Universal Formula for Smooth Safe Stabilization

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Abstract— **This paper formulates a safe and stabilizing control state feedback law for a control affine nonlinear system. We assume that there exist a known control Lyapunov function (CLF) and a control barrier function (CBF) that are compatible, i.e., there exists a control choice satisfying the conditions given by both the CLF and CBF at each given state. In contrast to the approach in the literature of finding a minimum-norm control using optimization on the feasible control set, we take a different approach by finding and combining different weighted centroids of the feasible control set. As a result, we can propose a control feedback law with guaranteed smoothness everywhere except at the origin, and with guaranteed continuity at the origin if small control property holds.**

I. Introduction

Recently, safety critical systems have gained attentions in control. In such systems, it is undesirable for the state trajectory to hit certain states at any point in time. This poses an extra difficulty in designing a controller because in addition to trying to stabilize the origin, one must make sure that the control signal will also not produce a trajectory that violate the state constraints. To address such a problem, motivated by the usage of CLF to certify stability, the concept of CBF is developed as a useful tool for guaranteeing safety. A CBF provides choices of control guaranteeing that the trajectory will not evolve to the unsafe states. Through the lens of this new concept, research in controls refocuses on the problem of not only stabilizing the system but also ensuring safety. Particularly, recent research topics, such as minimum-norm controller and control Lyapunov barrier function (CLBF), emerge with the ultimate aim of proposing a state feedback control law that satisfies both the safety and stability criteria given by the CLF and CBF.

Literature Review: This paper builds on three different bodies of literature. The first body of work is the literature on safe stabilization via the usage of CLF and CBF. Notably, there are two general approaches to how to exploit the available CLF and CBF. One way is to find the control through pointwise optimization over a feasible control set given by the CLF's and CBF's associated Lie derivative inequalities (see e.g., [1]–[3]). However, as pointed out in [4] with a counterexample, using this pointwise minimization method can result in a non-Lipschitz feedback controller. This can pose a problem for solution and uniqueness of the solution. To remedy this, [2], [3] sacrifice condition for stability for the guaranteed Lipschitzness of the controller. Another approach, as introduced by [5], is to combine a CLF and a CBF into a CLBF and then to use Sontag's universal formula for stabilization as its smooth control feedback law. However, there are problems such as the lack of guarantee that the newly constructed CLBF will satisfy the criterion required by the paper. The restrictiveness of CLBF is discussed further in [6]. Nevertheless, the smoothness of the controller is to be commended.

Next, we rely on the literature on smooth stabilization using a universal formula. The idea of smooth stabilization begins with [7], which shows that there exists a continuous control feedback for a control-affine system with a differentiable CLF. As briefly mentioned in [8], the idea can be extended to guarantee a smooth control feedback. However, the proof does not provide a construction of such feedback. This motivates [8] to use the analyticity of the root of a quadratic function with respect to its parameters to formulate the famous Sontag's universal formula for stabilization. As its name suggests, the formula does not take into account safety which would limit the feasible control option. It should be noted that there are extension to the universal formula in [9], [10], which consider constrained inputs of $u \in [0, \infty)$ and $u \in [0, 1]$. However, these constraints are static unlike constraints from CBF that change across each state. To the best of our knowledge, there is no universal formula for the latter case.

Last, our work fits in the literature of set-valued function selection. The feasible control set, given jointly by a CLF and a CBF, can be viewed as a set-valued function that changes across each state. Under the context of this paper, the idea behind selection is to pick a control from each feasible control set at each point to construct a single-valued function. One important selection theorem to note from [11], which is known as Michael's theorem, can be applied to our problem to show that a continuous feedback exists. However, the proof is not constructive. The work [12] and the book [13] use set-valued selection in controls context. In fact, in these works, it is shown that the minimum-norm controller are in fact continuous under lower semicontinuity of the control set. In addition, [13] discusses Lipschitz selection, but again does not give a constructive function. Another notable work is [14], which generalizes minimum-norm controller with a guide function, but once again, only continuity is guaranteed. Smooth selection for polytope set-valued function is explored in [15], [16], which suggests using the vertex of the polytope, but smoothness is only guaranteed almost everywhere, and if the result is applied in the context of controls, the control signal can be unnecessarily large.

Statement of Contributions: This paper considers the problem of safe stabilization for control affine systems. We first

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revisit the simpler problem of smooth stabilization under the same settings as what Sontag's universal formula stabilization, i.e., a smooth CLF is given. This is to lay a groundwork for when we later consider an additional requirement for safety from a smooth CBF. The goal of the paper is to construct a smooth feedback controller from the CLF and CBF. The contribution of this paper is threefold. The first contribution is the construction of an alternative formula for stabilization. The construction is based on the idea of finding a weighted centroid of the admissible control set, using the weights given by a probability density function of a normal distribution. With this formula, we are able to achieve exactly what Sontag's universal formula does. On top of that, we can extend the idea to when a CBF is also given. This brings us to the second and main contribution of this paper; we provide a formula for a feedback controller for safe stabilization, constructed from the given CLF and CBF. In doing so, we guarantee the smoothness of the controller. Also noteworthy, unlike a minimum-norm controller does, our proposed controller does not need to sacrifice the stability of the origin for the controller's Lipschitzness. The third contribution of this paper is the extension of Artstein's theorem for the existence of a smooth controller under the existence of a differentiable CLF. Particularly, we consider multiple control-affine inequalities in addition to the CLF. We provide an example to demonstrate our results. All proofs are omitted for reasons of space and will appear elsewhere.

