# Edge Centrality Matrix: Impact of Network Modification on Gramian Controllability Metrics

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*Abstract*— **Due to recent technological advances, performance enhancement of complex networked control systems by edge modification done according to their importance in the network is becoming increasingly feasible. Unlike the nodal case, edge characterization with respect to a given performance metric is a rather unexplored research area. In this work, we seek to address this problem by proposing a novel Gramian-based edge centrality matrix which characterizes all the possible edges in the network with respect to physically realizable energy-based performance metrics. We rigorously prove the relationship of the various edge centrality matrix for different performance metrics with the gradient of the controllability Gramian with respect to edge weights. Notable feature of our proposed edge characterization is that it exhibits the contribution of individual inputs. We then analyze the edge centrality matrix for directed ring and line networks. Finally, through numerical examples, we validate a structural property of proposed edge centrality matrix and demonstrate its utility in network edge modification.**

## I. Introduction

Networked control systems (network) are modern complex dynamical systems which are conglomeration of small subsystems often geographically separated by large distances and communicating over wireless channels. Complex networks find application in numerous domains, including social networks, biological systems, robotic systems, power networks, and transportation networks, and have become an important part of day-to-day human life. Hence, it is necessary that the network functions efficiently and resiliently in the event of external interference (disturbances, malicious attacks). As the destabilizing interference (energy) is applied at nodes and propagated in the network through edges, we have to understand how each node and edge influences the network. Influence of nodes and edges in the network based on topological properties has been quantified by various centrality measures. However, these topology based centrality measures neglect the dynamical (controllability, energy) effects of nodes and edges in the system. While energy based nodal centrality measures and its use for improving network performance are well developed in the literature, we cannot say the same for edges. In this work we seek to address this research gap by introducing analytically grounded energy based edge centrality measures and explore its various properties and uses.

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*Literature review:* In a complex network, the notion of centrality quantifies the relative importance of components (nodes/edges) in a network with respect to an appropriate performance/influence metric [1], [2]. Different performance metrics lead to different centrality measures [3]. These centrality measures are then used to characterize and order the network nodes/edges. Node based centrality measures are most commonly used for network characterization [2] due to its computational feasibility. This is because unlike in the case of edges, the network node set is relatively smaller in size. Based on the topological properties of the network some commonly used nodal centrality measures include degree [2], [4], closeness [2], betweeness [5], eigenvector [6], Katz [7], PageRank (Google) [8], percolation [9], cross-clique [10], Freeman [4], topological [11], Markov [12], hub and authority [13], routing [14], subgraph [15], and total communicability [16] centralities. In case of edges, various notions such as betweeness centrality [17], edge HITS centrality, and edge total communicability centrality [18] have been proposed. All the discussed centrality measures do not consider network dynamics and hence fail to capture the influence of energy propagation in the system.

Energy propagation in the system is related to the controllability of the dynamical system. While classical Kalman controllability only indicates the mere possibility of steering a system arbitrarily in the state space, it fails to quantify the energy required for steering [19], [20]. Works such as [21]–[23] establish a relationship between the abstract notion of Kalman controllability and physically realizable energy-based metrics through the controllability Gramian [20]. Different functions of the controllability Gramian produce different energy-based metrics. These Gramian-based metrics are used to order the nodes in network and then this ordered set is used to select control nodes by heuristic or deterministic process. For example, one can use trace of the controllability Gramian [23], [24], the trace of its inverse [23], its determinant [21], [25] or its minimum eigenvalue [22] as an performance metric. In [26], a novel notion of node centrality called the *2k-communicability* is used for time-varying placement of actuators in network. In contrast to the above stated works, [27] proposes a novel notion of Gramian-based edge centrality which explores the importance of edges in network from the perspective of energy propagation. But the proposed edge centrality [27] can only be applied to directed networks with non-negative edge weights. Also, the edge centrality measure in [27] is independent of the input locations and is not tightly bounded. This may sometimes lead to inaccurate ordering of edges.

