Network Modification using a Novel Gramian-based Edge Centrality

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Abstract— **Modifying the structure of man-made and natural networked systems has become increasingly feasible due to recent technological advances. This flexibility offers great opportunities to save resources and improve controllability and energy efficiency. In contrast (and dual) to the well-studied optimal actuator placement problem, this work focuses on improving network controllability by adding and/or re-weighting network edges while keeping the actuation structure fixed. First a novel energy-based edge centrality measure is proposed and then its relationship with the gradient (with respect to edge weights) of the trace of the controllability Gramian is rigorously characterized. Finally, a network modification algorithm based on the proposed measure is proposed and its efficacy in terms of computational complexity and controllability enhancement is numerically demonstrated.**

I. Introduction

Recent years have seen an unprecedented surge of interest from the scientific community in improving the controllability of complex dynamical networks, with applications that range from infrastructure and robotic to biological and social networks. Due to historical and practical reasons, complex network controllability has been largely focused on optimizing the location of the actuators (control inputs) while assuming that the network edges (whether fixed or time-varying) is given. Improving network controllability through the modification of the network edges, however, has been less studied. This work seeks to address this gap by introducing a computationally efficient and analytically grounded algorithm for network edge modification.

Literature review: While classical (Kalman) controllability is a binary notion (encoding the mere possibility of steering the network state arbitrarily in the state space) [1], [2], recent work has introduced various energy-based metrics for quantifying (and then optimizing) controllability [3]–[5]. These metrics often stem from the controllability Gramian due to its relationship with the minimal energy required for steering the network between any pair of states [6]. The optimal actuator scheduling problem then seeks to find the best location (nodes) for placing a limited number of actuators in order to maximize a metric of network controllability [3]–[5], [7]–[11].

An implicit but strong assumption in the optimal actuator scheduling problem is that the edge structure of the network is fixed (or given, if time-varying). If the existing edge structure of a network is not suitable for control, this may pose significant constraints on the best achievable degree of controllability via actuator scheduling. This has motivated recent works to study the dual problem of maximizing network controllability via edge modification. Minimal edge addition for structural controllability was studied in [12]. Edge addition in consensus networks is studied using alternating direction method of multipliers (ADMM) [13] in [14] and using spectral systemic performance measures and greedy algorithm in [15]. In [16], optimal perturbations of existing edges were computed by optimizing energy-based metrics using semidefinite programming (SDP) [17]. Using the procedure presented in [16], new edges can also be added in the network but the location of the edges should be given a priori. In [18], edge modifications were prescribed using energy-based metrics for networks with diagonal controllability Gramian. Finally, edge modifications using topological edge centrality measures was done in [19], [20].

Statement of contributions: In the aforementioned works, edge modifications were done for special cases (consensus networks and networks with diagonal controllability Gramian), only on existing edges (i.e., without the possibility of adding new edges), or using topological edge centrality (i.e., without considering network dynamics). The contributions of this work are threefold: (i) We derive a novel energy-based edge centrality measure using the controllability and observability Gramians. This edge centrality measure is computationally inexpensive to compute which makes it suitable for use in case of large networks. To the best of our knowledge, this is the first Gramian-based edge centrality measure that takes into account network dynamics and the propagation of energy over the network; (ii) We prove a relationship between the proposed edge centrality measure and the gradient of the trace of the controllability Gramian. On the basis of this relationship, (existing or potentially new) edges can be ranked in descending order of their centrality measure as a proxy for their relative impact on the trace of the controllability Gramian; and (iii) using this ordered list of edges, we propose a computationally efficient network modification algorithm by first selecting a small subset of edges with the largest centrality measures and then maximizing the trace of controllability Gramian (without approximations) over the selected subset. The latter step (which is in general computationally expensive) is feasible due to the small size of the selected subset of edges and can be solved using publicly available software. For this step, we impose budget constraints both on the number of added/modified edges and the total amount of weight added

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to them. Examples are provided to demonstrate the utility and efficiency of the proposed algorithm. For reasons of space, all proofs are omitted and will appear elsewhere.

