Frequency-driven market mechanisms for optimal dispatch in power networks

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Abstract

This paper studies real-time bidding mechanisms for economic dispatch and frequency regulation in electrical power networks described by topologies with edge-disjoint cycles. We consider a market administered by an independent system operator (ISO) where a group of strategic generators participate in a Bertrand game of competition. Generators bid prices at which they are willing to produce electricity. Each generator aims to maximize their profit, while the ISO seeks to minimize the total generation cost while respecting line flow limits and regulate the frequency of the system. We consider a continuous-time bidding process coupled with the swing dynamics of the network through the use of frequency as a feedback signal for the negotiation process. We analyze the stability of the resulting interconnected system, establishing frequency regulation and the convergence to a Nash equilibrium and optimal generation levels. Simulations illustrate our theoretical findings.

1 Introduction

Power generation dispatch is typically done in a hierarchical fashion, where the different layers are separated according to their time scales. Broadly, at the top layer, economic efficiency is ensured via market clearing and at the bottom layer, frequency control and regulation is achieved via primary and secondary controllers. However, the intermittent and uncertain nature of distributed energy resources (DERs) represents a major challenge to the current design. Of particular concern is the need to maintain both frequency regulation and cost efficiency of regulation reserves in the face of increasing fluctuations in renewables. This presents an opportunity to rethink the architecture and its hierarchical separation, with potential significant implications for efficiency, ancillary services, and resilience of the future power grid. To this end, we propose an integrated dynamic market mechanism which combines the real-time market and frequency regulation, allowing competitive market players, including renewable generation, to negotiate electricity prices while using the most recent information on grid frequency.

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Literature review: The combination of economic dispatch and frequency regulation has received increasing attention in recent years. Various works have sought to move beyond the traditional and compartmentalized hierarchical control layers to instead simultaneously achieve frequency stabilization and economic dispatch in power networks [Trip et al., 2016, Zhang and Papachristodoulou, 2015, Li et al., 2016] and microgrids [Cady et al., 2015, Dörfler et al., 2016]. Along this line of research, the various agents involved work cooperatively towards the satisfaction of a common goal. An alternative body of research has investigated the use of price-based incentives for economic generation- and demand-side management and frequency regulation [Alvarado et al., 2001, Shiltz et al., 2016, Stegink et al., 2017]. To achieve these goals, these works consider dynamic pricing mechanisms in conjunction with system dynamics of the power network. We also adopt this approach, with the key difference that here generators bid in the market and are, therefore, price-setters instead of price-takers. This viewpoint results in a Bertrand game of competition among the generators. The work [Cherukuri and Cortés, 2020] studied this type of games and established that iterative bidding can achieve convergence to an optimal allocation of power generation, without considering the effects on the dynamics of the power network. The underlying assumption was that generation setpoints could be commanded after convergence, which in practice poses a limitation, considering the fast time-scales at which DERs operate. This was extended in [Stegink et al., 2019], which considers the power network dynamics to develop a time-triggered hybrid implementation of bidding and power setpoint updates based on a simplified formulation of optimal dispatch with quadratic cost functions, no generator bounds, and no line flow limits. Our treatment here supersedes these limitations.We propose an online bidding scheme where the setpoints are updated continuously throughout time to better cope with fast changes in the network. The novelty of the present paper lies in the conjoined treatment of frequency regulation and optimal power dispatch, incorporating key elements of power systems operation and the competitive aspect among the generators.

Statement of contributions: We consider an electrical power network consisting of an independent system operator (ISO) and a group of competitive generators. The physical interconnection of buses and transmission lines is described by an undirected graph with edge-disjoint cycles, meaning that no two cycles have an edge in common. The integration of different layers at the same timescale might create unintended outcomes, since the mechanisms employed at the upper layers cannot be assumed to be in steady state when analyzing the mechanisms at the lower ones. Our technical presentation successfully addresses this challenge, providing a novel algorithmic solution that combines real-time markets with strategic players and frequency regulation. Each generator seeks to maximize its individual profit, while the ISO aims to solve the economic dispatch problem while respecting thermal line limits and regulate the frequency. Since the generators are not willing to share their cost functions, the ISO is unable to solve the economic dispatch problem. Instead, it has the generators compete in a bidding market where they submit bids to the ISO in the form of a price at which they are willing to produce electricity. In return, the ISO determines the power generations levels the generators have to meet. We analyze the underlying Bertrand game among the generators and characterize the Nash equilibria that correspond to optimal power dispatch termed *efficient Nash equilibria*. In particular, we establish the existence of such efficient Nash equilibria and provide a sufficient condition for its uniqueness. We also propose a Nash equilibrium seeking scheme in the form of a continuous-time bidding process that captures the interaction between the generators and the ISO. In this scheme, the generators adjust their bid based on their current bid and the production level that the ISO requests from them with the aim to maximize their profit. At the same time, the ISO adjusts the generation setpoints to minimize the total payment to the generators while taking the power balance and frequency deviation into account. Moreover, along the execution of the algorithm the nonnegativity constraints on the bids and power generation quantities are satisfied. The use of the local frequency error as a feedback signal in the negotiation process couples the ISO-generator coordination scheme with the swing dynamics of the power network. We show that each equilibrium of the interconnected system corresponds to an efficient Nash equilibrium, optimal generation levels and zero frequency regulation. We furthermore establish local convergence to such an equilibrium by invoking a suitable invariance principle for the closed-loop projected dynamical system. Finally, the numerical results on a 6-bus example show fast convergence of the closed-loop system to an optimal equilibrium, even under sudden changes of the load and the cost functions.

Notation: Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ be the set of real, nonnegative real, and positive real numbers, resp. We write $[n] := \{1, \dots, n\}$. The transpose operator is denoted by $\top$. We denote by $\mathbb{1}_n \in \mathbb{R}^n$ the n-dimensional vector whose elements are equal to 1. Given a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, its gradient and its Hessian evaluated at x is written as $\nabla f(x)$ and $\nabla^2 f(x)$, resp. A twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* on $S \subset \mathbb{R}^n$ if it is convex and, for some $\mu > 0$, its Hessian satisfies $\nabla^2 f(x) \succeq \mu I$ for all $x \in S$. The projection of a point $y \in \mathbb{R}^n$ onto a closed convex set $\mathcal{K} \subset \mathbb{R}^n$ is $\text{proj}_{\mathcal{K}}(y) = \arg\min_{z \in \mathcal{K}} \|z - y\|$. The projection of vector $v \in \mathbb{R}^n$ at a point $x \in \mathcal{K}$ with respect to $\mathcal{K}$ is $\Phi_{\mathcal{K}}(x, v) = \lim_{\delta \to 0^+} \big(\text{proj}_{\mathcal{K}}(x + \delta v) - x\big)/\delta$. For $A \in \mathbb{R}^{m \times n}$, the induced 2-norm is denoted by $\|A\|$. Given $v \in \mathbb{R}^n, \tau \in \mathbb{R}^{n \times n}$, we write $\|v\|_\tau := \sqrt{v^\top \tau v}$. Given a set of numbers $v_1, v_2, \dots, v_n \in \mathbb{R}$, $\text{col}(v_1, \dots, v_n)$ denotes the column vector $\begin{bmatrix} v_1, \dots, v_n \end{bmatrix}^\top$ and likewise $\text{diag}(v_1, \dots, v_n)$ denotes the $n \times n$ diagonal matrix with entries $v_1, \dots, v_n$ on the diagonal. For $u, v \in \mathbb{R}^n$ we write $u \perp v$ if $u^\top v = 0$. The complementarity conditions $u \geq 0, v \geq 0, u \perp v$ are denoted as $0 \leq u \perp v \geq 0$. The notations $\mathbf{sin}(.)$ and $\mathbf{cos}(.)$ are the element-wise sine and cosine functions, resp.