II. Problem Statement

Consider[1] a nonlinear control-affine system of the form

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The system vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be smooth, and $f(0) = 0$ so that the origin is an equilibrium of the unforced system. The goal of this paper is to find a control feedback $k : \mathbb{R}^m \to \mathbb{R}^n$ such that $u = k(x)$ guarantees both the global asymptotic stability of the origin and the safety of the trajectories for the closed-loop system. We address the problem of asymptotic stability and safety with a control Lyapunov function and a control barrier function, resp., whose definitions we recall next.

Definition 2.1: (Control Lyapunov Function): Given a function $\delta : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a **δ-relaxed Control Lyapunov Function (δ-CLF)** for the system (1) if

(i) V is proper, i.e., $\{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is a compact set for all $c > 0$;
(ii) V is positive definite;
(iii) For each $x \in \mathbb{R}^n \setminus \{0\}$, $\exists u \in \mathbb{R}^m$ such that

$$L_f V(x) + L_g V(x)u < \delta(x). \qquad \diamond \tag{2}$$

The standard notion of CLF (cf. [13, Section 3.3.1]) corresponds to $\delta(x) = 0$ in this definition when applied specifically to the system (1), and we refer to it by 0-CLF. One can indeed find in the literature different variations of this notion: for example, the right-hand side of the inequality (2) may include a negative definite function [17] or the function V might only be continuous, in which case the monotonic property in (iii) is expressed in terms of directional derivatives [18]. Throughout the paper, we focus the definition above, even though our results are generalizable to other cases as well. Similarly, we define a CBF as follows.

Definition 2.2: (Control Barrier Function [2], [3]): Given a open set $\mathcal{D} \subset \mathbb{R}^n$, a function $h : \mathbb{R}^n \to \mathbb{R}$, continuously differentiable on $\mathbb{R}^n \setminus \mathcal{D}$ is a **control barrier function (CBF)** for the system (1) if

(i) $h(x) = 0$ for all $x \in \partial\mathcal{D}$;
(ii) $h(x) < 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{D}$;
(iii) For each $x \in \mathbb{R}^n \setminus \mathcal{D}$, $\exists u \in \mathbb{R}^m$ such that

$$L_f h(x) + L_g h(x)u \leq \beta(-h(x)) \tag{3}$$

where β is a Lipschitz class-$\mathcal{K}$ function. $\diamond$

The set $\mathcal{D}$ is referred to as the *unsafe* set. The purpose of a CBF is to guarantee that all trajectories with an initial condition outside of $\mathcal{D}$ will not enter it. For instance, when the right-hand side of inequality (3) is zero, then the condition guarantees that the value of the function h does not increase along the trajectory. Therefore, if the initial condition is outside $\mathcal{D}$, where the value of h is negative, then the function will remain negative, and the trajectory will not enter the set $\mathcal{D}$. The function β in the right-hand side allows for the value of h to actually increase on points in the interior of $\mathbb{R}^n \setminus \mathcal{D}$. As a result, the trajectory still avoids the set $\mathcal{D}$.

We assume for the rest of the paper that the system (1) admits a 0-CLF and a CBF. With these functions, one can deduce particularly that, if there exists a feedback $u = k(x)$ satisfying inequalities (2) and (3), global asymptotic stability of the origin and safety of the trajectories can be guaranteed. This motivates us to introduce the concept of compatibility.

Definition 2.3: (Compatibility): We refer to a collection of inequalities of the form $a(x)+b(x)u < 0$ or $a(x)+b(x)u \leq 0$ as **(strictly) compatible on** $\mathcal{X} \subset \mathbb{R}^n$ if, for each $x \in \mathcal{X}$, there exists a corresponding $u \in \mathbb{R}^m$ satisfying all inequalities (strictly). We call a δ-CLF V and a CBF h compatible if their inequalities (2) and (3) are compatible on $\mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$. $\diamond$