Organization: The paper is organized as follows. Section II describes the considered optimal edge modification problem. In Section III, we describe a new edge centrality measure and discuss its properties which are relevant for optimal edge modification. Based on this centrality measure, in Section IV we describe a computationally efficient edge modification procedure and demonstrate its utility in Section V. We conclude the paper in Section VI.

Notation: We use $\mathbb{R}$ and $\mathbb{R}_{>0}$ to denote the sets of reals and positive reals, respectively. $(\cdot)^\top$ represents the transpose of a vector or matrix. For a matrix $\mathcal{W} \in \mathbb{R}^{n\times n}$, $\text{tr}\,(\mathcal{W})$ denotes its trace and $\text{vec}\,(\mathcal{W}) \in \mathbb{R}^{n^2}$ its column-wise vectorized form. For matrices $X, Y \in \mathbb{R}^{n\times n}$, their Frobenius inner product is given by $\langle X, Y\rangle_F = \text{tr}\left(X^\top Y\right)$. This induces the Frobenius norm $\|X\|_F = \sqrt{\langle X, X\rangle_F}$, which we use to express $\langle X, Y\rangle_F = \|X\|_F\|Y\|_F \cos\phi$, where $\phi = \angle\left(\text{vec}\,(X), \text{vec}\,(Y)\right)$ is the angle between the vectors $\text{vec}\,(X)$ and $\text{vec}\,(Y)$. For $j \in \{1, \ldots, n\}$, $e_j \in \mathbb{R}^n$ denotes the j^{th} canonical unit vector.

II. Problem Description

Consider a directed network of n nodes represented by the triplet $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}_A, w_A)$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the node set, $\mathcal{E}_A = \{\,(i,j) \mid i \in \mathcal{V}, j \in \mathcal{V}, i \neq j\,\}$ is the edge set, and $w_A : \mathcal{E}_A \mapsto \mathbb{R}_{\geq 0}$ is a weight function. The pair (i, j) denotes an edge directed from node i to node j. The nodal dynamics are given by the discrete-time dynamics

$$x\,(t+1) = Ax\,(t) + Bu\,(t)\,,\ t = \{0, \ldots, T-1\}, \qquad (1)$$

where $T > 0$ is a finite time horizon, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state and input vectors respectively. Here, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix defined by $a_{ij} = w_A\left[(i,j)\right] > 0$ if the edge $(j,i) \in \mathcal{E}_A$ else $a_{ij} = 0$, and $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_m \end{pmatrix} \in \mathbb{R}^{n\times m}$ is the actuator location (input) matrix. Here $b_i \in \{0,1\}^n$ are binary vectors with 0's everywhere except at one entry, signifying the presence of an actuator at that node.

The dynamical system (1) is controllable in $T-$steps if any arbitrary initial state $x\,(0) = x_0$ can be steered to any arbitrary final state $x\,(T) = x_T$ using a finite control input sequence $\{u\,(0)\,, u\,(1)\,, \ldots, u\,(T-1)\}$. Formally, (A, B) is controllable in $T-$steps if and only if the controllability Gramian [2] $\mathcal{W}_A\,(T)$ is nonsingular, where

$$\mathcal{W}_A\,(T) = \sum_{t=0}^{T-1} A^t BB^\top {A^t}^\top. \qquad (2)$$

This binary notion of controllability does not distinguish between systems that are more easily controllable than others. To do so, various notions of performance metrics based on the spectral properties of $\mathcal{W}_A$ have been proposed in the literature [3], [4], including tr$(\mathcal{W}_A)$, tr$(\mathcal{W}_A^{-1})^{-1}$, det$(\mathcal{W}_A)$, and $\lambda_{\min}(\mathcal{W}_A)$. Throughout the paper, we employ tr$(\mathcal{W}_A)$ as our measure of network controllability. This choice is motivated by the various advantages and disadvantages of the measures described above, see [10, Appendix B] for a detailed discussion. For large networks, tr$(\mathcal{W}_A)$ is not only computationally feasible to calculate (while the other metrics are not due to limited machine precision) but is also intrinsically less conservative than metrics that rely on the smallest eigenvalue of $\mathcal{W}_A$ and is analytically more tractable.