2 Power network model and dynamics

We consider an electrical power network consisting of n buses and m transmission lines. The network is represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes $\mathcal{V} = [n]$ represent buses and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ are the transmission lines connecting the buses. The edges are arbitrarily labeled with a unique identifier in $[m]$ and the ends of each edge are arbitrary labeled with '+' and '-'. The resulting labeled directed graph is denoted by $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d)$. The incidence matrix $D \in \mathbb{R}^{n \times m}$ of this directed graph is

$$D_{i\ell} = \begin{cases} +1 & \text{if } i \text{ is the positive end of edge } \ell, \\ -1 & \text{if } i \text{ is the negative end of edge } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Each bus k represents a control area and is assumed to have a load P_{dk} and N_k number of generators. We let $N = \sum_{k=1}^n N_k$ be the total number of generators and assign them a unique identity in $[N]$. Let the set of generators at node k be $G_k \subset [N]$ (this set is empty if there are no generators connected to bus k). The dynamics at the buses is assumed to be governed by the *swing equations* [Machowski et al., 2008],

$$\begin{aligned} \dot{\delta} &= \omega, \\ M\dot{\omega} &= -D\Gamma\, \mathbf{sin}(D^\top \delta) - A\omega + E_g P_g - P_d, \end{aligned} \tag{1}$$

with $P_d = \text{col}(P_{d1}, \dots, P_{dn})$. Here $E_g \in \mathbb{R}^{n \times N}$ whose columns are unit vectors indicating which generator belongs to which node. In addition, $\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m)$, where $\gamma_\ell = B_{kj} V_k V_j = B_{jk} V_k V_j$ and $\ell \in [m]$ corresponds to the edge between nodes i and j. Table 1 presents a list of symbols employed in the model (1). *We assume*[1] *that*

[1] All assumptions in the paper are italicized and are summarized on Table 2.

$\delta \in \mathbb{R}^n$	voltage phase angle
$\omega \in \mathbb{R}^n$	frequency deviation w.r.t. the nominal frequency
$P_g \in \mathbb{R}^N_{\geq 0}$	power generation
$P_d \in \mathbb{R}^n_{\geq 0}$	power load
$M \in \mathbb{R}^{n\times n}_{\geq 0}$	diagonal matrix of moments of inertia
$A \in \mathbb{R}^{n\times n}_{\geq 0}$	diagonal matrix of damping constants
$V_k \in \mathbb{R}_{>0}$	voltage magnitude at bus k
$B_{kj} \in \mathbb{R}_{>0}$	negative of the susceptance of line (k, j)

Table 1
Elements of swing equations (1).

at least one diagonal element of A is positive and the matrix M is invertible. Finally, we assume that the cycles in $(\mathcal{V}, \mathcal{E})$ are edge-disjoint, that is, no two cycles have an edge in common. Note that radial networks trivially satisfy this condition.

It is convenient to work with the voltage phase angle differences $\varphi = D^\top \delta \in \mathbb{R}^m$. Further, let $U(\varphi) = -\mathbb{1}_m^\top \Gamma \mathbf{cos}\,\varphi$, then the physical system (1) in the (φ, ω)-coordinates reads as

$$\begin{aligned} \dot{\varphi} &= D^\top \omega, \\ M\dot{\omega} &= -D\nabla U(\varphi) - A\omega + E_g P_g - P_d. \end{aligned} \tag{2}$$

3 Problem description

Here we formulate the problem statement, introduce the game-theoretic tools, and discuss the goals of the paper.

3.1 ISO-generator coordination

Taking as starting point the electrical power network model described in Section 2, here we outline the elements of the ISO-generator coordination problem following the exposition of [Cherukuri and Cortés, 2020]. Let $C_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the cost incurred by generator $i \in [N]$ in producing P_{gi} units of power. *We assume C_i is strongly convex on the domain $\mathbb{R}_{\geq 0}$ and satisfies $\nabla C_i(0) \geq 0$.* Given a load P_d and the total network cost

$$C(P_g) := \sum_{i\in[N]} C_i(P_{gi}), \tag{3}$$

the ISO seeks to solve the *economic dispatch (ED)* problem

$$\begin{aligned} \underset{(P_g,v)}{\text{minimize}} \quad & C(P_g), && \text{(4a)} \\ \text{subject to} \quad & Dv + P_d - E_g P_g = 0, && \text{(4b)} \\ & P_g \geq 0, && \text{(4c)} \\ & -v^b \leq v \leq v^b && \text{(4d)} \end{aligned}$$

and, at the same time, to regulate the frequency of the physical power network. Here $v \in \mathbb{R}^m$ represents the power flow between the buses and v^b captures flow restrictions. For radial networks, this vector corresponds to the thermal line limits. For more general networks, this represents a further restriction that allows satisfaction of the network flow constraint at the equilibrium of the interconnected system. We elaborate more on this design aspect in Section 5.2 below. *To guarantee feasibility of the nonlinear power flow equations, we assume $v^b < \Gamma\mathbb{1}_m$ element-wise. Furthermore, we assume the total load to be positive, i.e., $\mathbb{1}_n^\top P_d > 0$ such that (4) is feasible.* Since the constraints (4b), (4c), and (4d) are affine, Slater's condition holds implying that (4) has zero duality gap. We can also show that its primal-dual optimizer $(P_g^*, v^*, \lambda^*, \mu^*, \nu_-^*, \nu_+^*)$ where $\lambda^*, \mu^*, \nu_-^*, \nu_+^*$ are associated to constraints (4b), (4c), (4d) resp., is unique by exploiting strong convexity of C. *We assume that for the power injection $P_g = P_g^*$, there exists an equilibrium $(\bar{\varphi}, \bar{\omega})$ of (2) that satisfies $\bar{\varphi} \in (-\pi/2, \pi/2)^m$.* The latter assumption is standard and is referred to as the *security constraint* in the power systems literature [Machowski et al., 2008]. *We also assume that at the optimal generation P_g^*, at each generator bus, there exist at least two generators producing positive power.* The reason for such a condition will become clear in the next section where we discuss the existence of the efficient Nash equilibrium. Note that our formulation can also incorporate flexible demand, such as demand response elements. We can consider their contribution as "negative" generation, and as long as the convexity of the problem (4) is retained, the results extend to this generalized setup.

We note that the ISO cannot determine the optimizer of the ED problem (4) because generators are strategic and they do not reveal their cost functions to anyone. Instead, the ISO operates a market where each generator $i \in [N]$ submits a bid $b_i \in \mathbb{R}_{\geq 0}$ in the form of a price at which it is willing to provide power. Based on these bids, the ISO aims to find the power allocation that meets the load and minimizes the total payment to the generators. Thus instead of solving the ED problem (4) directly, the ISO considers, given a bid $b \in \mathbb{R}^N_{\geq 0}$, the convex optimization problem

$$\begin{aligned} \underset{(P_g,v)}{\text{minimize}} \quad & b^\top P_g, && \text{(5a)} \\ \text{subject to} \quad & Dv + P_d - E_g P_g = 0, && \text{(5b)} \\ & P_g \geq 0, && \text{(5c)} \\ & -v^b \leq v \leq v^b. && \text{(5d)} \end{aligned}$$

A major difference between (4) and (5) is that the latter problem is linear and may have multiple solutions. Let $P_g^{\text{opt}}(b)$ be the optimizer of (5) the ISO selects (this selection might not be unique) given bids b and broadcasts to the generators. Knowing the ISO's strategy, each generator i bids a quantity $b_i \geq 0$ to maximize its payoff

$$\Pi_i(b_i, P_{gi}^{\text{opt}}(b)) := P_{gi}^{\text{opt}}(b) b_i - C_i(P_{gi}^{\text{opt}}(b)), \tag{6}$$

where $P_{gi}^{\text{opt}}(b)$ is the i-th component of the optimizer $P_g^{\text{opt}}(b)$. This function is not necessarily continuous in the bid b. Since each generator is strategic, we analyze the market clearing and the dispatch process explained above using game theory [Başar and Oldser, 1982, Fudenberg and Tirole, 1991].

3.2 Inelastic electricity market game

We define the *inelastic electricity market game* as

- M is invertible; A is nonzero; edge-disjoint cycles in $\mathcal{G}$
- C_i strongly convex, $\nabla C_i(0) \geq 0$; flow capacity $v_b < \Gamma \mathbb{1}_m$
- at P_g^* security constraint holds and two generators at each generator bus have positive production

Table 2
Summary of assumptions. Detailed explanations are given in Section 2 and Section 3.

- Players: the set of generators $[N]$.
- Action: for each player i, the bid $b_i \in \mathbb{R}_{\geq 0}$.
- Payoff: for each player i, the payoff Π_i defined in (6).

