p}(t_k)]$ then

$$
\tilde{r}_k - t_k = r_k - t_k \le \frac{p_k}{R(t_k)} \le \frac{\bar{p}(t_k)}{R(t_k)} \le T_M(\bar{p}(t)), \tag{37}
$$

where the last inequality follows from (21). As a result of (37), we see that $h_{\mathrm{pf}}(\tilde{r}_k) \le 1$ and $h_{\mathrm{ch}}(\tilde{r}_k) \le 1$. Then, invoking Claim (b), we see that $\mathcal{L}_2(t_{k+1}) \le 1$, from which it follows that $\underline{p_{k+1}} \le \bar{p}(t_{k+1})$, which proves (i).

Now, we prove (ii) - the main idea here is that for each $k \in \mathbb{Z}_{\ge 0}$, either $\tilde{r}_k - t_k$ or $t_{k+1} - \tilde{r}_k$ is sufficiently large to guarantee (ii). To show this, we pick $\sigma_1 \in (0, 1)$ and partition the set $\mathbb{Z}_{\ge 0}$ into two subsets G and L,

$$
\begin{aligned}
G &= \{k \in \mathbb{Z}_{\ge 0} : \tilde{r}_k - t_k > \sigma_1 T_M(p_k)\}, \\
L &= \{k \in \mathbb{Z}_{\ge 0} : \tilde{r}_k - t_k \le \sigma_1 T_M(p_k)\}.
\end{aligned}
$$

Then, it is clear that $\{t_{k+1} - t_k\}_{k \in G}$ and $\{\tilde{r}_{k+1} - \tilde{r}_k\}_{k \in G}$ are uniformly lower bounded by $\sigma_1 T_M(1)$. Thus, all that remains is to handle the set L. Recall that the assumptions and the design are such that, for each $k \in \mathbb{Z}_{\ge 0}$, we guarantee $h_{\mathrm{pf}}(\tilde{r}_k) \le 1$ and $h_{\mathrm{ch}}(\tilde{r}_k) \le 1$ for $\tilde{r}_k \le t_k + T_M(p_k)$. As a result, and due to the fact that $\{p_k\}$ is upper bounded by $p^{\max}$, there exist $h^0_{\mathrm{pf}}, h^0_{\mathrm{ch}} \in (0, 1)$ such that $h_{\mathrm{pf}}(\tilde{r}_k) \le h^0_{\mathrm{pf}}$ and $h_{\mathrm{ch}}(\tilde{r}_k) \le h^0_{\mathrm{ch}}$ for all $k \in L$. Thus, from Claim (a) and (22), it is clear that for any $k \in L$, $t_{k+1} - \tilde{r}_k \ge T_L$, where T_L is a lower bound on the time it takes h_{pf} to evolve from h^0_{pf} to 1 and the time it takes h_{ch} to evolve from h^0_{ch} to 1. Finally, because both h^0_{pf} and h^0_{ch} are strictly less than 1, it follows $T_L > 0$, which proves (ii).

Regarding (iii), we have already seen that for any $k \in \mathbb{Z}_{\ge 0}$, $h_{\mathrm{pf}}(t) \le 1$ for all $t \in [t_k, \tilde{r}_k]$. Further, (22) also ensures that $h_{\mathrm{pf}}(t) \le 1$ for all $t \in [\tilde{r}_k, t_{k+1}]$. Therefore $h_{\mathrm{pf}}(t) \le 1$ $(V(x(t) \le V_d(t))$ for all $t \ge t_0$, which completes the proof. □

Proof of Lemma 5.3. From Lemma 3.2, we know $\Gamma_1(h_{\mathrm{pf}}(\tau_l), \epsilon(\tau_l)) \ge \Gamma_1(1, \epsilon_r(t))$. So, we need to show that $\Gamma_1(1, \epsilon_r(t)) \ge T_b(t)$ or, as per Lemma 3.2(iv), that $\bar{h}_{\mathrm{pf}}(T_b(t), 1, \epsilon_r(t)) \le 1$. Direct computation shows that this is indeed the case, which implies $h_{\mathrm{pf}}(s) \le 1$ for all $s \in [\tau_l, \tau_u]$ by the definition of Γ_1. The second claim follows from

$$
h_{\mathrm{ch}}(\tau_u) \le \bar{h}_{\mathrm{ch}}(T_b(t), 1, \epsilon_r(t), 0)
$$

$$= \|e^{AT_b(t)}\|_\infty e^{\frac{\beta}{2}T_b(t)}\epsilon_r(t) \leq e^{\bar{\mu}T_b(t)}\epsilon_r(t) \leq 1. \quad \square$$