In this work, while keeping the input structure unchanged, our focus is on improving controllability by modifying the network edges. The modification may involve perturbing the existing edges or adding new ones. Let the modification be represented by $\mathcal{G}_{\delta A} = (\mathcal{V}, \mathcal{E}_{\delta A}, w_{\delta A})$ with weighted adjacency matrix $\delta A = (\delta A_{ij})$. We assume that $\mathcal{G}_A$ as well as $\mathcal{G}_{\delta A}$ do not have any self-loops i.e., $a_{ii} = 0$ and $\delta A_{ii} = 0$.

In general, both the number of modified edges and the total added weight may be constrained. Let $N_{\max}$ and $w_{\max} \in \mathbb{R}_{>0}$ denote these bounds, respectively. Our problem of interest can then be formulated as

$$\begin{aligned} \max_{\delta A} \quad & \text{tr}\left(\mathcal{W}_{A+\delta A}\,(T)\right) \\ \text{s.t.} \quad & \text{card}\left(\mathcal{E}_{\delta A}\right) \leq N_{\max}, \quad \sum w_{\delta A}\left(\mathcal{E}_{\delta A}\right) \leq w_{\max}, \end{aligned} \qquad (3)$$

where $\mathcal{W}_{A+\delta A}\,(T)$ is the controllability Gramian of the modified network,

$$\mathcal{W}_{A+\delta A}\,(T) = \sum_{t=0}^{T-1} \left(A + \delta A\right)^t BB^\top {\left(A + \delta A\right)^t}^\top. \qquad (4)$$

The problem in (3) is a mixed-integer and non-convex optimization problem which becomes computationally intractable as the network size increases. In the following, we propose a computationally efficient approximation using a novel edge centrality measure and demonstrate its tight correlation with the exact solution of (3) both theoretically and in numerical examples.

III. Energy-Based Edge Centrality Measure

Given the focus of the paper on optimizing network structure, here we introduce a novel notion of edge centrality that seeks to quantify their importance in influencing network behavior. This notion builds on measures of the energy exchange between an individual node and the network, for which we employ the controllability and observability Gramians. We later explore how the proposed notion of edge centrality can be invoked to address the network modification problem (3) by showing its connection with the gradient of the objective function.

A. Node-Network Interactions

As the input matrix B is kept unchanged, the energy input to the network from external sources is fixed. Due to the original structure of the network, the energy supplied may get accumulated at some nodes. At the same time, some nodes which are efficient at distributing energy may not receive adequate energy. When the structure of the network is modified, a redistribution of the energy exchange takes place in the system. Intuitively, the modified edges should connect (or strengthen existing connections from) nodes which accumulate energy to the nodes which are in a good position for distributing it. These properties can be quantified by the influence of the network on a node and the influence of a node on the network, respectively, as given next following the exposition in [9].

Node-to-Network Influence: The influence of a particular node on the network is quantified by considering the particular node as the only input node and computing the trace of the resultant controllability Gramian. For the t^{th} time step with $1 \leq t \leq T$, the influence of the j^{th} node on the network is denoted by $p_j^{(t)}$ and is computed as follows,

$$p_j^{(t)} = \text{tr}\left(\mathcal{W}_j^{(t)}\right) = \sum_{u=1}^{n} \sum_{k=0}^{t-1} e_u^\top A^k e_j e_j^\top {A^k}^\top e_u,$$
$$\mathcal{W}_j^{(t)} = \sum_{k=0}^{t-1} A^k e_j e_j^\top {A^k}^\top. \tag{5}$$

Clearly $p_j^{(t)} \geq 1$. If $p_j^{(t)} = 1$ for all $t > 1$, then it can be shown that node j is a sink [9] and thus accumulates energy.

Network-to-Node Influence: In parallel to the above definition, the influence of the network on a particular node is quantified by considering the particular node as the only output node and computing the trace of the resultant *observability* Gramian [2]. For the t^{th} time step with $1 \leq t \leq T$, the influence of the network on the i^{th} node is denoted by $q_i^{(t)}$ and is computed as follows,

$$q_i^{(t)} = \text{tr}\left(\mathcal{M}_i^{(t)}\right) = \sum_{v=1}^{n} \sum_{k=0}^{t-1} e_v^\top {A^k}^\top e_i e_i^\top A^k e_v, \tag{6}$$
$$\mathcal{M}_i^{(t)} = \sum_{k=0}^{t-1} {A^k}^\top e_i e_i^\top A^k.$$

Similarly, $q_i^{(t)} \geq 1$ and if $q_i^{(t)} = 1$ for all $t > 1$, node i is a source [9] and thus does not accumulate any energy.