In the sequel we interchangeably use the notation $b \in \mathbb{R}^N_{\geq 0}$ and $(b_i, b_{-i}) \in \mathbb{R}^N_{\geq 0}$ for the bid vector, where $b_{-i} \in \mathbb{R}^{N-1}_{\geq 0}$ represents the bids of all players except i. We note that the payoff of generator i not only depends on the bids of the other players but also on the optimizer $P_g^{\text{opt}}(b)$ the ISO selects. Therefore, the concept of a Nash equilibrium is defined slightly differently compared to the usual one.

Definition 3.1 *(Nash equilibrium [Cherukuri and Cortés, 2020]): A bid profile $b^* \in \mathbb{R}^N_{\geq 0}$ is a* Nash equilibrium *of the inelastic electricity market game if there exists an optimizer $P_g^{opt}(b^*)$ of (5) such that*

$$\Pi_i(b_i, P_{gi}^{opt}(b_i, b_{-i}^*)) \leq \Pi_i(b_i^*, P_{gi}^{opt}(b^*))$$

for all $i \in [N]$, for all $b_i \in \mathbb{R}_{\geq 0}$ with $b_i \neq b_i^*$, and all optimizers $P_{gi}^{opt}(b_i, b_{-i}^*)$ of (5) given bids (b_i, b_{-i}^*).

We are interested in bid profiles for which the optimizer of (4) is also a solution to (5). This is captured next.

Definition 3.2 *(Efficient bid and efficient Nash equilibrium): An* efficient bid *of the inelastic electricity market is a bid $b^* \in \mathbb{R}^N_{\geq 0}$ for which the optimizer P_g^* of (4) is also an optimizer of (5) given bids $b = b^*$ and*

$$P_{gi}^* = \underset{P_{gi} \geq 0}{\arg\max} \{P_{gi} b_i^* - C_i(P_{gi})\} \quad \text{for each } i \in [N]. \quad (7)$$

A bid $b^* \in \mathbb{R}^N_{\geq 0}$ is an efficient Nash equilibrium *if it is an efficient bid and a Nash equilibrium.*

At the efficient Nash equilibrium, the optimizer of the ED problem coincides with the production levels that maximize the individual profits (6) of the generators.

3.3 Paper goals: control design for stable interconnection

Given the problem setup, neither the ISO nor the individual strategic generators are able to determine the efficient Nash equilibrium a priori. As a first objective, we are interested in designing a Nash equilibrium seeking mechanism in the form of a bidding process where the generators coordinate with the ISO to dynamically update their bids and production levels, while respecting the nonnegativity constraints throughout its execution. Our second objective is the characterization of the stability properties of the interconnection of the bidding process with the physical dynamics of the power network.

4 Existence and uniqueness of Nash equilibria

Here we establish existence of an efficient Nash equilibrium and provide a condition for its uniqueness. We start by providing a characterization of a set of efficient Nash equilibria.

Proposition 4.1 *(Characterization of efficient Nash equilibria): Let $(P_g^*, v^*, \lambda^*, \mu^*, \nu_-^*, \nu_+^*)$ be a primal-dual optimizer of (4), i.e., $P_g^* \in \mathbb{R}^N, v^* \in \mathbb{R}^m, \lambda^* \in \mathbb{R}^n, \mu^* \in \mathbb{R}^N, \nu_-^* \in \mathbb{R}^m, \nu_+^* \in \mathbb{R}^m$ satisfy the Karush-Kuhn-Tucker (KKT) conditions*

$$0 = \nabla C(P_g^*) - E_g^\top \lambda^* - \mu^*, \quad (8a)$$

$$0 = -D^\top \lambda^* - \nu_+^* + \nu_-^*, \quad (8b)$$

$$0 = Dv^* - E_g P_g^* + P_d, \quad (8c)$$

$$0 \leq P_g^* \perp \mu^* \geq 0, \quad (8d)$$

$$0 \leq (v^b - v^*) \perp \nu_+^* \geq 0, \quad 0 \leq (v^b + v^*) \perp \nu_-^* \geq 0. \quad (8e)$$

Suppose that at each node k, $P_{gi}^* > 0$ for at least two distinct generators $i \in G_k$ or $P_{gi}^* = 0, \forall i \in G_k$. Then, any $b^* \in \mathbb{R}^N_{\geq 0}$ satisfying $E_g^\top \lambda^* \leq b^* \leq \nabla C(P_g^*)$ is an efficient Nash equilibrium of the inelastic electricity market game.

PROOF. Let $(P_g^*, v^*, \lambda^*, \mu^*, \nu_-^*, \nu_+^*)$ satisfy (8). Note that since the objective function of (4) is strongly convex, for any optimizer $(\bar{P}_g, \bar{v})$ of (4), we have $\bar{P}_g = P_g^*$. From (8a), $E_g^\top \lambda^* \leq \nabla C(P_g^*)$. Fix $b^* \in \mathbb{R}^N_{\geq 0}$ satisfying $E_g^\top \lambda^* \leq b^* \leq \nabla C(P_g^*)$. We will now prove that b^* is efficient. Define $\hat{\mu}^* := b^* - E_g^\top \lambda^*$ and note that $(P_g^*, v^*, \lambda^*, \hat{\mu}^*, \nu_-^*, \nu_+^*)$ satisfies

$$\begin{aligned} b^* &= E_g^\top \lambda^* + \hat{\mu}^*, & Dv^* - E_g P_g^* + P_d = 0, \\ 0 &\leq P_g^* \perp \hat{\mu}^* \geq 0, & 0 = -D^\top \lambda^* - \nu_+^* + \nu_-^* \end{aligned} \quad (9)$$

and (8e). We note that Slater's condition holds for (5) and its KKT conditions are given by (9). Consequently, (P_g^*, v^*) is a primal optimizer of (5). In addition, the bid b^* satisfies

$$P_{gi}^* = \underset{P_{gi} \geq 0}{\arg\max} \{P_{gi} b_i^* - C_i(P_{gi})\} \quad \text{for each } i \in [N]. \quad (10)$$

This is true as for each $i \in [N]$, the following optimality conditions: $\nabla C_i(P_{gi}^*) = b_i^* + \eta_i^*$, and $0 \leq P_{gi}^* \perp \eta_i^* \geq 0$, are satisfied for $\eta_i^* = \nabla C_i(P_{gi}^*) - b_i^*$. Note that in the above set of conditions, $P_{gi}^* \eta_i^* = 0$ because if $P_{gi}^* > 0$, then $\nabla C_i(P_{gi}^*) = \lambda_k^* = b_i^*$ for $i \in G_k$. Thus, we have established that b^* is efficient. In the remainder of the proof we show that b^* is a Nash equilibrium. Suppose generator i deviates from the bid b_i^*. We distinguish between two cases. Suppose first that $b_i > b_i^*$, then by replacing b^* by (b_i, b_{-i}^*) in (5) and checking the optimality conditions, we obtain $P_{gi}^{\text{opt}}(b_i, b_{-i}^*) = 0$ as, by assumption, there is at least one other generator j at node k such that $b_j^* = \lambda_k^* < b_i$. Without loss of generality assume that $P_{gi}^* > 0$ since otherwise $\Pi_i(b_i^*, P_{gi}^*) = \Pi_i(b_i, P_{gi}^{\text{opt}}(b_i, b_{-i}^*))$. For $P_{gi}^* > 0$, we have $b_i^* = \nabla C_i(P_{gi}^*)$ and therefore $\nabla C_i(P_{gi}) \leq b_i^*$ for all $P_{gi} \in [0, P_{gi}^*]$. As a result

$$\Pi_i(b_i, P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*)) = C(0) \leq \Pi_i(b_i^*, P_{gi}^*).$$

This shows that a bid $b_i > b_i^*$ does not increase its payoff. Suppose now that $b_i < b_i^*$, then

$$\begin{aligned}
&\Pi_i(b_i, P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*)) = b_i P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*)) - C_i(P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*))\\
&\leq b_i^* P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*)) - C_i(P_{gi}^{\mathrm{opt}}(b_i, b_{-i}^*))\\
&\leq b_i^* P_{gi}^* - C_i(P_{gi}^*) = \Pi_i(b_i^*, P_{gi}^*),
\end{aligned}$$

where the second inequality follows from (10) as b^* is efficient. Hence, each generator i has no incentive to deviate from bid b_i^* given b_{-i}^*. We conclude that b^* is an efficient Nash equilibrium of the inelastic electricity market game. □

The proof of Proposition 4.1 shows that if $P_{gi}^* > 0$, then generator i's efficient Nash equilibrium bid b_i^* is equal to the (unique) Lagrange multiplier λ_k^* associated to the power balance (4b) at bus k where $i \in G_k$. In the other case that $P_{gi}^* = 0, i \in G_k$, generator i's Nash equilibrium bid is larger than or equal to λ_k^*. This represents the case that generator i's marginal costs at zero power production is larger than or equal to the market clearing price at that node, and hence generator i is not willing to produce any electricity in that case. The underlying assumption in Proposition 4.1 is that at least two generators have a positive production at the optimal generation levels. We assume this condition holds for the remainder of the paper unless stated otherwise. This assumption is satisfied even if there is one generator at some node provided this generator produces zero power at the optimum. Note that if there is only one generator at some node and at the optimum it produces positive power, then it is possible that the line limits are reached at the optimum and the generator can increase its bid to an arbitrary large value and still get positive power allocated at the market clearing dispatch. In such a case an efficient Nash bid will not exist. The assumption prevents this situation.