*Statement of contributions:* In the aforementioned literature, the majority of works explore node/edge centrality measures only from the network topological perspective and neglect the dynamical effects. The works which consider the effect of system dynamics are either limited to the nodal case or limited to directed network with non-negative weights or independent of the input locations. In view of this, the contributions of this work are threefold: (i) We propose a novel Gramian-based edge centrality matrix which is a map of edge centrality measures for existing as well as virtual edges between any two nodes of the network. The edge centrality measures are computed with respect to a given performance metric and is derived for network with discrete-time dynamics. Notable features of our proposed edge centrality matrix are its independence from the underlying topological properties (like sign of edge weights, symmetry etc.) of the network and its additive nature in the input space. This helps us to quantify the effect of individual inputs of network with respect to a given performance metric; (ii) We derive edge centrality matrix for different performance metrics and rigorously prove their relationship with the gradient of the Gramian with respect to the edge weights. We also discuss various interpretations of the edge centrality matrix with physically realizable energy-based system properties; (iii) We analyze the edge centrality matrix for special type of networks namely the directed line network and directed ring network. We rigorously prove that perturbing *'only'* existing edges can cause first-order changes in different performance metrics. Finally, through a numerical examples we verify and demonstrate the use of edge centrality matrix for enhancing performance of network by edge modification. For space reasons, all proofs are omitted and will appear elsewhere.

*Notation:* We use $\mathbb{R}$ to denote the set of reals. For $j \in \{1, \dots, n\}$, $e_j \in \mathbb{R}^n$ denotes the $j^{\text{th}}$ canonical unit vector. $(\cdot)^\top$ represents the transpose of a vector or matrix. For a vector $x$, we use $\|x\|$ to denote its norm. For a matrix $X$, we denote the element at the $(i,j)^{th}$ place as $X(i,j)$. For the same matrix $X$, we use $X = (x_{ij})$ to denote $X(i,j) = x_{ij}$. For a matrix $\mathcal{W}$, we use $\operatorname{tr}(\mathcal{W})$ for its trace. For a symmetric matrix $X$, we use $\lambda_{\min}(X)$ to denote its smallest eigenvalue. We use $I$ to represent the identity matrix of appropriate dimension. Let $\mathcal{X}$ be an edge set then we denote $(i,j)$ as its element such that it is an edge directed from node $i$ to node $j$ i.e., $i \longrightarrow j$. If we have $A$ as the weighted adjacency matrix of a network graph then it is undirected if $A = A^\top$.

## II. Problem Motivation

Consider a network of $n$ nodes represented by the triplet $\mathcal{G}_A = (\mathcal{V}, \mathcal{E}_A, w_A)$, where $\mathcal{V} = \{1, 2, \dots, n\}$ is the node set, $\mathcal{E}_A = \{\, (i,j) \mid i \in \mathcal{V}, j \in \mathcal{V} \,\}$ is the edge set, and $w_A : \mathcal{E}_A \mapsto \mathbb{R}$ is a weight function. The network dynamics is described by the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \{0, \dots, T-1\}, \quad (1)$$

where $T > 0$ is a finite time horizon, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are state and input vectors respectively. Here, $A = (a_{ji}) \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix defined by $a_{ji} = w_A[(i,j)] \neq 0$ if the edge $(i,j) \in \mathcal{E}_A$ else $a_{ji} = 0$. The matrix $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_m \end{pmatrix} \in \mathbb{R}^{n \times m}$ is the input location matrix. Here $b_i \in \{0,1\}^n$ are binary vectors with 0's everywhere except at one entry, signifying the presence of an input at that node. We assume that the input structure is known and the system $(A, B)$ is controllable for $T = n$. The input $u$ can be a known control input or a unknown disturbance/malicious input.

Controllability is the property that describes the effect on the network state that is achievable through the use of inputs. Formally, the controllability of (1) is the ability to steer the state from an initial condition $x(0) = x_0$ to any arbitrary final condition $x(T) = x_T$ in $T-$steps by appropriately selecting the control input sequence $\{u(0), u(1), \dots u(T-1)\}$. The controllability of (1) can be assessed in a number of ways: here, we employ the controllability Gramian,

$$\mathcal{W}(T) = \sum_{t=0}^{T-1} A^t B B^\top A^{t\top}. \quad (2)$$

The system with matrices $(A, B)$ is controllable if the matrix $\mathcal{W}(T)$ is symmetric positive definite [20]. This is a qualitative test that does not capture the input effort required to actually steer the system state. To address this, one can instead employ controllability metrics based on the Gramian, including $\operatorname{tr}(\mathcal{W})$, $-\operatorname{tr}(\mathcal{W}^{-1})$, $\det(\mathcal{W})$, $\log\det(\mathcal{W})$, and $\lambda_{\min}(\mathcal{W})$.