B. Edge Centrality Measure

Here, we combine the energy exchange notions described above to define a new edge centrality measure which is, interestingly, closely related to the problem in (3). Centralities are measures used to quantify the influence and importance of nodes and edges in a network. Various node and edge centrality measures based on the topology of the network have been proposed, see, e.g., [21], [22]. For the case of node centrality, energy (Gramian)-based measures have also been given [9], [10]. However, to the best of our knowledge, energy-based notions of edge centrality have not been proposed.

Intuitively, the influence of an edge in a network is related to the nodes it connects and the extent to which it facilities the energy distribution in the network. If an edge connects an energy-rich node (one with high q_i) to a node which facilitates energy distribution (one with high p_j), then the edge has more influence on the energy distribution in the network. Thus, we propose

$$c_{ij}^{(t)} = q_i^{(t)} p_j^{(t)}. \tag{7}$$

as a measure of the centrality of the edge directed from node i to node j at time t. For the complete time horizon T, the edge centrality is defined as,

$$c_{ij} = \sum_{t=1}^{T-1} c_{ij}^{(t)} \tag{8}$$

Clearly $c_{ij}^{(t)} \geq 1$, so the minimum attainable centrality measure is $T-1$, which has a simple intuitive interpretation. If $c_{ij}^{(t)} = 1$ for $t \geq 1$, then the corresponding edge is directed from a source node i to a sink node j and thus constitutes a minimally-influential interconnection, cf. Figure 1.

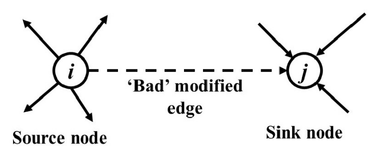

Fig. 1: An example of a minimally-influential edge with $c_{ij} = T - 1$.

Figure 2 shows the histogram of the correlation coefficients between the elements of the gradient of the objective function in (3) (with respect to all the edge weights $\{a_{ji}\}_{i,j}$) and $\{c_{ij}\}_{i,j}$ for 10^3 random Erdős-Rényi networks. As the plot shows, there exists a remarkable average correlation coefficient of $R \simeq 0.9$, with p-value smaller than machine epsilon in all cases, showing extreme statistical significance. These results prompt our investigation of the relationship between c_{ij} and Problem (3), which we carry out next.

C. Relationship Between Edge Centrality and the Gradient of the Trace of the Controllability Gramian

Consider the addition of an edge directed from node i to node j. The corresponding adjacency matrix δA will have only one non-zero element, so $\delta A = e_j e_i^\top$. Let,

$$C_j^{(t)} = \begin{pmatrix} A^{t-1}e_j & A^{t-2}e_j & \cdots & Ae_j & e_j \end{pmatrix},$$
$$O_i^{(t)} = \begin{pmatrix} e_i & A^\top e_i & \cdots & {A^{t-2}}^\top e_i & {A^{t-1}}^\top e_i \end{pmatrix}^\top. \tag{9}$$

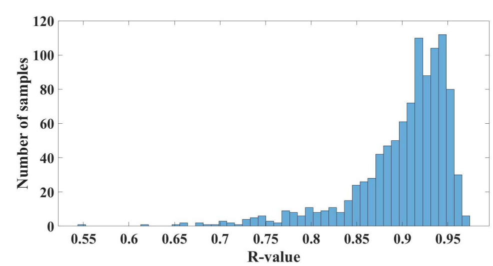

Fig. 2: Histogram of the correlation coefficient (R-value) between the values of $\{c_{ij}\}_{i,j}$ and $\{\frac{\partial \text{tr}(\mathcal{W}_A)}{\partial a_{ji}}\}_{i,j}$ for 10^3 random networks with 25 node, 8 inputs and 0.2 as edge existence probability, suggesting that the former can be used as a proxy for the latter.

The next result relates the gradient of the trace of the controllability Gramian to the matrices $C_j^{(t)}$ and $O_i^{(t)}$ in (9).