[1]Throughout the paper, we use the following notation. We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ the set of natural, real, and nonnegative real numbers, resp. We write $\mathcal{P}(\bar{\mathcal{S}})$, $\mathrm{int}(\mathcal{S})$, $\bar{S}_c$ and $\partial\mathcal{S}$ for the power set, the interior, the closure, and the boundary of the set $\mathcal{S}$, resp. For $n \in \mathbb{N}$, we use the notation $[n]$ to denote the set $\{1, \ldots, n\}$. Given $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. I_m and 0_m denote the identity matrix of size m and the m-dimensional zero vector, resp. For a function $\psi : \mathbb{R}^n \to \mathbb{R}$, we use $\mathrm{supp}(\psi)$ as the support of the function, i.e., the set $\{x \in \mathbb{R}^n \mid \psi(x) \neq 0\}$. We use the notation $\mathcal{U} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ to denote a set-valued map, which is equivalent to $\mathcal{U} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$, i.e., a map that assigns a subset in $\mathbb{R}^m$ to each point in $\mathbb{R}^n$. We say a function $k : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^l$ on an open set $\mathcal{X} \subseteq \mathbb{R}^n$ if all its derivatives up to l-th order are continuous on $\mathcal{X}$, and we loosely call the function smooth on $\mathcal{X}$. For functions $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$, we use $g \circ f : \mathcal{X} \to \mathcal{Z}$ to denote its composition, i.e., $g \circ f(x) = g(f(x))$. Also, we use $\times$ to denote the Cartesian product, and we extend its usage for pointwise Cartesian product for functions with same domain; for example, $f : \mathcal{X} \to \mathcal{Y}$ and $h : \mathcal{X} \to \mathcal{Z}$ has the Cartesian product $f \times h : \mathcal{X} \to \mathcal{Y} \times \mathcal{Z}$ where $(f \times h)(x) = f(x) \times h(x)$. We use the notation $L_f V$ to denote the Lie derivative of a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. A function $\beta : \mathbb{R} \to \mathbb{R}$ is of class-$\mathcal{K}$ if $\beta(0) = 0$ and β is strictly increasing. A function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(0) = 0$ and $V(x) > 0$ otherwise.

Given a 0-CLF V and a CBF h, we want to emphasize the importance of their compatibility. If they are not compatible, this means that there exists at least a state x where there is no control u that can satisfy inequalities (2) and (3) simultaneously, and either stability or safety needs to be sacrificed. In the literature, it is common to sacrifice stability by allowing $\delta(x) \neq 0$ to ensure compatibility. We come back to this point later in Section VI.

On top of compatibility, it is also important that the control as a feedback function is at least Lipschitz continuous, to guarantee the existence and uniqueness of solutions. This motivates the formulation of the main problem.

Problem 2.4: (Feedback Safe Stabilization): Assume we are given a 0-CLF $V : \mathbb{R}^n \to \mathbb{R}$ and a CBF $h : \mathbb{R}^n \to \mathbb{R}$ for the system (1) which are compatible. Find a smooth control feedback $k : \mathbb{R}^m \to \mathbb{R}^n$ such that $u = k(x)$ satisfies both inequalities (2) and (3) for all $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$.

In what follows, we show that the problem above has a solution when the 0-CLF and a CBF are strictly compatible, and provide a constructive formula for the feedback.

III. Existence of a Smooth Control Feedback

In this section, we examine the existence of the solution to Problem 2.4. The work [8] shows that it is possible to construct a smooth control feedback when stability is our only concern. Prior to providing its formula for universal stabilization, the work suggests that the existence of a smooth feedback controller can be derived through extending Artstein's Theorem for the existence of a continuous feedback controller (cf. [7]) by considering a partition of unity. Incidentally, using the same concept, we can extend Artstein's Theorem even further to when more than one control-affine inequalities are considered.

Proposition 3.1: (Extension of Artstein's Theorem): Consider a collection of n inequalities of the form $a(x)+b(x)u < 0$ or $a(x)+b(x)u \leq 0$ each defined on the domain $\mathcal{X}_i \subseteq \mathbb{R}^n$ with $a_i : \mathcal{X}_i \to \mathbb{R}$ and $b_i : \mathcal{X}_i \to \mathbb{R}^m$ continuous. If the inequalities are strictly compatible on $\cap_{i\in[n]}\mathcal{X}_i$, then there exists a $\mathcal{C}^\infty$ selection function $k : \cap_{i\in[n]}\mathcal{X}_i \to \mathbb{R}^m$ such that $u = k(x)$ satisfy all the inequalities for all $x \in \cap_{i\in[n]}\mathcal{X}_i$. □

In the context of safe stabilization, Proposition 3.1 provides a non-constructive statement suggesting that there exists a smooth control feedback for a given 0-CLF and any number of CBFs as long as they are all strictly compatible.