The aforementioned controllability metrics also have physically realizable interpretations. The energy to control a dynamical system as well as its output response are physically realizable quantities and are of great practical importance. The average energy required to take a dynamical system from $0-$state to a desired final state $x_f$ over infinite time is $x_f^\top \mathcal{W}^{-1} x_f$ [22]. For $x_i = \frac{x_f}{\|x_f\|}$, $x_i^\top \mathcal{W}^{-1} x_i = \frac{1}{\lambda_i}$ is called the *eigen-energy* [28] i.e., the minimum energy required to move the system in the direction $x_i$. If $(\lambda_i, x_i)$ are smallest eigenvalue-eigenvector pair, then $\frac{1}{\lambda_i}$ represents the energy required to steer the system in the most difficult direction $x_i$ [23]. Now if $x_i = e_i$, then $x_i^\top \mathcal{W}^{-1} x_i = \left[\mathcal{W}^{-1}\right]_{ii}$ is called the *nodal-energy* of the $i^{\text{th}}-$node. The $i^{\text{th}}-$nodal energy is defined as the energy required to drive the state of node $i$ from 0 to 1 while leaving the final states of the other nodes to the $0-$state [29]. If the output $y = e_i^\top x = x_i$, then $e_i^\top W e_i$ is the square of the $\mathcal{H}_2$ norm of the system. The square of the $\mathcal{H}_2$ norm is the energy in the output response of the system to an unit impulse input or it is the expected root mean square value of the output response to a white noise excitation input [29], [30]. Thus if the nodal energy is high then the nodal state is more robust against input disturbances. We refer the interested reader to [22], [23], [26], [29] for a more detailed discussion.

Our goal is to study the effect of changes in the network structure on its controllability properties (while maintaining intact the input structure). This analysis is motivated by two complementary types of scenarios, one where we might be interested in making the network more easily controllable, to facilitate the action of a defender, and another one where we seek to make the network more difficult to control, to obstruct the action of an attacker. It is also possible that both scenarios occur concurrently, where the input nodes are a combination of known control (defender) input nodes and malicious (attacker) input nodes, denoted by matrices $B_d$ and $B_a$, respectively. This has application in tackling practical problems such as mitigating effect of malicious attacks at input nodes or network edges, suppressing output response at particular nodes caused due to malicious inputs etc. In our study, we measure controllability by means of the Gramian-based performance metrics described above. By changes in network structure, we mean modifying the weights in existing edges or adding new edges of suitable weight.

## III. First-Order Dependence of Gramian-Based Controllability Metrics on Edge Weights

To analyze the effect of perturbation of network edge weights on the performance metrics, we first need to reformulate the gradient of Gramian in an appropriate form. In Theorem 3.1 we rigorously establish a novel expression for the gradient Gramian with respect to the elements of matrix $A$.

*Theorem 3.1: (Reformulated gradient of Gramian with respect to edge weights).* Consider $A \in \mathbb{R}^{n \times n}$, constant matrix $P = P^\top \in \mathbb{R}^{n \times n}$ and $H = \left(h_1 \dots h_k \dots h_m\right) \in \mathbb{R}^{n \times m}$. If $\phi(A) = \sum_{t=0}^{T-1} A^t H H^\top A^{t^\top}$ then

$$\frac{\partial}{\partial a_{ji}} \operatorname{tr}\left(P \phi(A)\right) = 2 \sum_{k=1}^{m} \sum_{t=1}^{T-1} \operatorname{tr}\left(\overline{C}_k^{(t)} \overline{O}_k^{(t)} e_i e_j^\top\right), \qquad (3)$$

where

$$\overline{C}_k^{(t)} = \left(A^{t-1^\top} P A^t h_k \quad \cdots \quad A^\top P A^t h_k \quad P A^t h_k\right),$$

$$\overline{O}_k^{(t)} = \left(h_k \quad A h_k \quad \cdots \quad A^{t-2} h_k \quad A^{t-1} h_k\right)^\top. \qquad (4)$$

Next, in Corollary 3.2, we derive expressions for the gradient of different functions of the controllability Gramian using Theorem 3.1.