$\square$

Proof of Lemma 5.4. The fact that $\tilde{B}_j$ is an interval follows directly from the definition (24). If $\bar{\pi}_j > 0$, then at any time $t \in I_j$, $\bar{p}(t) = \bar{\pi}_j > 0$. Thus, if $\psi^{\tau_l}(t) = 0$ for some $t \in I_j$,

$$\begin{aligned} &\mathcal{P}_j - R_j(t - \theta_j) = \mathcal{P}_j - R_j(t + T_j - \theta_{j+1}) < 1 \\ &\implies (\theta_{j+1} - t) < \frac{1}{R_j} + \left(T_j - \frac{\mathcal{P}_j}{R_j}\right) < \frac{2}{R_j}, \end{aligned}$$

where the last inequality follows from the optimality of $\mathcal{D}_s(\theta_j, \tau_l(\theta_j))$ because otherwise, if $R_jT_j - \mathcal{P}_j \geq 1$, then the optimality of $\mathcal{P}_j$ would imply that $\mathcal{P}_j = \mathcal{P}_j + 1$, which is a contradiction. This proves the result. $\square$

Proof of Lemma 5.5. Notice from the definition of $\epsilon(t)$ in (9) and (6) that for any $k \in \mathbb{Z}_{\geq 0}$ and $s \in [r_k, r_{k+1})$

$$\epsilon(s) = \frac{\|e^{A(s-t_k)}\|_\infty e^{(\beta/2)(s-t_k)}\epsilon(t_k^-)}{2^{p_k}} \leq \frac{e^{\bar{\mu}(s-t_k)}\epsilon(t_k^-)}{2^{p_k}},$$

which when recursively used gives us

$$\epsilon(\tau_l(t)) \leq \frac{e^{\bar{\mu}(\tau_l(t)-t)}\epsilon(t)}{2^{(\mathcal{B}(t,\tau_l(t))/n)}},$$

where $\mathcal{B}(t, \tau_l(t))$ is the total number of bits communicated (transmitted and received) during the time interval $[t, \tau_l(t)]$. In other words, for any $t \geq t_0$, if

$$\mathcal{B}(t, \tau_l(t)) \geq n \log_2 \left(\frac{e^{\bar{\mu}(\tau_l(t)-t)}\epsilon(t)}{\epsilon_r(t)}\right)$$

ensures that $\epsilon(\tau_l(t)) \leq \epsilon_r(\tau_l(t))$. Initially, $\mathcal{L}_3(t_0, \epsilon(t_0)) \leq 0$ ensures that there is enough bit capacity, i.e., $\mathcal{B}(t_0, \theta_{j_1}) \leq \hat{\mathcal{D}}_s(t_0, \theta_{j_1})$ to ensure the inequality holds. Lemma 5.3 guarantees that for any $k \in \mathbb{Z}_{>0}$, if $\epsilon(\theta_{j_k}) \leq \epsilon_r(\theta_{j_k})$ then $\epsilon(\theta_{j_k+1}) \leq 1$. Then by induction and the use of the fact that $\mathcal{L}_3(\theta_{j_k+1}, 1) \leq 0$ for each $k \in \mathbb{Z}_{\geq 0}$ we deduce that there exists a transmission policy that ensures $\epsilon(\theta_{j_k}) \leq \epsilon_r(\theta_{j_k})$ for each $k \in \mathbb{Z}_{>0}$. Consequently, by invoking Lemma 5.3 again, there exists a control policy that ensures $h_{\mathrm{pf}}(s) \leq 1$ for all $s \in (\theta_{j_k}, \theta_{j_k+1}]$ for each $k \in \mathbb{Z}_{>0}$. $\square$

Proof of Theorem 5.6. Notice that (28) ensures that for any $k \in \mathbb{Z}_{>0}$, $\psi^{\tau_l}(t_k) \geq 1$. Now, notice from (29) that for any $k \in \mathbb{Z}_{>0}$, $\tilde{r}_k > r_k$ if and only if $\psi^{\tau_l}(r_k) = 0$ and $\bar{p}(r_k) \geq 1$. That is, $\tilde{r}_k > r_k$ if and only if $r_k \in (\tau_1, \tau_2]$, an artificial blackout interval. In all other cases, $\tilde{r}_k = r_k$. Thus, it follows from Lemma 5.4 that $\tilde{r}_k - r_k \leq \frac{2}{R(r_k)}$ for all $k \in \mathbb{Z}_{>0}$. Hence, for all $k \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} &\tilde{r}_k - t_k = (\tilde{r}_k - r_k) + (r_k - t_k) \leq \frac{2}{R(r_k)} + \frac{p_k}{R(t_k)} \\ &\implies \tilde{r}_k - t_k \leq \begin{cases} \frac{p_k}{R(t_k)}, & \text{if } \tilde{r}_k = r_k \\ \frac{(p_k+2)}{\min\{R(t_k), R(r_k)\}}, & \text{if } \tilde{r}_k > r_k. \end{cases} \end{aligned}$$

In either case, it follows from (27) that $\tilde{r}_k - t_k \leq T_M(p_k) \leq T_M(\psi^{\tau_l}(t_k))$ for all $k \in \mathbb{Z}_{>0}$. Thus, claims (a) and (b) in the proof of Theorem 5.1 hold here also.