Corollary 3.2: (Existence of Smooth Safe Stabilizing Control Feedback): For the system (1) with continuous f and g, Let the system admit a 0-CLF and a CBF that are compatible. If $L_g h(x) = 0_m \implies L_f h(x) < \beta(-h(x))$, then there exists a control feedback $k : \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\}) \to \mathbb{R}^m$, $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$ such that $u = k(x)$ satisfies both (2) and (3) for each $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$, and hence, global asymptotically safely stabilizes the closed-loop system (1) when $\delta = 0$. □

This result shows that there is a solution to Problem 2.4 under the additional assumption that $L_g h(x) = 0_m \implies L_f h(x) < \beta(-h(x))$. Although we have not yet worked out a counterexample, we suspect there is one when the condition does not hold. Therefore, we require this condition to hold. Note that in the literature (e.g., minimum-norm controller [2], [3]), it is common to require that $L_g h(x) \neq 0$, a condition stricter than what we require. The result above does not give a constructive formula. Here, we follow [8] and shall construct a formula based on the available 0-CLF and CBF. As a result, the smoothness of the controller will rely heavily on the smoothness of the mentioned functions. As opposed to finding a $\mathcal{C}^\infty$ function given only a continuously differentiable 0-CLF and a continuously differentiable CBF, much like Sontag's universal formula in [8], we will require that the two functions are $\mathcal{C}^{l+1}$ for which we will give a $\mathcal{C}^l$ control feedback formula.

Sontag's universal formula provides, in addition to the smoothness of the controller, continuity at the origin under the small control property of the admissible control set. We shall do the same here for our formula, so we provide the following definition.

Definition 3.3: (Small Control Property): Given an admissible control set-valued function $\mathcal{U} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. **Small control property holds with** $\mathcal{U}$ if for every $\epsilon > 0$, there exists $\omega > 0$ so that there exists $\|u\| < \epsilon$ such that $u \in \mathcal{U}(x)$ for all $\|x\| < \omega$. ⋄

Note that the usual definition of small control property is tied directly to the control set associated with inequality (2) of a 0-CLF. However, since we deal in this paper not only with a 0-CLF, but also an additional condition from a CBF, we give above definition to suit our purpose.

IV. Alternative Universal Formula for Smooth Stabilization

In finding a constructive control feedback formula, one might want to begin by building on Sontag's universal formula for stabilization (cf. [8]) because it already handles smooth stabilization by satisfying inequality (2). We will briefly mention here the trouble in generalizing the formula to the case with multiple inequalities. Sontag's formula relies on the fact that the roots of a quadratic function behave analytically with respect to the function's parameters and the fact that function must be decreasing at one of the roots. To guarantee the satisfaction of inequality (2), a function quadratic in u is constructed so that the condition for the function decreasing at the root is precisely inequality (2). The root is the desired input, and can be computed using the quadratic formula. To develop a universal formula for the case with two inequalities (2) and (3) to satisfy, one would have to consider a 2-dimensional quadratic function, and find a formula to the root at which its function's derivative describes both inequalities. Unlike the one-dimensional case, finding such a root can be problematic and its analyticity with respect to the function's parameters is unclear. This motivates the alternative formula proposed here.

Assume $V : \mathbb{R}^n \to \mathbb{R}$ is a 0-CLF for the system (1). The inequality (2) limits the choice of admissible controls for stabilization. Because of the dependency on x, we write the admissible inputs as a set-valued function $\mathcal{U}_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$,

$$\mathcal{U}_1(x) = \{u \in \mathbb{R}^m \mid L_f V(x) + L_g V(x)u < 0\}.$$

Clearly, if we select $u = k(x) \in \mathcal{U}_1(x)$ for each $x \in \mathbb{R}^n \setminus \{0\}$, then (2) is satisfied. However, this is not enough to guarantee stabilization, because the continuity properties of the control might not be enough to guarantee the existence and uniqueness of solutions. To do so, we need a control feedback that is at least Lipschitz continuous. To define this, we rely on the function $\mu : \mathcal{P}(\mathbb{R}^m) \times \mathbb{R}_{\geq 0} \to \mathbb{R}^m$,

$$\mu(\mathcal{S}, \sigma) = \frac{\int_{\mathcal{S}} u \exp(-u^\top u/(2\sigma^2)) du}{\int_{\mathcal{S}} \exp(-u^\top u/(2\sigma^2)) du}. \tag{4}$$

The function can be interpreted as the mean of a set $\mathcal{S}$ with weights from a zero-mean, σ^2-variance Gaussian probability density distribution. We are now ready to propose our alternative universal formula for stabilization.

Proposition 4.1: (Alternative Universal Formula for Smooth Stabilization): Let $V : \mathbb{R}^n \to \mathbb{R}$ be a 0-CLF for the system (1) and $\sigma : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^l$ positive definite function. If $L_f V$ and $L_g V$ are $\mathcal{C}^l$, then $\mu \circ (\mathcal{U}_1 \times \sigma)$ is $\mathcal{C}^l$ on $\mathbb{R}^n \setminus \{0\}$. In addition, the feedback control $u(x) = \mu(\mathcal{U}_1(x), \sigma(x))$ satisfies (2) for each $x \in \mathbb{R}^n \setminus \{0\}$ and hence globally asymptotically stabilizes the origin. Furthermore, if the small control property holds with $\mathcal{U}_1$, the function k is also continuous at the origin. □

Proposition 4.1 provides an alternative design to Sontag's universal formula for exploiting the existence of a 0-CLF for global feedback stabilization. Furthermore, as we show in the forthcoming discussion, the construction behind the proposed formula can be extended to accommodate satisfaction of an additional inequality, particularly inequality (3) corresponding to a CBF.