*Corollary 3.2: (Gradient of functions of Gramian).* Consider the network dynamics (1) with $BB^\top = \sum_{k=1}^{m} b_k b_k^\top$ and the controllability Gramian defined in (2). For $P = P^\top \in \mathbb{R}^{n \times n}$, let

$$\Theta_P = \sum_{k=1}^{m} \sum_{t=1}^{T-1} \overline{C}_k^{(t)} \overline{O}_k^{(t)} \qquad (5)$$

with $\overline{C}_k^{(t)}$, $\overline{O}_k^{(t)}$ as defined in Theorem 3.1 with $H = B$. Then for at some $A = A_0$ the following are true:

1) $\frac{\partial}{\partial a_{ji}} \operatorname{tr}(\mathcal{W}) = 2\Theta_I(j,i)$ for $P = I$.
2) $\frac{\partial}{\partial a_{ji}} \log\det(\mathcal{W}) = 2\Theta_{W^{-1}}(j,i)$ for $P = \mathcal{W}^{-1}$.
3) $\frac{\partial}{\partial a_{ji}} \{-\operatorname{tr}\left(\mathcal{W}^{-1}\right)\} = 2\Theta_{W^{-2}}(j,i)$ for $P = \mathcal{W}^{-2}$.
4) $\frac{\partial}{\partial a_{ji}} \lambda_{\min}(\mathcal{W}) = 2\Theta_{V_{\min}}(j,i)$ for $P = V_{\min}$. $V_{\min} = v_{\min} v_{\min}^\top$ where $v_{\min}$ is the eigenvector of $\mathcal{W}$ corresponding to the eigenvalue $\lambda_{\min}(\mathcal{W})$.

Corollary 3.2 shows us that with appropriate choice of matrix $P = P^\top$, the gradient of various performance can be expressed in the format (5).

## IV. Edge Centrality and Its Properties

Here, we build on the results of Section III to propose a novel notion of edge centrality. We discuss some of its properties and then characterize its structure for directed ring and line networks.

### A. The Edge Centrality Matrix

The performance metrics are the functions of controllability Gramian. In Section III, we derived expressions for the first-order change in different performances metrics with respect to edge weight perturbation. The quantity $\Theta_P$ in (5) is a matrix whose $(j,i)^{\text{th}}$ element represents the change in a given performance metric when the weight of the edge $i \longrightarrow j$ is perturbed. We call $\Theta_P(j,i)$ as the centrality of edge $i \longrightarrow j$ in the network and matrix $\Theta_P$ as the *edge centrality matrix* with respect to a given performance metric. From (5), $\Theta_P$ is additive in the input space as $\Theta_P = \sum_{k=1}^{m} \Theta_P^k$, where $\Theta_P^k = \sum_{t=1}^{T-1} \overline{C}_k^{(t)} \overline{O}_k^{(t)}$. We call $\Theta_P^k$ as the $k^{\text{th}}-$input edge centrality matrix for a given performance metric. So, using Corollary 3.2, we can construct edge centrality matrices for the complete input set ($\Theta_P$) as well as individual inputs ($\Theta_P^k$) for different performance metrics, cf. Section II. Given the energy interpretations associated to these notions, the edge centrality matrices encode first-order changes in physically realizable quantities whenever we perturb the network.