Theorem 3.1: (Gradient of trace of Gramian). For the network dynamics (1),

$$\frac{\partial}{\partial a_{ji}} \text{tr}\left(\mathcal{W}_A\left(T\right)\right) = 2\sum_{t=0}^{T-1} \text{tr}\left(BB^\top {A^t}^\top C_j^{(t)} O_i^{(t)}\right), \quad (10)$$

where the controllability Gramian $\mathcal{W}_A\left(T\right)$ is defined in (2).

Theorem 3.1 establishes an important relationship between the gradient of the trace of the controllability gramian $\mathcal{W}_A(T)$ of the original network with input matrix B, the controllability Gramian of the network with node j as the only input node, and the observability Gramian of the network with node i as the only output node. This result serves as a basis for establishing a relationship between the centrality measure $c_{ij}^{(t)}$ in (7) and the gradient of the trace of Gramian, as derived next.

For any edge $(i,j) \in \{1,\ldots,n\}^2$ and $t \in \{1,\ldots,T-1\}$, define

$$g_{ij}^{(t)} = 2\text{tr}\left(BB^\top {A^t}^\top C_j^{(t)} O_i^{(t)}\right)$$

so that $\frac{\partial}{\partial a_{ji}} \text{tr}\left(\mathcal{W}_A\left(T\right)\right) = \sum_{t=1}^{T-1} g_{ij}^{(t)}$. Further, let

$$\begin{aligned}
\phi_{ij}^{(t)} &= \angle\left(\text{vec}\left(A^t BB^\top\right), \text{vec}\left(C_j^{(t)} O_i^{(t)}\right)\right), \\
X_j^{(t)} &= {C_j^{(t)}}^\top C_j^{(t)}, \quad Y_i^{(t)} = O_i^{(t)} {O_i^{(t)}}^\top, \\
r_j^{(t)} &= \min\{t,n\}, \quad s^{(t)} = \sqrt{\frac{\text{tr}\left({X_j^{(t)}}^2\right)}{r_t^{(t)}} - \left(\frac{p_j^{(t)}}{r_j^{(t)}}\right)^2}, \\
\Psi_{ij}^{(t)} &= \|A^t BB^\top\|_F \cos\phi_{ij}^{(t)}. \qquad (11)
\end{aligned}$$

Then, for any $t = 1,\ldots,T-1$, the relationship between $g_{ij}^{(t)}$ and the $c_{ij}^{(t)}$ can be stated as follows.

Theorem 3.2: (Edge centrality-based bounds on the gradient of trace of Gramian). For any $t \in \{1,\ldots,T-1\}$,

$${\underline{g}_{ij}^{(t)}}^2 \le {g_{ij}^{(t)}}^2 \le {\overline{g}_{ij}^{(t)}}^2, \quad (12)$$

where

$$\begin{aligned}
{\underline{g}_{ij}^{(t)}}^2 &= 4{\Psi_{ij}^{(t)}}^2\left(\frac{c_{ij}^{(t)}}{r_j^{(t)}} - q_i^{(t)} s^{(t)} \sqrt{r_j^{(t)} - 1}\right), \\
{\overline{g}_{ij}^{(t)}}^2 &= 4{\Psi_{ij}^{(t)}}^2\left(\frac{c_{ij}^{(t)}}{r_j^{(t)}} + q_i^{(t)} s^{(t)} \sqrt{r_j^{(t)} - 1}\right). \qquad \square
\end{aligned}$$

The significance of Theorem 3.2 is in showing that the gradient of (the t^{th} summand of) the trace of the controllability Gramian is bounded by functions dependent on the centrality measure $c_{ij}^{(t)}$. According to (12), ${g_{ij}^{(t)}}^2$ belongs to an interval whose mid-point is proportional to $c_{ij}^{(t)}$, explaining the high correlation between the gradient of tr$(\mathcal{W}_A)$ and c_{ij} observed in Figure 2. Nevertheless, we note that the length of this interval is also a function of the location (i,j) of the modified edge and therefore the analysis of their relationship is also required for obtaining a complete characterization of the relationship between the gradient of tr$(\mathcal{W}_A)$ and c_{ij}. Hence, a characterization of the tightness of this bound is still an open problem for future research.