V. Universal Formula for Smooth Safe Stabilization

In this section we build on the developments of Section IV to deal with an additional inequality from a CBF. Before we move on to give our formula, we first define here a useful auxiliary function. The following function $s : \mathbb{R} \to [0, 1]$ is $\mathcal{C}^\infty$, (cf. [19, Eq (3) in Lemma 1.10]),

$$s(t) = \begin{cases} 0, & t \leq 0 \\ \left(1 + \frac{e^{1/t}}{e^{1/(1-t)}}\right)^{-1}, & 0 < t < 1 \\ 1, & t \geq 1 \end{cases} \tag{5}$$

The function above is strictly increasing from 0 to 1 in the interval $(0, 1)$. Notably, the function is *flat* (derivatives with respect to t of all order are zeros) at $t = 0$ and $t = 1$. As a result, the function is particularly useful for smoothly transitioning from one function to another in a convex set. Also, we denote with a shorthand notation the following

$$\begin{aligned} \mu_{[1,2]}(x) &= \mu(\mathcal{U}_1(x) \cap \mathcal{U}_2(x), \sigma(x)), \\ \mu_1(x) &= \mu(\mathcal{U}_1(x), \sigma(x)), \\ \mu_2(x) &= \mu(\mathcal{U}_2(x), \sigma(x)). \end{aligned}$$

where $\mathcal{U}_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\mathcal{U}_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ are defined as

$$\begin{aligned} \mathcal{U}_1(x) &= \{u \in \mathbb{R}^m \mid \text{Ineq. (2) holds}\}, \\ \mathcal{U}_2(x) &= \{u \in \mathbb{R}^m \mid \text{Ineq. (3) holds}\}. \end{aligned}$$

With above, we are ready to give our main result.

Theorem 5.1: (Universal Formula for Smooth Safe Stabilization): Let $V : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a 0-CLF and a CBF that are compatible for the system (1). Also let $\sigma : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^l$ positive definite function. Define

$$k(x) = s(\rho(x))(\mu_1(x) + \mu_2(x)) + [1 - s(\rho(x))]\mu_{[1,2]}(x) \tag{6}$$

where $\rho(x) = \frac{L_g V(x) L_g h(x)^\top}{\|L_g V(x)\| \|L_g h(x)\|}$. If the following hold,

(i) $L_f V$, $L_g V$, $L_f h$, $L_g h$, and β are $\mathcal{C}^l$ on $\mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$;
(ii) $L_g h(x) = 0_m \implies L_f h(x) < \beta(-h(x))$,

then k is $\mathcal{C}^l$ on $\mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$. In addition, the feedback control $u = k(x)$ satisfies both (2) and (3) for each $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$ and hence, global asymptotically safely stabilizes the closed-loop system (1). Furthermore, if small control property holds with $\mathcal{U}_1 \cap \mathcal{U}_2$, the function k is also continuous at the origin. □

Theorem 5.1 gives a constructive formula for a strict compatible pair of 0-CLF and CBF. The control feedback given by (6) will achieve smooth safe stabilization. The main idea behind the construction of the controller is to exploit weighted centroid $\mu_{[1,2]}$, which is already smooth almost everywhere. Note this is done by examining the closed form solution (cf. [20], [21]). For the place where it is not smooth, we can conveniently transition the controller into $\mu_1 + \mu_2$ because both $\mu_{[1,2]}$ and $\mu_1 + \mu_2$ become either μ_1 or μ_2 there. Using the property of the weighted centroid, the proposed controller can satisfy both control inequality constraints.

VI. Discussion on δ and compatibility

In this section, we have a further discussion on compatibility and how we can obtain it through using a δ-CLF. Our results rely heavily on the compatibility of the given 0-CLF and CBF. Here, we consider the scenario where compatibility does not hold for the 0-CLF and the CBF with the property $L_g h(x) = 0 \implies L_f h < 0$, i.e., there exists an x where there exists no u satisfying both inequalities (2) and (3). Particularly, we can reason with contraposition that this scenario must occur when the unsafe set $\mathcal{D}$ is bounded because there can be no smooth feedback control as proven in [6]. In any case, the question remains: what can be done when compatibility does not hold.