The previous observations lead to the identification of the same sufficient condition as in [Cherukuri and Cortés, 2020] (which only establishes the existence of one specific efficient Nash equilibrium) to guarantee the uniqueness of the efficient Nash equilibrium, which we state here for completeness.

Corollary 4.2 *(Uniqueness of the efficient Nash equilibrium [Cherukuri and Cortés, 2020]): Let* $(P_g^*, v^*, \lambda^*, \mu^*, \nu_-^*, \nu_+^*)$ *be a primal-dual optimizer of* (4) *and suppose that* $P_g^* \in \mathbb{R}^N_{>0}$, *then* $b^* = \nabla C(P_g^*) = E_g^\top \lambda^*$ *is the unique efficient Nash equilibrium of the inelastic electricity market game.*

Remark 4.3 *(Any efficient Nash equilibrium characterized by Proposition 4.1 is positive):* Under our assumption that, at each node $k \in [n]$, $P_{gi}^* > 0, i \in G_k$ for at least two generators, it follows from the optimality conditions (8) that, for such P_{gi}^*, we have $\nabla C_i(P_{gi}^*) > 0$ by the strict convexity of C_i and the assumption $\nabla C_i(0) \geq 0$. This implies that $\lambda^* \in \mathbb{R}^n_{>0}$ and therefore also $b^* \in \mathbb{R}^N_{>0}$. •

We note that under our assumption, the set of efficient Nash equilibrium is not necessarily unique. We will see later that as a consequence, the set of equilibria of the interconnected system (12) need not be a singleton.

5 Interconnection of bid update scheme with power network dynamics

In this section we introduce a Nash equilibrium seeking mechanism between the generators and the ISO. Each generator dynamically updates its bid based on the power generation setpoint received from the ISO, while the ISO changes the power generation setpoints depending on the generator bids and the frequency of the network. This update mechanism of the bids and the setpoints is written as a continuous-time dynamical system. We assume that each generator can only communicate with the ISO and is not aware of the number of other generators participating, their respective cost functions, or the load at its own bus. This assumption on the communication architecture is reasonable as in an electricity market, each participating generator submits bids to the ISO and the ISO declares the generator setpoints upon market clearing. We study the interconnection of the proposed online bidding process with the power system dynamics and establish local convergence to an efficient Nash equilibrium, optimal power dispatch, and zero frequency deviation.

5.1 *Price-bidding mechanism*

In our design, each generator $i \in [N]$ changes its bid $b_i \geq 0$ according to the projected dynamical system

$$\tau_{b_i} \dot{b}_i = \Phi_{\mathbb{R}_{\geq 0}}(b_i, P_{gi} - \nabla C_i^*(b_i)), \tag{11a}$$

with gain $\tau_{b_i} > 0$. The projection operator in the above dynamics ensures that trajectories starting in the nonnegative orthant remain there (see notations for the exact definition). The map $C_i^* : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denotes the *convex conjugate* of the cost function C_i and is defined as $C_i^*(b_i) := \max_{P_{gi} \geq 0}\{b_i P_{gi} - C_i(P_{gi})\}$. The map C_i^* is convex and continuously differentiable on the domain $\mathbb{R}_{\geq 0}$ and strictly convex on the domain $[\nabla C_i(0), \infty)$ [Hiriart-Urruty and Lemaréchal, 2013, Section I.6] . Moreover, $\nabla C_i^*(b_i) = \arg\max_{P_{gi} \geq 0}\{b_i P_{gi} - C_i(P_{gi})\}$ for all $b_i \geq 0$.

The motivation behind the update law (11a) is as follows. Given the bid $b_i > 0$, generator i seeks to produce power that maximizes its profit, which is given by $P_{gi}^{\mathrm{des}} = \nabla C_i^*(b_i) = \arg\max_{P_{gi} \geq 0}\{b_i P_{gi} - C_i(P_{gi})\}$. However, if the ISO requests more power from the generator compared to its desired quantity, i.e., $P_{gi} > P_{gi}^{\mathrm{des}}$, then i will increase its bid to increase its profit. On the other hand if $P_{gi} < P_{gi}^{\mathrm{des}}$, then i will decrease its bid. For the ISO, we also provide an update law which depends on the generator bids and the network frequency. This involves seeking a primal-dual optimizer of (5) or, equivalently, finding a saddle-point of the augmented Lagrangian

$$\begin{aligned}
\mathcal{L}(P_g, v, \lambda) = b^\top P_g + \lambda^\top (Dv + P_d - E_g P_g)\\
+ (1/2)\|Dv + P_d - E_g P_g\|_\Xi^2,
\end{aligned}$$

where $\Xi \in \mathbb{R}^{n \times n}$ is a positive definite matrix. By writing the associated projected saddle-point dynamics (see e.g., [Cherukuri et al., 2017, Goebel, 2017]), the ISO dynamics takes the form

$$\begin{aligned}\tau_g \dot{P}_g &= \Phi_{\mathbb{R}^N_{\geq 0}}(P_g, E_g^\top(\lambda+\Xi(Dv + P_d - E_g P_g) - \sigma^2\omega) - b),\\ \tau_v \dot{v} &= \Phi_{[-v^b, v^b]}(v, -D^\top(\lambda + \Xi(Dv + P_d - E_g P_g)),\\ \tau_\lambda \dot{\lambda} &= Dv + P_d - E_g P_g, \end{aligned} \tag{11b}$$

with design parameters $\sigma \in \mathbb{R}_{>0}$ and diagonal positive definite matrices $\tau_\lambda \in \mathbb{R}^{n\times n}$, $\tau_v \in \mathbb{R}^{m\times m}$, and $\tau_g \in \mathbb{R}^{N\times N}$. Bearing in mind the ISO's second objective of driving the frequency deviation to zero, we add the feedback signal $-\sigma^2\omega$ to adjust the generation based on the frequency deviation in the grid. The dynamics (11b) can be interpreted as follows. If generator i bids higher than the Lagrange multiplier λ_k for $i \in G_k$ (which can be interpreted as a price) associated with the power balance constraint (5b) at node k, then the power generation setpoint at node i is decreased, and vice versa. The terms $\Xi(Dv + P_d - E_g P_g)$ and $-\sigma^2\omega$ in (11b) help to compensate for the supply-demand mismatch in the network.

In the following, we analyze the equilibria and the stability of the interconnection of the physical power network dynamics (2) with the bidding process (11). We assume that the bids and power generations are initialized within the feasible domain, i.e., $b(0) \geq 0, P_g(0) \geq 0$.

5.2 *Equilibrium analysis of the interconnected system*

The closed-loop system comprises the ISO-generator bidding scheme (11) and the power network dynamics (2),

$$\dot{\varphi} = D^\top\omega, \tag{12a}$$
$$M\dot{\omega} = -D\Gamma \,\mathbf{sin}\,\varphi - A\omega + E_g P_g - P_d, \tag{12b}$$
$$\tau_b \dot{b} = \Phi_{\mathbb{R}^N_{\geq 0}}(b, P_g - \nabla C^*(b)), \tag{12c}$$
$$\tau_g \dot{P}_g = \Phi_{\mathbb{R}^N_{\geq 0}}(P_g, E_g^\top(\lambda + \Xi(Dv + P_d - E_g P_g) - \sigma^2\omega) - b), \tag{12d}$$
$$\tau_v \dot{v} = \Phi_{[-v^b, v^b]}(v, -D^\top(\lambda+\Xi(Dv+P_d-E_g P_g))), \tag{12e}$$
$$\tau_\lambda \dot{\lambda} = Dv + P_d - E_g P_g, \tag{12f}$$

where $C^*(b) := \sum_{i\in[N]} C_i^*(b_i)$, $\tau_b = \mathrm{diag}(\tau_{b_1}, \ldots, \tau_{b_N}) \in \mathbb{R}^{N\times N}$. We note that the implementation of this interconnected dynamics is distributed, in the sense that it only requires for each bus to interact with its neighboring buses, and for each generator to interact with the ISO, and therefore scales up with the size of the system. To be specific, each generator can maintain and update its own generation and bid dynamics and one computing agent at each bus updates the Lagrange multiplier associated to that bus and the flow variables of edges directed towards it. One can verify that with such assignment of variables, the dynamics can be implemented without the need for global information.