To compute the gradients of the performance metrics directly, one would need to compute $\frac{\partial \mathcal{W}}{\partial a_{ji}}$. Now, if the discrete-time Lyapunov equation, cf. [20],

$$A \mathcal{W} A^\top - \mathcal{W} + BB^\top = 0$$

is used then $\frac{\partial \mathcal{W}}{\partial a_{ji}}$ is the solution to

$$A \frac{\partial \mathcal{W}}{\partial a_{ji}} A^\top - \frac{\partial \mathcal{W}}{\partial a_{ji}} + \frac{\partial A}{\partial a_{ji}} \mathcal{W} A^\top + A \mathcal{W} \left(\frac{\partial A}{\partial a_{ji}}\right)^\top = 0.$$

For a $n-$node network, to construct the edge centrality matrix for any performance metric, the discrete-time Lyapunov equation needs to be solved $n^2$ times. Moreover, the Lyapunov equation has a valid solution only if the matrix $A$ is stable [20]. In contrast, Corollary 3.2 computes complete edge centrality matrices for various performance metrics using simple matrix multiplications. Thus, Corollary 3.2 offers us a computationally efficient way to determine the importance of each edge in the network with respect to a given Gramian-based performance metric.

*Remark 4.1: (Significance of proposed edge centrality matrices).* Our proposed edge centrality matrices encode the first-order effect of edge perturbation on a given performance metric. A notable fact regarding our proposed edge centrality matrices is that they offer the system designer additional flexibility to examine the effects of edge perturbation due to *individual inputs*, or *a subset of inputs*, or *the complete set of inputs*. This may lead to better and efficient network modifications, resulting in energy and resources savings. Also, we can compute these edge centrality matrices for different performance metrics in a computationally cheap manner, which is an additional advantage. •

### B. Directed Line and Ring Networks

In general, given a network and a set of actuator nodes, it is difficult to characterize the structure of the edge centrality matrices a priori. Here, we show that this can however be achieved for particular classes of directed networks, specifically, line and ring networks. Formally, given $n$ nodes, the weighted adjacency matrix for the directed line network is

$$a_{i,i-1} \neq 0 \text{ for } i = 2, 3, \ldots, n; \tag{6a}$$

and $a_{ij} = 0$ otherwise, and the weighted adjacency matrix for the directed ring network is

$$a_{i,i-1} \neq 0 \text{ for } i = 2, 3, \ldots, n;\ a_{1n} \neq 0; \tag{6b}$$

and $a_{ij} = 0$ otherwise.

It should be noted that the directed line network is controllable with a single input at node 1. If multiple inputs are used for actuation then the directed line network is controllable only if node 1 is one of the input locations. However the directed ring network is controllable with a single or multiple inputs at any node. The feature of the directed line/ring networks is that they admit a diagonal controllability Gramian. In [29], the expression of the Gramian is given for the case $B = e_1$. In Proposition 4.2 next, we give expression of Gramian for single or multiple inputs applied at any arbitrary nodes.

*Proposition 4.2: (Controllability Gramian of directed line and ring networks).* Consider directed line/ring networks without self-loops as in (6) with any arbitrary input $b = e_k$ and Gramian $\mathcal{W}_k$. Then for a given time horizon $T$, $\mathcal{W}_k$ is a diagonal matrix.

It should be noted that the Gramian is diagonal but positive semi-definite matrix for $T < n$. Also for the case $T > n$, we have given only the logical pathway of the proof as the expressions for $v_t$ are more complicated and irrelevant in our current discourse.

Observe that from Proposition 4.2, for directed line networks the controllability Gramian is positive definite if and only if node$-1$ is one of the input nodes else it is positive semi-definite. Thus, directed line networks are controllable if and only if $e_1$ is one of the inputs. For ring networks, the controllability Gramian is always positive definite for any number of inputs applied at any arbitrary nodes.

We rely on Proposition 4.2 to show next that, for directed line and ring networks, the structure of the edge centrality matrices $\Theta_P$ and $\Theta_P^k$ is same as that of the corresponding adjacency matrix. By this we meant that, if $A(j,i) = 0$ then $\Theta_P(j,i) = 0$ and if $A(j,i) \neq 0$ then $\Theta_P(j,i)$ may or may not be 0 (i.e., $\Theta_P$ may have non-zero entries only at places where the matrix $A$ has non-zero entries).

*Theorem 4.3: (Structural property of edge centrality matrices for directed line and ring networks).* Consider a controllable $n-$node directed line/ring network $A$ with dynamics (1) and a single input $b_k = e_k$ or multiple inputs. If $\Theta_P$ is as defined in (5) and $P$ is a constant diagonal matrix, then $\Theta_P$, $\Theta_P^k$ and $A$ have the same structure.