To answer the question, we review the literature to see how the problem of incompatibility is dealt. We recall from the literature the minimum-norm controller (cf. [2], [3]). Given a 0-CLF and a CBF, a minimum-norm controller $k_{\min} : \mathbb{R}^m \to \mathbb{R}^n$ is computed through solving pointwise the following quadratic programming

$$\begin{aligned} \begin{bmatrix} k_{\min}(x) & \delta(x) \end{bmatrix}^\top = & \underset{\tilde{u} = [u^\top, w]^\top \in \mathbb{R}^{m+1}}{\operatorname{argmin}} \tilde{u}^\top \tilde{u} \\ & \text{s.t. } L_f V(x) + \begin{bmatrix} L_g V(x) & -1 \end{bmatrix} \tilde{u} < 0, \\ & \quad\;\; L_g h(x) + \begin{bmatrix} L_g h(x) & 0 \end{bmatrix} \tilde{u} \leq \beta(-h(x)). \end{aligned}$$

Notice that the first inequality constraint is no longer inequality (2) from the 0-CLF because of the relaxation input w. Instead, it represents the inequality (2) associated with a δ-CLF. Thus, stability of the origin is no longer guaranteed.

The reason behind this sacrifice is to guarantee the feasibility of the quadratic programming. By introducing a relaxation input w, there exists a $\tilde{u}$ satisfying the inequality constraints for each x because the coefficient of $\tilde{u}$ from the two inequalities are always linearly independent.

We can integrate the idea of introducing a relaxation input to obtain compatibility into our universal formula. First, we redefine $\mathcal{U}_1$ and $\mathcal{U}_2$ appropriately with the relaxation input

$$\begin{aligned}\mathcal{U}_1(x) &= \{\tilde{u} \mid L_f V(x) + \begin{bmatrix} L_g V(x) & -1 \end{bmatrix} \tilde{u} < 0\},\\ \mathcal{U}_2(x) &= \{\tilde{u} \mid L_f h(x) + \begin{bmatrix} L_g h(x) & 0 \end{bmatrix} \tilde{u} \le \beta(-h(x))\}.\end{aligned}$$

where $\tilde{u} = \begin{bmatrix} u & w \end{bmatrix}^\top \in \mathbb{R}^{m+1}$. With $\mu_{[1,2]}$, μ_1, and μ_2 redefined with the above set-valued functions, we find a smooth control feedback,

Proposition 6.1: (Exploiting δ for Compatibility): Let the system (1) $V : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ be a $0-CLF$ and a CBF that are not necessarily compatible. If the following assumptions hold,

(i) $L_f V$, $L_g V$, $L_f h$, $L_g h$, and β are $\mathcal{C}^l$;
(ii) $L_g h(x) = 0 \implies L_f h(x) < \beta(-h(x))$,

then with a $\mathcal{C}^l$ positive definite function $\sigma : \mathbb{R}^n \to \mathbb{R}$, and ρ redefined as

$$\rho(x) = \frac{\begin{bmatrix} L_g V(x) & -1 \end{bmatrix} \begin{bmatrix} L_g h(x) & 0 \end{bmatrix}^\top}{\| \begin{bmatrix} L_g V(x) & -1 \end{bmatrix} \| \| \begin{bmatrix} L_g h(x) & 0 \end{bmatrix} \|},$$

the function k defined as in (6) is $\mathcal{C}^l$ on $\mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$. In addition, the control feedback $u = \begin{bmatrix} I_m & 0_m \end{bmatrix} k(x)$ satisfies the inequality (3) for each $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$ and hence, guarantees the safety of the trajectories for the closed-loop system (1). Also, the control feedback $u = \begin{bmatrix} I_m & 0_m \end{bmatrix} k(x)$ satisfies the inequality (3) associated with the δ-CLF where the relaxation function given by $\delta(x) = \begin{bmatrix} 0_m^\top & 1 \end{bmatrix} k(x)$, for each $x \in \mathbb{R}^n \setminus (\mathcal{D} \cup \{0\})$. Furthermore, if small control property holds with $\mathcal{U}_1 \cap \mathcal{U}_2$, the function k is also continuous at the origin. □

Proposition 6.1 provides a feedback controller for when the system (1) admits a CLF and a CBF that are not compatible. The idea is to introduce a relaxation input w and then apply the universal formula. As a result, we can find that the new set of inequalities are compatible. The downside of introducing a relaxation input is that we no longer guarantee the stability of the origin. We satisfy instead the condition for a δ-CLF. However, because δ is a smooth function, it will be upper bounded on any compact domain. Regardless, it must be noted again that stability is no longer guaranteed. Nevertheless, this method eliminates the need to search for a compatible δ-CLF and has been proven useful in the literature.