D. Computational Aspects of Proposed Edge Centrality

Here we show how the redundancy in the calculations of $\{p_j\}_j$ and $\{q_i\}_i$ can be exploited to enhance the computational efficiency of calculating $\{c_{ij}\}_{i,j}$. We then compare this computational complexity to that of calculating $\{\frac{\partial \text{tr}(\mathcal{W}_A)}{\partial a_{ij}}\}_{i,j}$.

Let $\widehat{p}_j^{(t)}$ be a vector of diagonal elements of $\mathcal{W}_j^{(t)}$ in (5) and

$$\mathcal{H}^{(t)} = \begin{pmatrix}\widehat{p}_1^{(t)} & \ldots & \widehat{p}_j^{(t)} & \ldots & \widehat{p}_n^{(t)}\end{pmatrix} \in \mathbb{R}^{n\times n}. \quad (13)$$

The next result shows the relationship among $p_j^{(t)}$, $q_i^{(t)}$, and $\mathcal{H}^{(t)}$.

Proposition 3.3: (Relation between $p_j^{(t)}$ and $q_i^{(t)}$). Let $\mathcal{H}^{(t)}$ be defined as in (13). Then,

$$p_j^{(t)} = \sum_{i=1}^{n} \mathcal{H}^{(t)}(i,j), \qquad q_i^{(t)} = \sum_{j=1}^{n} \mathcal{H}^{(t)}(i,j). \qquad \square$$

The computation of $\{c_{ij}\}_{i,j}$ can be simplified based on Proposition 3.3, as follows. Let

$$\begin{aligned}
p^{(t)} &= \begin{pmatrix}p_1^{(t)} & p_2^{(t)} & \ldots & p_j^{(t)} & \ldots & p_{n-1}^{(t)} & p_n^{(t)}\end{pmatrix}^\top, \\
q^{(t)} &= \begin{pmatrix}q_1^{(t)} & q_2^{(t)} & \ldots & q_i^{(t)} & \ldots & q_{n-1}^{(t)} & q_n^{(t)}\end{pmatrix}^\top,
\end{aligned}$$

and define

$$\Theta^{(t)} = p^{(t)} {q^{(t)}}^\top \in \mathbb{R}^{n\times n}, \quad \Theta = \sum_{t=1}^{T-1} \Theta^{(t)}. \quad (14)$$

Then, it is straightforward to check that $c_{ij} = \Theta_{ji}$.

One can see that the computation of g_{ij} for any (i,j) and the calculation of either of p_j or q_i incur the same order of computational complexity. However, if (3) is to be solved by directly calculating the gradient $\frac{\partial}{\partial a_{ji}}\text{tr}(\mathcal{W}_A)$ of the objective function with respect to all the edges, the computation is to be done for $\left(n^2-n\right)$ edges (recall that we do not include self-loops) whereas for the proposed measure c_{ij}, the computation is to be done only for n nodes.

IV. Edge Modification Procedure

Building on the results of Section III, we next propose an edge modification procedure as a computationally efficient near-optimal approximation to (3). Without considering self-loops, the problem is to modify $N_{\max} \ll n^2 - n$ edges and compute their respective weights, giving a total of $2\left(n^2-n\right)$ ($\left(n^2-n\right)$ binary (edge selection) variables and $\left(n^2-n\right)$ continuous (edge weight)) variables. This quadratic growth with network size prohibits the direct solution of (3) for large networks, cf. Section III-D.

Our proposed relaxation is to restrict the possible edge choices to some selected search space N_S such that $N_{\max} <$

Algorithm 1 The Proposed Edge Modification Procedure

Input: $A, B, n, T, N_{\max}, w_{\max}, N_S, w_{ub}$

Output: η, w

1: For each node j and for each t with $1 \leq t \leq T-1$, compute $\mathcal{W}_j^{(t)}$ as in (5)

2: Form $\mathcal{H}^{(t)}$ as in (13)

3: Compute $p^{(t)}, q^{(t)}, \Theta^{(t)}$ and Θ as in (14)

4: Rank all the edges according to decreasing c_{ij} (Θ_{ji})