Next observe that, by the construction of $t \mapsto \psi^{\tau_l}(t)$ in (25), we have $\hat{\mathcal{D}}_s(\tilde{r}_k, \tau_l) \geq \hat{\mathcal{D}}_s(t_k, \tau_l) - np_k$. Next, noting that

$$\epsilon(\tilde{r}_k) = \|e^{A\tilde{\Delta}_k}\|_\infty e^{\frac{\beta}{2}\tilde{\Delta}_k}\frac{\epsilon(t_k)}{2^{p_k}} \leq e^{\bar{\mu}\tilde{\Delta}_k}\frac{\epsilon(t_k)}{2^{p_k}},$$

we have

$$
\begin{aligned}
& n\log_2\left(\frac{e^{\bar{\mu}(\tau_l-\tilde{r}_k)}\epsilon(\tilde{r}_k)}{\epsilon_r}\right) \leq n\log_2\left(\frac{e^{\bar{\mu}(\tau_l-t_k)}\epsilon(t_k)}{\epsilon_r}\right) - np_k \\
& \leq \sigma_1\hat{\mathcal{D}}_s(t_k,\tau_l) - np_k \leq \sigma_1(\hat{\mathcal{D}}_s(t_k,\tau_l) - np_k) \\
& \leq \sigma_1\hat{\mathcal{D}}_s(\tilde{r}_k,\tau_l),
\end{aligned}
$$

where the second inequality follows from $\tilde{\mathcal{L}}_3(t_k) \leq 0$ and the third inequality follows from $\sigma_1 \in (0,1)$. Therefore, $\tilde{\mathcal{L}}_3(\tilde{r}_k) \leq 0$. Thus, using induction, the proposed transmission policy ensures that by the beginning of the next blackout, $t = \tau_l$, $\epsilon(\tau_l) \leq \epsilon_r$. Lemma 5.3 then implies that, at the end of blackout, we have $h_{\mathsf{ch}}(\tau_u) \leq 1$ and $h_{\mathsf{pf}}(s) \leq 1$ for all $s \in [\tau_l,\tau_u]$. Hence, claim (i) follows as in the proof of Theorem 5.1(i) and using induction over the sequence of blackout slots.

Claim (ii) also follows by arguments analogous to the proof of Theorem 5.1(ii). Finally, we prove (iii). Notice (28) ensures that $\tilde{\mathcal{L}}_1(t_k) \leq 1$ for any $k \in \mathbb{Z}_{>0}$, which as a consequence of Lemma 3.2(iv) means that $h_{\mathsf{pf}}(t) \leq 1$ for all $t \in [t_k,\tilde{r}_k]$ for any $k \in \mathbb{Z}_{>0}$. Now, for $t \in [\tilde{r}_k, t_{k+1})$ for $k \in \mathbb{Z}_{\geq 0}$, there are three cases. *Case I:* $\psi^{\tau_l}(t) \geq 1$. In this case, $h_{\mathsf{pf}}(t) \leq 1$ because $\tilde{\mathcal{L}}_1(t) < 1$. *Case II:* $\psi^{\tau_l}(t) = 0$ and $\bar{p}(t) \geq 1$, which corresponds to a time during an artificial blackout $(\tau_1,\tau_2]$. Recall from Lemma 5.4 that $\tau_2 - \tau_1 \leq 2/R(\tau_1)$, which using (27) then implies $\tau_2 - \tau_1 \leq T_M(\psi^{\tau_l}(\tau_1^-))$. Next, by design (29), $\tilde{r}_k \notin (\tau_1,\tau_2]$ and hence $\tilde{r}_k < \tau_1$ and no transmission is in progress during $(\tau_1,\tau_2]$, which must mean $\tilde{\mathcal{L}}_1(\tau_1^-) < 1$. Lemma 3.2(iv) then implies $\Gamma_1(h_{\mathsf{pf}}(\tau_1),\epsilon(\tau_1)) \geq T_M(\psi^{\tau_l}(\tau_1^-)) \geq \tau_2 - \tau_1$. Therefore, $h_{\mathsf{pf}}(t) \leq 1$ for all $t \in [\tau_1,\tau_2)$. *Case III:* $\psi^{\tau_l}(t) = \bar{p}(t) = 0$, which corresponds to a time in a channel blackout slot. We have already seen in the proof of (i) that the proposed transmission policy ensures $h_{\mathsf{pf}}(s) \leq 1$ for all $s \in [\tau_l,\tau_u]$ for any channel black out $[\tau_l,\tau_u]$. Therefore, $h_{\mathsf{pf}}(t) \leq 1$ $(V(x(t)) \leq V_d(t))$ for $t \geq t_0$. □