VII. Numerical Example

In this section, we apply our results to an example. Consider a unicycle dynamics subjected to a drift with the following dynamics,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ -y \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

For the states, we write $z = \begin{bmatrix} x & y & \theta \end{bmatrix}^\top$. One can check that $V(z) = \frac{1}{2}\|z\|^2$ is a 0-CLF. Next, suppose unsafe states are given by the set $\mathcal{D} = \{z \in \mathbb{R}^3 \mid y > (2x+1)^2 + 1\}$. We use the following CBF candidate.

$$h(z) = y - (2x+1)^2 - 1$$

We find that $h(z) = 0$ on the boundary of the unsafe set and $h(z) < 0$ on $\mathbb{R}^n \setminus \mathcal{D}$. Next we check if the function is a CBF by checking if there exists a u satisfying (3) for each x. We pick the simplest class-$\mathcal{K}$ function for $\beta(-h(z)) = -kh(z)$ where k is a positive constant. Now we evaluate,

$$L_f h(z) + L_g h(z) u = -y - 4(2x+1)u\cos\theta + u\sin\theta.$$

For $k \ge 1$, we find that

$$\begin{aligned}-y &\le \begin{cases} -ky, & y < 0 \\ 0, & 0 \le y < (2x+1)^2 + 1 \end{cases}\\ &< \begin{cases} -ky + k((2x+1)^2+1), & y < 0 \\ -k(y - (2x+1)^2 - 1), & 0 \le y < (2x+1)^2 + 1 \end{cases}\\ &< -kh(z),\ \forall z \in \mathbb{R}^3 \setminus \mathcal{D}.\end{aligned}$$

Therefore, for $k \ge 1$, $u = 0$ satisfies (3) on $\mathbb{R}^3 \setminus \mathcal{D}$, and h is a CBF. We pick $k = 5$ for the simulation. Also note here that with $L_f h(z) < -kh(z)$, we can immediately satisfy the assumption $L_g h(z) = 0 \implies L_f h(z) < -kh(z)$.

Next, we examine the compatibility of the 0-CLF and CBF. First, it is clear that $u = 0$ satisfy both (2) and (3) for $y \ne 0,\ y \notin \mathcal{D}$. Then for $y = 0$, we can find that

$$\begin{aligned}L_g V(z) &= \begin{bmatrix} x\cos\theta & \theta \end{bmatrix}\\ L_g h(z) &= \begin{bmatrix} -(8x+4)\cos\theta + \sin\theta & 0 \end{bmatrix}.\end{aligned}$$

We only need to consider when these two vectors are linear dependent because otherwise there always exists a control that can satisfy both (2) and (3). As such, we consider when $\theta = 0$. For compatibility, we need a u satisfying

$$xu < 0,\ -(8x+4)u \le 0$$

In other words, we want $0 < (7x - 4)u$. Clearly, a u with the same sign as $7x - 4$ exists, and we can pick it arbitrarily small, so that small control property holds.

With initial conditions of $\begin{bmatrix} -1 & 2 & \pi \end{bmatrix}^\top$ and $\begin{bmatrix} 0 & 2 & \pi \end{bmatrix}^\top$, we simulate our proposed controller given by (6) with $\sigma(z) = 1 - \exp(z^\top z)$. The resulting trajectory is shown in Figure 1 in a thin solid curve. In comparison to Sontag's universal formula, given in dashed curve, our proposed controller results in a trajectory that avoids the unsafe states as predicted. In addition, the minimum-norm controller discussed Section VI in is plotted in a dotted line for comparison. For this controller, a relaxation input is introduced to guarantee Lipschitzness of the controller. Also, we add a negative definite function, $-0.1z^\top z$, on the right hand side of the CLF inequality (2) to "force" control effort; otherwise, the minimum-norm controller will be identically zero. For its plot, although the trajectory appears to converge towards the origin, there is no real guarantee that it will do so. This is not to mention that the controller is not differentiable at some point along the trajectory, which can be an undesirable property. In contrast, the control signals of our proposed formula are plotted in Figure 2. As guaranteed by Theorem 5.1, the signals are smooth and go to zero

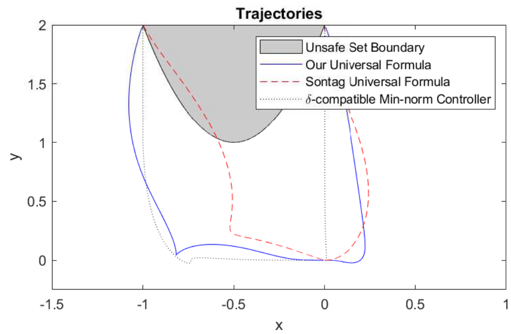

Fig. 1. Trajectories for different types of controllers with two different initial conditions. Using Sontag's universal feedback formula results in a trajectory that violates the state constraint because it does not take safety into account. Both the minimum-norm and our proposed controller produce safe trajectories that progress towards the origin (however, there is no guarantee that the minimum-norm controller will reach it).