It should be noted that in Theorem 4.3, for the directed line network case if we use single input then it should be placed at $e_1$ and if we use multiple inputs then $e_1$ should be one of the inputs. From Theorem 4.3, it is clear that in the case of directed line/ring networks, only the existing edges contribute to the various edge centrality matrices. Thus adding a new edge to directed line/ring networks will not cause any first-order change in the performance metrics, and can therefore be ruled out as a significant way of impacting the Gramian-based controllability metrics described in Section II.

## V. Numerical Examples

In this section, through numerical examples we show how the derived edge centrality matrices can be used for network modification. In the first example we consider line and ring networks which have a special structure and demonstrate validity of Theorem 4.3. In the second example, we consider a general $10-$node network and exhibit usage of edge centrality matrices for network modification for two different Gramian-based performance metrics.

### A. Analysis of Line and Ring Networks

Consider a $5-$node line/ring network with its parameters as in Figure 1. The network is a directed line network if $a_{15} = 0$ and a directed ring network if $a_{15} = 0.5$. In both cases, we take $T = n$ and illustrate the result obtained in Theorem 4.3.

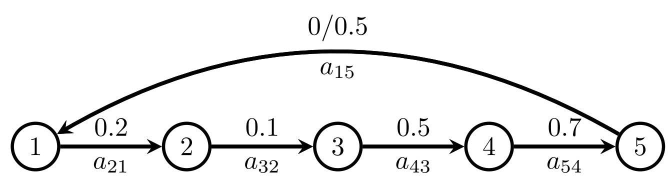


**Fig. 1:** Example of a $5-$node directed line/ring network. For line network $a_{15} = 0$ and for ring network $a_{15} = 0.5$.

*1) Directed line network:* The system has only one input at node 1 i.e., $b = e_1$. The Gramian for the system is $\mathcal{W} = \text{diagonal}\{1, 0.04, 0.0004, 0.0001, 0.00005\}$. As per the Corollary 3.2, different edge centrality matrices for different performance metrics are stated in Table I. The eigenvector corresponding to the smallest eigenvalue is $v_{\min} = e_5$.

| Mt \ Edge | $a_{21}$ | $a_{32}$ | $a_{43}$ | $a_{54}$ |
|---|---|---|---|---|
| $\text{tr}(\mathcal{W})$ | 0.4055 | 0.0011 | 0.0006 | 0.0001 |
| $\log\det(\mathcal{W})$ | 40 | 60 | 8 | 2.9 |
| $-\text{tr}(\mathcal{W}^{-1})$ | 329330 | 658160 | 121630 | 58310 |
| $\lambda_{\min}(\mathcal{W})$ | 0.0005 | 0.00098 | 0.000196 | 0.00014 |

**TABLE I:** Different edge centrality matrices for directed line network and input $b = e_1$. Mt = Metric.

From Table I, we see that the change in $\text{tr}(\mathcal{W})$ is more if the weights of edges closer to the input are changed. Interestingly, the largest change in $\log\det(\mathcal{W})/\text{tr}(\mathcal{W}^{-1})/\lambda_{\min}(\mathcal{W})$ is caused by the change in edge weight between nodes $2 \longrightarrow 3$ which is one edge away from the input.

*2) Directed ring network:* In case of the directed ring network, we consider input at node 3 i.e., $b = e_3$. The Gramian for the system is $\mathcal{W} = \text{diagonal}\{0.0306, 0.0012, 1, 0.25, 0.1225\}$. The different edge centrality matrices for different $P$ are stated in Table II. The eigenvector corresponding to the smallest eigenvalue is $v_{\min} = e_2$.

| Mt \ Edge | $a_{21}$ | $a_{32}$ | $a_{43}$ | $a_{54}$ | $a_{15}$ |
|---|---|---|---|---|---|
| $\text{tr}\,(\mathcal{W})$ | 0.12 | 0 | 1.6174 | 0.441 | 0.1274 |
| $\log\det(\mathcal{W})$ | 10 | 0 | 16 | 8.57 | 8 |
| $-\text{tr}\,(\mathcal{W}^{-1})$ | 8163 | 0 | 3445 | 2449 | 3396 |
| $\lambda_{\min}\,(\mathcal{W})$ | 0.0122 | 0 | 0.0049 | 0.0035 | 0.0049 |

**TABLE II:** Different edge centrality matrices for directed ring network and input $b = e_3$.

Interestingly, the edge incoming in the input node$-3$ has no first order influence in the network i.e., change in the edge $2 \longrightarrow 3$ has no first-order effect on the any of the controllability metrics.