5: Select the first N_S edges and form the set $\mathcal{E}_S$

6: For each element in $\mathcal{E}_S$, construct the matrices in Δ_k

7: Solve the (small-scale) optimization problem (15)

$N_S < n^2 - n$, thus reducing the number of optimization variables to $2N_S$. This search space can be obtained by choosing the N_S edges with the largest values of c_{ij} (cf. Section III-B). Let $\mathcal{E}_S$ be the set of N_S selected edges. For each $1 \leq k \leq N_S$, if (i,j) is the corresponding element in $\mathcal{E}_S$, define $\Delta_k \in \mathbb{R}^{n \times n}$ such that $(\Delta_k)_{ij} = 1$ while other entries of Δ_k are zero. Then, problem (3) over this smaller search space is given by

$$\begin{aligned} \max_{\eta, w} \quad & \operatorname{tr}\left(\mathcal{W}_{A+\delta A}\left(T\right)\right), \qquad (15) \\ \text{s.t.} \quad & \delta A = \sum_{k=1}^{N_S} \eta_k w_k \Delta_k, \\ & \eta = \begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_k \dots & \eta_{N_S} \end{pmatrix}^\top \in \{0,1\}^{N_S}, \\ & w = \begin{pmatrix} w_1 & w_2 & \dots & w_k & \dots & w_{N_S} \end{pmatrix}^\top, \\ & 0 \leq w_k \leq w_{ub}, \sum_{k=1}^{N_S} \eta_k \leq N_{\max}, \sum_{k=1}^{N_S} \eta_k w_k \leq w_{\max}, \end{aligned}$$

where $\mathcal{W}_{A+\delta A}(T)$ is given in (4), η is the binary edge selection vector, and w is the edge weight vector. The pseudo-code for the resulting edge modification procedure is stated in Algorithm 1. The MINLP (15) is computationally tractable and can be solved using publicly available software such as the OPTI TOOLBOX [23] for MATLAB which uses BONMIN solver [24] for integer variables. The identification of criteria for optimally selecting N_S is an interesting open problem.

V. Numerical Examples

In this section, two examples are studied to show the efficacy of the proposed edge modification procedure.

Example 1: Consider a network with $n = 10$ nodes, adjacency matrix A whose entries are all zero except for

$$\begin{aligned} & a_{12} = 0.69,\ a_{18} = 0.36,\ a_{1,10} = 1.24,\ a_{23} = 0.20, \\ & a_{25} = 0.02,\ a_{26} = 0.87,\ a_{27} = 0.64,\ a_{37} = 0.37, \\ & a_{46} = 0.76,\ a_{51} = 0.66,\ a_{76} = 0.99,\ a_{84} = 0.50, \\ & a_{91} = o.52, \quad a_{10,9} = 0.74, \end{aligned}$$

The constraints on the optimal edge modification problem (15) are $T = n, N_{\max} = 3$, $w_{\max} = 1$, and $w_{ub} = 0.4$. Three approaches are then compared in seeking a (approximate) solution to (15), as follows.

1) *Approach 1:* Using Algorithm 1 with $N_S = 10$.
2) *Approach 2:* The value of c_{ij} is computed for all the edges (cf. Section III-D) and ranked in descending order. The three edges with the largest values of c_{ij} are then selected and their weights are augmented by $w_1 = 0.4$, $w_2 = 0.4$, and $w_3 = 0.2$, respectively.
3) *Approach 3:* The edges are selected by exhaustively searching the edge space and fixing the location and weight of one edge at a time. In other words, the maximum value of w_{ub} is added to all $n^2 - n$ non-self-loop edges in the network (one at a time), the one with the largest $\operatorname{tr}(\mathcal{W}_{A+\delta A}) - \operatorname{tr}(\mathcal{W}_A)$ is selected, and its weight is increased by w_{ub}. The process is then repeated (with the updated A) until all the $w_{\max}$ is distributed.

The results are tabulated in Table I.