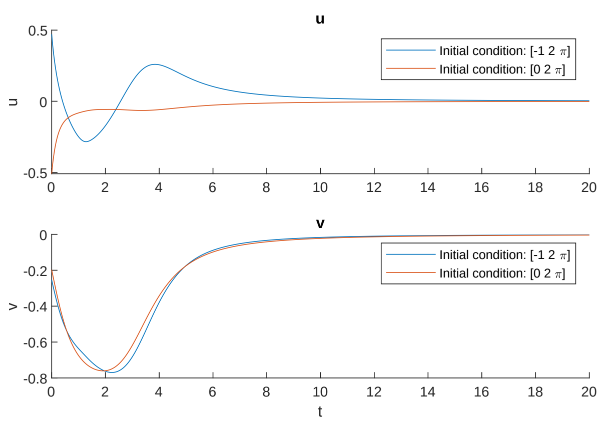

Fig. 2. The control inputs along trajectory using our control feedback formula (6) with the two different initial conditions. Both control inputs appear smooth in simulation as predicted. In addition, because the small control property holds, the control both converges to zero as the state converges towards the origin.

VIII. Conclusions

We have formulated a feedback controller for safe stabilization. Given a CLF and a CBF, the formula considers the associated admissible control set, and calculates the weighted centroids with normal distribution weights of different sets. By combining the centroids in a smooth way, the given feedback controller retains the smoothness property of the CLF-CBF pair. Also, by manipulating the "standard deviation," the controller can be found continuous at the origin when the small control property holds. In future work, we plan to work on improving our results in two different ways. First, we wish to extend the formula to the case with multiple CBFs (which should exist as predicted by Proposition 3.1). Second, we plan to refine the concept of compatibility to a system level as opposed to defining it as a pointwise property. In particular, we notice that even if compatibility does not hold for a state, there may still exist a control that drives the trajectory to where compatibility holds.

References

[1] B. Morris, M. J. Powell, and A. D. Ames, "Sufficient conditions for the Lipschitz continuity of QP-based multi-objective control of humanoid robots," in *IEEE Conf. on Decision and Control*, (Florence, Italy), pp. 2920–2926, Dec 2013.
[2] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.
[3] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, "Robustness of control barrier functions for safety critical control," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 54 – 61, 2015.
[4] B. J. Morris, M. J. Powell, and A. D. Ames, "Continuity and smoothness properties of nonlinear optimization-based feedback controllers," in *IEEE Conf. on Decision and Control*, (Osaka, Japan), pp. 151–158, Dec 2015.
[5] M. Z. Romdlony and B. Jayawardhana, "Stabilization with guaranteed safety using control Lyapunov-barrier function," *Automatica*, vol. 66, pp. 39 – 47, 2016.
[6] P. Braun and C. M. Kellett, "On (the existence of) control Lyapunov barrier functions," 2017.
[7] Z. Artstein, "Stabilization with relaxed controls," *Nonlinear Analysis*, vol. 7, pp. 1163–1173, 1983.
[8] E. D. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization," *Systems & Control Letters*, vol. 13, no. 2, pp. 117 – 123, 1989.
[9] Y. Lin and E. D. Sontag, "A universal formula for stabilization with bounded controls," *Systems & Control Letters*, vol. 16, no. 6, pp. 393 – 397, 1991.
[10] Y. Lin and E. D. Sontag, "Control-Lyapunov universal formulas for restricted inputs," *Control-Theory and Advanced Technology*, vol. 10, pp. 1981–2004, 1995.
[11] E. Michael, "Continuous selections. i," *Annals of Mathematics*, vol. 63, no. 2, pp. 361–382, 1956.
[12] R. A. Freeman and P. V. Kokotovic, "Inverse optimality in robust stabilization," *SIAM Journal on Control and Optimization*, vol. 34, no. 4, pp. 1365–1391, 1996.
[13] R. A. Freeman and P. V. Kototovic, *Robust Nonlinear Control Design: State-space and Lyapunov Techniques*. Cambridge, MA, USA: Birkhauser Boston Inc., 1996.
[14] H. Yuqing and H. Jianda, "Generalized point wise min-norm control based on control Lyapunov functions," in *2007 Chinese Control Conference*, pp. 404–408, July 2006.
[15] D. T. Luc and P. H. Dien, "Differentiable selection of optimal solutions in parametric linear programming," *Proceedings of the American Mathematical Society*, vol. 125, no. 3, pp. 883–892, 1997.
[16] D. T. Luc, "Smooth representation of a parametric polyhedral convex set with application to sensitivity in optimization," *Proceedings of the American Mathematical Society*, vol. 125, no. 2, pp. 555–567, 1997.
[17] E. D. Sontag, "Control-Lyapunov functions," in *Open Problems in Mathematical Systems and Control Theory* (V. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems, eds.), pp. 211–216, Springer London, 1999.
[18] E. D. Sontag and H. J. Sussmann, "General classes of control-Lyapunov functions," in *Stability Theory* (R. Jeltsch and M. Mansour, eds.), pp. 87–96, Birkhäuser Basel, 1996.
[19] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*. No. 94 in Graduate Texts in Mathematics, New York: Springer, 2 ed., 1989.
[20] G. M. Tallis, "The moment generating function of the truncated multi-normal distribution," *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 23, no. 1, pp. 223–229, 1961.
[21] G. M. Tallis, "Plane truncation in normal populations," *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 27, no. 2, pp. 301–307, 1965.