If multiple inputs are used in the directed ring network say $B = \begin{pmatrix} e_2 & e_4 \end{pmatrix}$ then the result is as tabulated in Table III. The Gramian for the system is $\mathcal{W} = \text{diagonal}\{0.123, 0.005, 0.01, 1.002, 0.491\}$. The eigenvector corresponding to the smallest eigenvalue is $v_{\min} = e_3$.

| Mt \ Edge | $a_{21}$ | $a_{32}$ | $a_{43}$ | $a_{54}$ | $a_{15}$ |
|---|---|---|---|---|---|
| $\text{tr}\,(\mathcal{W})$ | 0.050 | 0.282 | 0.016 | 1.77 | 0.511 |
| $\log\det(\mathcal{W})$ | 0.098 | 20.15 | 0.03 | 5.74 | 4.04 |
| $-\text{tr}\,(\mathcal{W}^{-1})$ | 4.9 | 1990 | 1 | 30.5 | 34.5 |
| $\lambda_{\min}\,(\mathcal{W})$ | 0.0005 | 0.201 | 0 | 0.0001 | 0.0002 |

**TABLE III:** Different edge centrality matrices for directed ring network and multiple inputs $B = (e_2 \;\; e_4)$.

*3) Undirected line/ring networks:* In case of undirected line/ring networks, numerical experiments show that the result of Theorem 4.3 does not hold.

### *B. Edge Modification in a 10−node Directed Network*

Consider a $10-$network with its parameters as shown in Figure 2. The network has 4 input at nodes $\{4, 5, 6, 8\}$. The system $(A, B)$ is controllable. We show how the influence matrices can be used for network modification for different types of performance metrics.

1) In this case, we divide the input nodes as defender input nodes $\{4, 5, 6\}$ represented by $B_d = (e_4 \;\; e_5 \;\; e_6)$ and attacker input node 8 represented by $B_a = e_8$. Our objective is to increase the influence of the defender input $B_d$ and decrease influence of attacker input $B_a$ simultaneously. We take trace of Gramian as the performance metric. So, we have to increase $f_d = \text{tr}\,(\mathcal{W}_d)$ and decrease $f_a = \text{tr}\,(\mathcal{W}_a)$. Now, initially $f_d^0 = 7.335$ and $f_a^0 = 1.305$. For

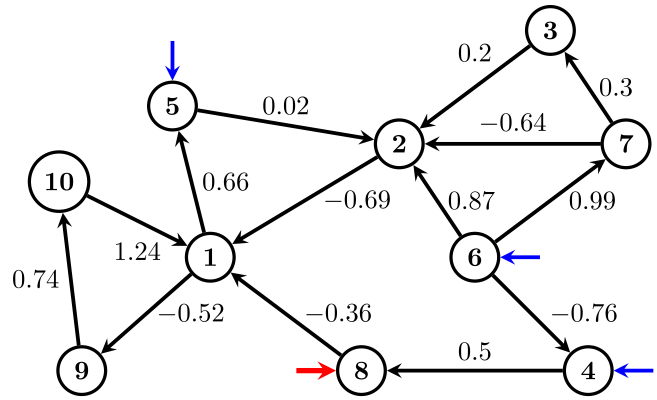


**Fig. 2:** A $10-$node directed network. The blue and red arrows are the input nodes of the system

$T = n$, constructing $\Theta_I^{\{4,5,6\}}$ and $\Theta_I^8$. From the influence matrix for the defender inputs, the highest edge centrality of 3.68 is for the edge $6 \longrightarrow 2$. For the attacker input case the lowest edge centrality is of $-1.69$ for the edge $8 \longrightarrow 1$. Both edges $6 \longrightarrow 2, 8 \longrightarrow 1$ already exists in the network and their strengthening will result in improvement of performance. Choosing weight of 0.2 for each edge gives the new performance metrics, $f_d = 8.214$ and $f_a = 1.06$ which demonstrates improvement in performance. It should be noted that here we have computed the edge centrality matrices for a subset of inputs.