We next investigate the equilibria of (12). In particular, we are interested in equilibria that correspond simultaneously to an efficient Nash equilibrium, economic dispatch, and frequency regulation, as specified next.

Definition 5.1 *(Efficient equilibrium): An equilibrium $\bar{x} = \mathrm{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{v}, \bar{\lambda})$ of (12) is* efficient *if $\bar{\omega} = 0$, $\bar{b}$ is an efficient Nash equilibrium, and $(\bar{P}_g, \bar{v})$ is an optimizer of (4).*

The next result shows that all equilibria of (12) are efficient.

Proposition 5.2 *(Equilibria are efficient): Any equilibrium $\bar{x} = \mathrm{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{v}, \bar{\lambda})$ of (12) is efficient.*

PROOF. Let $\bar{x}$ be an equilibrium of (12), then there exist $\bar{\mu}_b, \bar{\mu}_g \in \mathbb{R}^N$ and $\bar{\nu}_+, \bar{\nu}_- \in \mathbb{R}^m$ such that

$$0 = D^\top\bar{\omega}, \tag{13a}$$
$$0 = -D\Gamma\,\mathbf{sin}\,\bar{\varphi} - A\bar{\omega} + E_g\bar{P}_g - P_d, \tag{13b}$$
$$0 = \bar{P}_g - \nabla C^*(\bar{b}) + \bar{\mu}_b, \tag{13c}$$
$$\begin{aligned}0 = E_g^\top\bar{\lambda} - \bar{b} + E_g^\top\Xi(D\bar{v} + P_d - E_g\bar{P}_g))\\ - \sigma^2 E_g^\top\bar{\omega} + \bar{\mu}_g,\end{aligned} \tag{13d}$$
$$0 = -D^\top(\bar{\lambda} + \Xi(D\bar{v} + P_d - E_g\bar{P}_g)) - \bar{\nu}_+ + \bar{\nu}_-, \tag{13e}$$
$$0 = D\bar{v} + P_d - E_g\bar{P}_g, \tag{13f}$$
$$0 \leq \bar{b} \perp \bar{\mu}_b \geq 0, \quad 0 \leq \bar{P}_g \perp \bar{\mu}_g \geq 0, \tag{13g}$$
$$0 \leq v^b - \bar{v} \perp \bar{\nu}_+ \geq 0, \quad 0 \leq v^b + \bar{v} \perp \bar{\nu}_- \geq 0. \tag{13h}$$

We first show that $\bar{\omega} = 0$. From (13a) it follows that $\bar{\omega} = \mathbb{1}_n\omega_s$ for some $\omega_s \in \mathbb{R}$. Then by pre-multiplying (13b) by $\mathbb{1}_n^\top$ and using (13f) we obtain $\mathbb{1}_n^\top A\mathbb{1}_n\omega_s = 0$, which implies that $\bar{\omega} = \mathbb{1}_n\omega_s = 0$. We prove next that $(\bar{P}_g, \bar{v})$ is a primal optimizer of (4). We claim that $\bar{\mu}_b = 0$ since, by contradiction, if $\bar{\mu}_{bi} > 0$ for some $i \in [N]$, then $\bar{b}_i = 0$ and therefore $0 = \bar{P}_{gi} - \nabla C_i^*(\bar{b}_i) + \bar{\mu}_{bi} = \bar{P}_{gi} + \bar{\mu}_{bi} > 0$, see also Remark 4.3. Therefore, (13c) implies that $\bar{P}_g = \nabla C^*(\bar{b}) = \arg\max_{P_g\geq 0}\{P_g^\top\bar{b} - C(P_g)\}$ and thus satisfies the optimality conditions

$$\nabla C(\bar{P}_g) = \bar{b} + \bar{\eta},\ 0 \leq \bar{P}_g \perp \bar{\eta} \geq 0, \text{ for some } \bar{\eta}. \tag{14}$$

Let us define $\bar{\mu} = \bar{b} + \bar{\eta} - E_g^\top\bar{\lambda} \geq 0$ where the inequality holds by (13d). By (13g) and (14) we have $\bar{P}_g^\top\bar{\mu} = \bar{P}_g^\top(\bar{b} - E_g^\top\bar{\lambda}) = P_g^\top\bar{\mu}_g = 0$. Hence, $(\bar{P}_g, \bar{v}, \bar{\lambda}, \bar{\mu}, \bar{\nu}_+, \bar{\nu}_-)$ satisfies

$$\begin{aligned}\nabla C(\bar{P}_g) = E_g^\top\bar{\lambda} + \bar{\mu}, \qquad D\bar{v} + P_d - E_g\bar{P}_g = 0,\\ 0 \leq \bar{P}_g \perp \bar{\mu} \geq 0, \quad -D^\top\bar{\lambda} - \bar{\nu}_+ + \bar{\nu}_- = 0,\end{aligned} \tag{15}$$

and (13h) implying that $(\bar{P}_g, \bar{v}, \bar{\lambda}, \bar{\mu}, \bar{\nu}_+, \bar{\nu}_-)$ is a primal-dual optimizer of (4). Furthermore, (14) implies $\bar{b} \leq \nabla C(\bar{P}_g)$ and thus, by Proposition 4.1, $\bar{b}$ is an efficient Nash equilibrium. Hence, $\bar{x}$ is an efficient equilibrium of (12). □

An important observation from the proof of Proposition 5.2 is that, at the equilibrium,

$$\begin{aligned}-D\Gamma\,\mathbf{sin}\,\bar{\varphi} + E_g\bar{P}_g - P_d = 0,\\ -D\bar{v} + E_g\bar{P}_g - P_d = 0,\end{aligned}$$

implying that $D(\bar{v} - \Gamma\,\mathbf{sin}\,\bar{\varphi}) = 0$. This fact ensures satisfaction of thermal line limits at the equilibrium, provided the vector v^b is chosen appropriately in the ED problem (4). In the case of a tree network, choosing v^b equal to the thermal line limits ensures this, since the expression $D(\bar{v} - \Gamma \mathbf{sin}\, \bar{\varphi}) = 0$ implies $\bar{v} = \Gamma \mathbf{sin}\, \bar{\varphi}$ and due to the constraint, $-v^b \leq \bar{v} \leq v^b < \Gamma \mathbb{1}_m$, we conclude that the physical power flow also satisfies the thermal limits, i.e., $-v^b \leq \Gamma \mathbf{sin}\, \bar{\varphi} \leq v^b$.

For general networks with edge-disjoint cycles, one needs more care when selecting v^b, as in this case $\bar{v}$ need not be equal to $\Gamma \mathbf{sin}\, \bar{\varphi}$. The next result shows how to choose v^b smaller than thermal line limits by a precise amount in this scenario to ensure that the power flow at the equilibrium, $\Gamma \mathbf{sin}\, \bar{\varphi}$, satisfies the thermal limits. The result adapts the formal statements and discussion provided in [Zholbaryssov and Domínguez-García, 2020, Section V] for our setup. We include the proof for completeness of exposition.