Property		*Approach 1*	*Approach 2*	*Approach 3*
Edges	Edge 1	$1 \to 6$	$1 \to 6$	$2 \to 6$
	Edge 2	$1 \to 10$	$1 \to 10$	$7 \to 6$
	Edge 3	$1 \to 9$	$1 \to 9$	$2 \to 7$
Weights	w_1	0.2	0.4	0.4
	w_2	0.4	0.4	0.4
	w_3	0.4	0.2	0.2
Initial objective value		9.23	9.23	9.23
Final objective value		26.92	26.54	39.74
Percentage increase		199.7%	187.6%	330.7%
Initial $\lambda_{\min}(\mathcal{W}_A)$		0.0003	0.0003	0.0003
Final $\lambda_{\min}(\mathcal{W}_A)$		0.0004	0.0003	0.0001

TABLE I: Comparison of performance.

It is notable that *Approach 1* resulted in assigning the smallest weight to the edge with the largest edge centrality, signifying the importance of steps 5-7 in Algorithm 1. Table I also shows slight improvement in $\lambda_{\min}(\mathcal{W}_A)$ as a result of applying the proposed algorithm. Therefore, even though Algorithm 1 seeks to maximize $\operatorname{tr}(\mathcal{W}_A)$, it also maintains (and even improves) worst-case controllability. Finally, although the exhaustive search procedure (Approach 3) results in the maximum increase in controllability, the procedure is not scalable as n grows (in our simulations, it becomes infeasible for $n = 25$). The original and modified graphs (following Approach 1) are shown in Figure 3.

Example 2: In this example, the proposed edge modification procedure was implemented on 10^3 random Erdős-Rényi networks [25]. The considered networks have $n = 25$ nodes, 8 inputs and the probability of the existence of each edge is 0.2, independently of the other edges. The maximum number of edges to be modified is $N_{\max} = 3$, budget constraint on added weight is $w_{\max} = 1$, the maximum weight to be added to each edge is $w_{ub} = 0.4$, and Algorithm 1 is applied on each network with $N_S = 15, T = n$. For each network, we compute the percentage of increase in controllability and show the histogram of the resulting values in Figure 4. The mean percentage increase in the performance is about 3000%, showing the utility of the proposed edge centrality measure and edge modification algorithm.

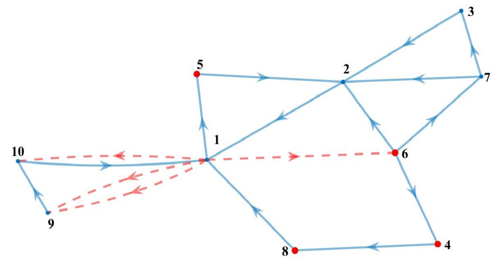

Fig. 3: The result of the proposed edge modification algorithm on the network of Example 1. Red sized nodes (4, 5, 6, 8) represent the actuator locations, while solid and dashed edges represent original and modified edges, respectively. The two (dashed-)edges directed from node 1 to node 9 indicate the strengthening of an existing edge.

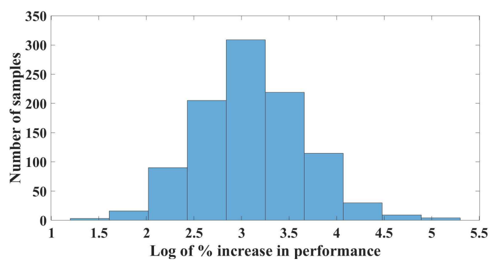

Fig. 4: The histogram of $\log_{10}$(%increase in tr($\mathcal{W}_A$)) for 10^3 Erdős-Rényi random networks following the proposed edge modification algorithm.

VI. Conclusions and Future Work

In this work, we studied the optimal edge modification problem where the location and weights of the edges in a network can be modified in order to improve network controllability. First, we introduced a novel energy-based edge centrality measure and established its relation with the gradient of the trace of the controllability Gramian (used as the measure of network controllability). Thus, we proposed to use the former as a (tight) proxy for the latter and showed its advantages in terms of computational complexity. This proxy was then used to design a sub-optimal but computationally efficient edge modification algorithm, the utility and efficiency of which were illustrated using numerical simulations. Future work will include the extension of the proposed edge centrality measure and its analytical properties to time-varying networks and networks with negative edge weights, the analytical characterization of the suboptimality gap of the proposed edge modification algorithm, its extension to ensure the resulting network structure retains stability properties, and its application to real-world networks.

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