2) In this case, we consider that the system has 4 inputs at $\{4, 5, 6, 8\}$ described by $B = (e_4 \;\; e_5 \;\; e_6 \;\; e_8)$. The objective is to decrease output response at node 2 due to the input at node 6. So, $f = e_2^\top \mathcal{W}_6 e_2$ where $\mathcal{W}_6$ is the controllability Gramian for the input at node 6. For $T = n$, $f^0 = 1.162$. With $P = e_2 e_2^\top$, we construct the edge centrality matrix for node 6. From the $6^{\text{th}}-$node influence matrix, we have the complete edge centrality map of all the possible edges in the network. We see that the edge $6 \longrightarrow 2$ has an influence of $+1.74$ and the edge $7 \longrightarrow 2$ has an influence of $-1.256$. Thus we have two options either we decrease weight of edge $6 \longrightarrow 2$ or increase $7 \longrightarrow 2$ or do both simultaneously. We modify edges $6 \longrightarrow 2$ and $7 \longrightarrow 2$ simultaneously with weight 0.2. The new performance is $f = 0.643$ showing an improvement. A white noise input at node 6 is shown in Figure 3 and the corresponding response comparison for a time period of 50 seconds is shown in Figure 4. It should be noted that this example shows how our proposed edge centrality matrix captures the effects of individual inputs.

Here we may select edges different from those used above for modification depending upon their edge centrality measures and constraints. Thus, the edge centrality matrix for complete or individual or subset of inputs offers a plethora of choices for edge selection. Developing an algorithmic method for edge modification using edge centrality matrices is part of our future work.

## VI. Conclusions

We have proposed a novel method to characterize edges in a network using the edge centrality matrix. For a given

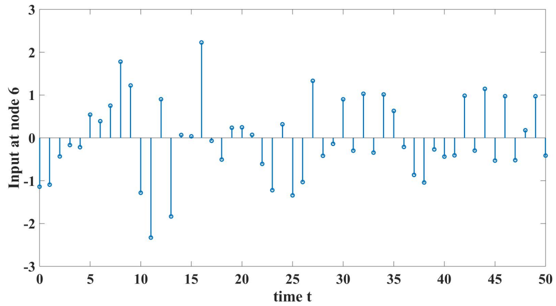


**Fig. 3:** Input at node 6.

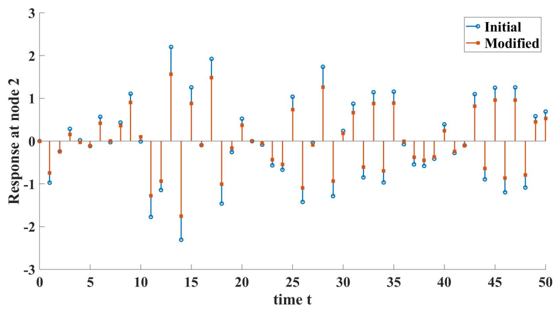


**Fig. 4:** Performance comparison between original and modified network.

performance metric, this concept provides a measure of edge centrality measure for all possible edges in the network. With rigorous analysis, we prove that edge centrality matrix for a given performance metric can be derived from the gradient of the controllability Gramian with respect to the edge weights. The analysis also led to an edge characterization scheme with respect to individual inputs. Next, we relate the edge centrality matrix with energy based physically realizable performance metrics. We also show that for special types of networks namely the directed line and ring networks the edge centrality matrix has the same structure as the weighted network adjacency matrix. Finally, through numerical examples we validate the structural property of edge centrality matrices for directed line/ring networks and demonstrate its usage in heuristic edge modification. Future work will study the development of an algorithmic approach to modify edges in the network using the edge centrality matrices with guarantees on convergence and optimality, along with the design of distributed schemes for the computation of the proposed centrality measures.

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