Proposition 5.3 *(Power flow limits hold at the equilibrium): Let $v^t \in \mathbb{R}^m_{\geq 0}$ be the thermal line limits for the network. Let C be the number of cycles. For each $k \in [C]$, let $(\mathcal{V}_k, \mathcal{E}_k) \subset (\mathcal{V}^d, \mathcal{E}^d)$ denote the subset of vertices and directed edges contained in cycle k, and define*

$$\mathcal{B}_k := \{v \in \mathbb{R}^{|\mathcal{E}^d|} \,|\, -v^t_{ij} + \beta^*_k \leq v_{ij} \leq v^t_{ij} - \beta^*_k, \forall (i,j) \in \mathcal{E}_k\},$$

where

$$\begin{aligned} \beta^*_k &:= \frac{\zeta^M_k}{2} - \frac{\zeta^m_k}{2} \sin(\psi_k), & \psi_k &:= \frac{\pi}{2(d_k - 1)}, \\ \zeta^M_k &:= \max_{(i,j) \in \mathcal{E}_k} v^t_{ij}, & \zeta^m_k &:= \min_{(i,j) \in \mathcal{E}_k} v^t_{ij}, \end{aligned} \tag{16}$$

and d_k is the number of edges in cycle k.

If a flow vector $v \in \mathbb{R}^{|\mathcal{E}^d|}$ satisfies: (a) the thermal limits for all edges not belonging to a cycle, (b) $v \in \cap_{k \in [C]} \mathcal{B}_k$, and (c) $D(v - \Gamma \sin \varphi) = 0$, then $-v^t \leq \Gamma \sin \varphi \leq v^t$, that is, the physical flow satisfies the thermal limits.

PROOF. Note that edges in $\mathcal{E}_k$ are directed according to the labels attached to the edges in $\mathcal{E}^d$. Recall that we obtained $(\mathcal{V}^d, \mathcal{E}^d)$ by assigning an arbitrary direction to each edge in $(\mathcal{V}, \mathcal{E})$. For each cycle k, let $\vec{\mathcal{E}}_k \subset \mathcal{V}_k \times \mathcal{V}_k$ be the set of directed edges that form a cyclic path covering all vertices in $\mathcal{V}_k$ once. Further, define the vector $\mathfrak{n}_k \in \{-1, 1, 0\}^{|\mathcal{E}^d|}$ as

$$(\mathfrak{n}_k)_{ij} := \begin{cases} 1, & \text{if } (i,j) \in \vec{\mathcal{E}}_k, \\ -1, & \text{if } (j,i) \in \vec{\mathcal{E}}_k, \\ 0, & \text{otherwise.} \end{cases}$$

This vector codifies the (mis-)alignment of the directed edges in $\mathcal{E}^d$ and $\vec{\mathcal{E}}_k$. For each k, define functions

$$h_k(v) := \sum_{(i,j) \in \mathcal{E}_k} \arcsin((\mathfrak{n}_k)_{ij} v_{ij} / \gamma_{ij}) \tag{17a}$$

$$\mu^m_k(v) := \max_{(i,j) \in \mathcal{E}_k} (-v^t_{ij} - (\mathfrak{n}_k)_{ij} v_{ij}), \tag{17b}$$

$$\mu^M_k(v) := \min_{(i,j) \in \mathcal{E}_k} (v^t_{ij} - (\mathfrak{n}_k)_{ij} v_{ij}). \tag{17c}$$

The function h_k sums the voltage angle differences across each edge of the cycle k given the flows v. Further, μ^m_k and μ^M_k keep track of the maximum allowable up-shift or down-shift of the flows v till any one of the line limits in cycle k become active. According to [Zholbaryssov and Domínguez-García, 2020, Proposition 3], the set $\mathcal{B}_k$ defined in the statement is equivalent to the following set

$$\begin{aligned} \mathcal{F}_k := \{v \in \mathbb{R}^{|\mathcal{E}^d|} \,|\, & h_k(v + \mathfrak{n}_k \mu^M_k(v)) \geq 0, \ \mu^M_k(v) \geq \beta^*_k, \\ & h_k(v + \mathfrak{n}_k \mu^m_k(v)) \leq 0, \ \mu^m_k(v) \leq -\beta^*_k\}. \end{aligned}$$

By the equivalence of $\mathcal{B}_k$ and $\mathcal{F}_k$, if the vector v belongs to the set $\cap_{k \in [C]} \mathcal{B}_k$, then we have

$$h_k(v + \mathfrak{n}_k \mu^M_k(v)) \geq 0, \quad \text{and} \quad h_k(v + \mathfrak{n}_k \mu^m_k(v)) \leq 0, \tag{18}$$

for all $k \in [C]$. Note that for each k, the function $\mu \mapsto h_k(v + \mathfrak{n}_k \mu)$ is monotonically increasing. Thus, (18) implies that for each k, there exists a unique $\bar{\mu}_k \in [\mu^m_k(v), \mu^M_k(v)]$ such that $h_k(v + \mathfrak{n}_k \bar{\mu}_k) = 0$. Due to the fact that all cycles are edge-disjoint, the expression $D(v - \Gamma \mathbf{sin}\, \varphi) = 0$ implies that $v = \Gamma \mathbf{sin}\, \varphi + \mathfrak{N} \mu^*$ for some $\mu^* \in \mathbb{R}^C$, where columns of the matrix $\mathfrak{N} \in \{-1, 1, 0\}^{|\mathcal{E}^d| \times C}$ consist of vectors $\mathfrak{n}_k$, $k \in [C]$. This implies that for all k,

$$h_k(v - \mathfrak{n}_k \mu^*_k) = h_k(v - \mathfrak{N} \mu^*) = h_k(\Gamma \mathbf{sin}\, \varphi) = 0, \tag{19}$$

where, the first of the above equality follows from the edge-disjoint property of the set of cycles and the last equality is due to the fact that adding angle differences across all edges in a cycle gives zero.

From the uniqueness of $\bar{\mu}_k \in [\mu^m_k(v), \mu^M_k(v)]$ such that $h_k(v + \mathfrak{n}_k \bar{\mu}_k) = 0$ and equation (19), we conclude that $-\mu^*_k \in [\mu^m_k(v), \mu^M_k(v)]$ for all k. This implies that the vector $\Gamma \mathbf{sin}\, \varphi$ belongs to the hyperrectangle $[v + \mu^m_k(v), v + \mu^M_k(v)]$ for all k. As a consequence, by definitions (17b) and (17c), we conclude that $\Gamma \sin \varphi$ satisfies flow constraints. □

According to Proposition 5.3, we select v^b for a general network with edge-disjoint cycles as follows. For an edge (i,j) that does not belong to any cycle, we let $v^b_{ij} = v^t_{ij}$. If (i,j) belongs to cycle $k \in [C]$, then we set $v^b_{ij} = v^t_{ij} - \beta^*_k$, where β^*_k is given in (16). This choice ensures that, as in the case of a tree network, the physical power flow satisfies the thermal line limits at the equilibrium. Note that if power flow constraints can be ensured by some other mechanism, then the assumption of edge-disjoint cycles can be dropped.

5.3 Convergence analysis

In this section we establish the local asymptotic convergence of (12) to an efficient equilibrium.

Theorem 5.4 *(Convergence of the closed-loop system (12)): Consider the subset of (efficient) equilibria,*

$$\begin{aligned} \mathcal{X} := \{\bar{x} = \operatorname{col}(\bar{\varphi}, \bar{\omega}, \bar{b}, \bar{P}_g, \bar{v}, \bar{\lambda}) : & \ \bar{x} \text{ is an equilibrium of } (12) \\ & \text{and } \bar{\varphi} \in (-\pi/2, \pi/2)^m\}. \end{aligned}$$

Then, $\mathcal{X}$ is locally asymptotically stable under (12)*. Moreover, the convergence of trajectories is to a point.*

PROOF. Our proof strategy to show local convergence to $\mathcal{X}$ is based on applying Theorem A.1, which is a special case of the invariance principle stated in [Brogliato and Goeleven, 2005] adapted for complementarity systems. To this end, we rewrite the projected dynamical system (12) as the equivalent complementarity system (20), see also [Brogliato et al., 2006, Theorem 1], [van der Schaft and Schumacher, 1998] for more details,

$$\dot{\varphi} = D^\top \omega, \tag{20a}$$
$$M\dot{\omega} = -D\Gamma \,\mathbf{sin}\, \varphi - A\omega + E_g P_g - P_d, \tag{20b}$$
$$\tau_b \dot{b} = P_g - \nabla C^*(b) + \mu_b, \tag{20c}$$
$$\tau_g \dot{P}_g = E_g^\top(\lambda + \Xi(Dv + P_d - E_g P_g)) - \sigma^2\omega) - b + \mu_g, \tag{20d}$$
$$\tau_v \dot{v} = -D^\top(\lambda + \Xi(Dv + P_d - E_g P_g)) - \nu_+ + \nu_-, \tag{20e}$$
$$\tau_\lambda \dot{\lambda} = Dv + P_d - E_g P_g, \tag{20f}$$
$$0 \le b \perp \mu_b \ge 0, \quad 0 \le P_g \perp \mu_g \ge 0, \tag{20g}$$
$$0 \le v^b - v \perp \nu_+ \ge 0, \quad 0 \le v^b + v \perp \nu_- \ge 0, \tag{20h}$$

where $\mu_b, \mu_g \in \mathbb{R}^N$ and $\nu_+, \nu_- \in \mathbb{R}^m$. We can equivalently write (20) in the compact form

$$\dot{x} = F(x) + C^\top \Lambda, \tag{21a}$$
$$0 \le Cx + d \perp \Lambda \ge 0, \tag{21b}$$

with $x = \mathrm{col}(\varphi, \omega, b, P_g, v, \lambda), \Lambda = \mathrm{col}(\mu_b, \mu_g, \nu_-, \nu_+)$, and

$$F(x) = \begin{bmatrix} D^\top \omega \\ M^{-1}(-D\nabla U(\varphi) - A\omega + E_g P_g - P_d) \\ \tau_b^{-1}(P_g - \nabla C^*(b)) \\ \tau_g^{-1}(E_g^\top(\lambda + \Xi(Dv + P_d - E_g P_g)) - \sigma^2\omega) - b) \\ \tau_v^{-1} D^\top(-\lambda - \Xi(Dv + P_d - E_g P_g)) \\ \tau_\lambda^{-1}(Dv + P_d - E_g P_g) \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \tau_b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_g^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_v^{-1} & 0 \\ 0 & 0 & 0 & 0 & -\tau_v^{-1} & 0 \end{bmatrix}, \quad d = 0. \tag{22}$$

Note that F is Lipschitz continuous (here we observe that, since C is continuously differentiable and μ-strongly convex on $\mathbb{R}_{\ge 0}$, C^* is $\frac{1}{\mu}$-Lipschitz continuous on $\mathbb{R}_{\ge 0}$). For the equivalence of the projected dynamical system (12) and the complementarity system (20) to hold, we consider absolutely continuous solutions $t \mapsto x(t)$ that satisfy (20) almost everywhere (in time) in the sense of Lebesgue measure. Further, we consider (unique) solutions of (21) that are *slow*, i.e., at each t, Λ satisfies (21b) and is such that $\dot{x}(t)$ is of minimal norm, cf. [Brogliato et al., 2006].

Let $\bar{x} \in \mathcal{X}$ be arbitrary and fixed for the remainder of the proof. For aesthetic reasons we first consider the case where $\sigma = 1$ in (12d) or (20d) and later we explain how to generalize the convergence result. Consider the function V defined by

$$V(x) = U(\varphi) - (\varphi - \bar{\varphi})^\top \nabla U(\bar{\varphi}) - U(\bar{\varphi}) + \frac{1}{2}||x - \bar{x}||_\tau^2$$

with $\tau = \mathrm{blockdiag}(0, M, \tau_b, \tau_g, \tau_v, \tau_\lambda)$ and the map $\varphi \mapsto U(\varphi) = -\mathbb{1}_m^\top \Gamma \,\mathbf{cos}\, \varphi$. The function V was proposed in [Stegink et al., 2017] to analyze the stability of a power system dynamics coupled with a different market mechanism. Note that $V(\bar{x}) = 0, \nabla V(\bar{x}) = 0$ and, since $\bar{\varphi} \in (-\pi/2, \pi/2)^m$, $\nabla^2 V(\bar{x}) \succeq 0$. Consequently, there exists a compact level set Ψ of V around $\bar{x}$. We show now that the two conditions of Theorem A.1 are satisfied.

Condition (I): For C given in (22) and $d = 0$ the polyhedron (A.2) takes the form

$$K = \{x = \mathrm{col}(\varphi, \omega, b, P_g, v, \lambda) : b \ge 0, P_g \ge 0, v \in [-v^b, v^b]\}.$$

Consequently, for all $x \in \partial K \cap \Psi$ we have

$$x - \nabla V(x) = \begin{bmatrix} \varphi - \nabla U(\varphi) + \nabla U(\bar{\varphi}) \\ \omega - M\omega \\ b - \tau_b(b - \bar{b}) \\ P_g - \tau_g(P_g - \bar{P}_g) \\ v - \tau_v(v - \bar{v}) \\ \lambda - \tau_\lambda(\lambda - \bar{\lambda}) \end{bmatrix} \in K,$$

where the inclusion holds because of the definition of K and the property that the elements $b - \tau_b(b - \bar{b})$, $P_g - \tau_g(P_g - \bar{P}_g)$, and $v - \tau_v(v - \bar{v})$ are convex combinations provided that the time constants belong to the interval $(0, 1]$. We can assume this property for the time constants without loss of generality since otherwise we can consider a scaling of V, say $W(x) = \alpha V(x)$ with $\alpha = 1/\tau_{\max}$ and $\tau_{\max}$ being the maximum of all the diagonal terms in the matrices τ_b, τ_g, τ_v. Then, $x - \nabla W(x) \in K$ for all $x \in \partial K \cap \Psi$. We continue the proof with V as the candidate Lyapunov function.

Condition (II): Since $\bar{x} \in \mathcal{X}$ there exists $\bar{\Lambda}$ such that $F(\bar{x}) + C^\top \bar{\Lambda} = 0$. As a result, for each $x \in K$ we have

$$\begin{aligned}
&\langle \nabla V(x), F(x) \rangle = \langle \nabla V(x), F(x) - F(\bar{x}) - C^\top \bar{\Lambda} \rangle \\
&= (\nabla U(\varphi) - \nabla U(\bar{\varphi}))^\top D^\top \omega \\
&+ \omega^\top(-D(\nabla U(\varphi) - \nabla U(\bar{\varphi})) - A\omega + E_g(P_g - \bar{P}_g)) \\
&+ (b - \bar{b})^\top(P_g - \nabla C^*(b) - \bar{P}_g + \nabla C^*(\bar{b}) - \bar{\mu}_b) \\
&+ (P_g - \bar{P}_g)^\top(E_g^\top(\lambda - \bar{\lambda} + \Xi(D(v - \bar{v}) + E_g(\bar{P}_g - P_g)) \\
&- \omega) - \bar{\mu}_g - b + \bar{b}) + (\lambda - \bar{\lambda})^\top(D(v - \bar{v}) + E_g(\bar{P}_g - P_g)) \\
&+ (v - \bar{v})^\top D^\top(-\lambda + \bar{\lambda} - \Xi(D(v - \bar{v}) - E_g(P_g - \bar{P}_g))) \\
&+ (v - \bar{v})^\top(\bar{\nu}_+ - \bar{\nu}_-) \\
&= -\omega^\top A\omega - (b - \bar{b})^\top(\nabla C^*(b) - \nabla C^*(\bar{b})) \\
&- \|D(v - \bar{v}) - E_g(P_g - \bar{P}_g)\|_\Xi^2 - (b - \bar{b})^\top \bar{\mu}_b \\
&- (P_g - \bar{P}_g)^\top \bar{\mu}_g + (v - \bar{v})^\top(\bar{\nu}_+ - \bar{\nu}_-) \le 0,
\end{aligned} \tag{23}$$

where the inequality holds because C^* is convex, $\bar{b}^\top \bar{\mu}_b = 0, \bar{P}_g^\top \bar{\mu}_g = 0$ and $\bar{\mu}_b, \bar{\mu}_g, b, P_g \geq 0$. Hence, the second condition of Theorem A.1 is satisfied. Note that above we have used the following reasoning

$$\begin{aligned}(v-\bar{v})^\top(\bar{\nu}_+ - \bar{\nu}_-) &= (v - v^b + v^b - \bar{v})^\top(\bar{\nu}_+ - \bar{\nu}_-)\\ &= (v-v^b)^\top \bar{\nu}_+ - (v+v^b)^\top \bar{\nu}_- \leq 0,\end{aligned}$$

since $(v^b - \bar{v})^\top \bar{\nu}_+ = 0, (v^b + \bar{v})^\top \bar{\nu}_- = 0$ and $\bar{\nu}_+, \bar{\nu}_- \geq 0$.

Invariance of Ψ: We note that (23) does not necessarily imply that Ψ is forward invariant. We show this next. Observe that for each x, Λ satisfying $0 \leq Cx \perp \Lambda \geq 0$ we have

$$\begin{aligned}&\langle \nabla V(x), F(x) + C^\top \Lambda\rangle = \langle \nabla V(x), F(x)\rangle\\ &+ \langle \nabla V(x), C^\top \Lambda\rangle \leq \langle \nabla V(x), C^\top \Lambda\rangle \qquad (24)\\ &= (b-\bar{b})^\top \mu_b + (P_g - \bar{P}_g)^\top \mu_g + (v-\bar{v})^\top(\nu_- - \nu_+)\\ &= -\bar{b}^\top \mu_b - \bar{P}_g^\top \mu_g + (\bar{v} - v^b)^\top \nu_+ - (\bar{v}+v^b)^\top \nu_- \leq 0.\end{aligned}$$

Hence, V is non-increasing along trajectories initialized in $K \cap \Psi$ and so, Ψ is forward invariant as it is a level set.

Largest invariant set: Define

$$E = \{x \in K \cap \Psi \ : \ \langle F(x), \nabla V(x)\rangle = 0\}$$

and denote the largest invariant subset of E by $\mathcal{M}$. By (23) we note that each $x \in \mathcal{M}$ satisfies $\omega = 0, (Dv + P_d - E_g P_g) = 0$ and, $b_i = \bar{b}_i > 0$ for each $i \in [n]$ with $\bar{P}_{gi} > 0$ as C_i^* is strictly convex around such $\bar{b}_i$ (note that if $\bar{b}_i = 0$, then $0 = \bar{P}_{gi} - \nabla C_i^*(\bar{b}_i) + \bar{\mu}_{bi} = \bar{P}_{gi} + \bar{\mu}_{bi} > 0$, which results in a contradiction). Now consider a trajectory $t \mapsto x(t)$ that starts in $\mathcal{M}$ and by definition, remains there at all times. For i with $P_{gi} > 0$ since $b_i(t) = \bar{b}_i$ for all t, we get from (12c) that $P_{gi}(t) = \bar{P}_{gi} > 0$ for all t and from (12d) that $b_i(t) = \lambda_k(t) = \bar{\lambda}_k$ for $i \in G_k$ for all t. Since the initial condition of the trajectory was arbitrary, these properties hold for all points in $\mathcal{M}$. Next note that for each $x \in \mathcal{M}$ and $i \in [N]$ with $\bar{P}_{gi} = 0$, we have $\nabla C_i^*(b_i) = \nabla C_i^*(\bar{b}_i) = 0$ by the convexity of C_i. Using the definition of C_i^*, we get $P_{gi} = \bar{P}_{gi} = 0$. That is, $P_{gi} = \bar{P}_{gi}$ for all i. Finally, using all the facts, one concludes that each point in $\mathcal{M}$ is an equilibrium and so $\mathcal{M} \subset \mathcal{X}$. Thus, from the invariance principle, each trajectory initialized in Ψ converges to $\mathcal{X}$. Moreover, from (24), we deduce that $\bar{x}$ is stable. Since this equilibrium has been chosen arbitrarily, we conclude that every point in $\mathcal{X}$ is Lyapunov stable, implying that convergence of the trajectories is to a point.

The proof for $\sigma > 0, \sigma \neq 1$ proceeds in the same way as before except that we appropriately scale the Lyapunov function. Specifically, we define the Lyapunov function V with $\tau = \text{blockdiag}(0, M, \sigma\tau_b, \sigma\tau_g, \sigma\tau_v, \sigma\tau_\lambda) > 0$. □

Theorem 5.4 together with Proposition 5.2 ensure *social welfare-maximization*, i.e., the generators' setpoints converge to the optimizer of the ED problem (the same one the ISO would dispatch had it all the information available), and *incentive-compatibility*, i.e., the generators are willing to produce the optimal generation at the converged bid and do not want to deviate from this bid.

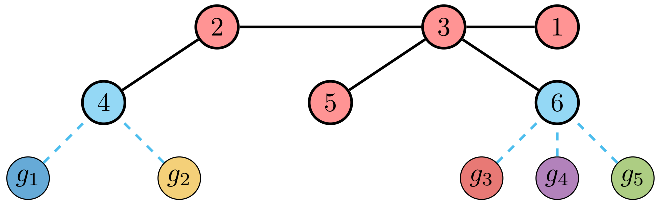


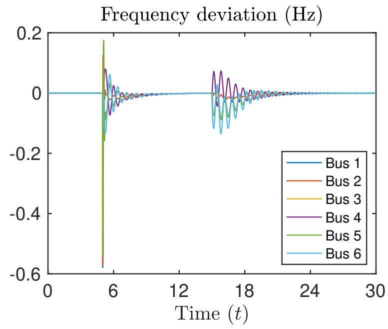
Figure 1. Schematic of a 6-bus power network. Each solid edge represents a transmission line and red nodes represent loads. Nodes 4 and 6 represent generator nodes which are connected to 2 and 3 producers, resp., where the different colors match the ones used in Figures 2b and 2c.

Regarding implementation, we envision that, under our scheme, the market would be cleared when the dynamics settles down and gets close to an efficient Nash equilibrium. The simulations show that the proposed algorithm handles well changes in the load. Hence, for a every consecutive dispatch event, one could continue implementing the algorithm from the previous dispatch setpoint.

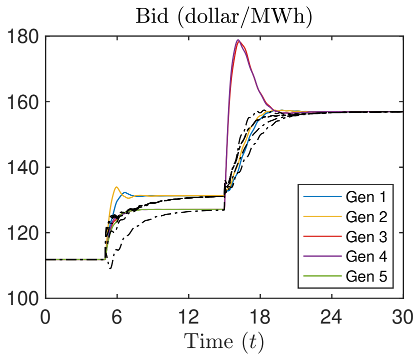
6 Simulations

We simulate the closed-loop dynamics (12) for the modified 6-bus model illustrated in Figure 1. We assume quadratic costs for all generators g_1, g_2, g_3, g_4, g_5 which are of the form $C_i(P_{gi}) = \frac{1}{2} q_i P_{gi}^2 + c_i P_{gi}$ with $(q_1, q_2, q_3, q_4, q_5) = (1.7, 4.6, 4, 5, 3)$ and $(c_1, c_2, c_3, c_4, c_5) = (5, 20, 25, 25, 25)$. In this 6-bus model, nodes 4 and 6 correspond to generator nodes while the other nodes are load nodes and have no power generation. There are two generators g_1, g_2 presents at node 4, and 3 generators g_3, g_4, g_5 present at node 6. The flow capacity of the line $(3, 6)$ is 70 MW and that of all other lines is 200 MW. We choose $M_4 = 5.22, M_6 = 3.98$ for generator nodes and $M_i \ll 1$ for the load nodes. We set $A_i \in [1.4, 2], V_i \in [1, 1.06]$ for all $i \in [n]$, $\Xi = \rho I$ with $\rho = 160$, and $\sigma = 14.1$. Further, we let $(\tau_b, \tau_g, \tau_v, \tau_\lambda) = (0.14\mathbb{1}_5, 0.56\mathbb{1}_5, 0.56\mathbb{1}_5, 0.007\mathbb{1}_6)$. At $t = 0\,$s, the load (in MW's) is given by $P_d = (13.5, 90, 44, 0, 3.3, 0)$. The system (12) is initialized at steady state at the optimal generation level $(P_{g1}, P_{g2}, P_{g3}, P_{g4}, P_{g5}) = (62.83, 19.96, 21.70, 17.36, 28.94)$. Figure 2 shows the evolution of the system (20). At $t = 5\,$s the loads are increased to $P_d = (16, 93, 47, 8, 4.5, 10)$ and the trajectories converge to a new efficient equilibrium with optimal power generation levels $(P_{g1}, P_{g2}, P_{g3}, P_{g4}, P_{g5}) = (74.27, 24.18, 25.54, 20.43, 34.06)$. Furthermore, at steady state the generators $1, 2, 3, 4, 5$ bid equal to the synchronized Lagrange multiplier (also the respective locational marginal costs) and thus, by Proposition 4.1, we know that this corresponds to an efficient Nash equilibrium. At steady state, the line $(3, 6)$ carries the maximum allowable flow. As seen in Figure 2(e), the virtual and the physical power flow are constrained for this line at 70 MW. As a consequence, the bids of generators g_1 and g_2 are different from the bids of generators g_3, g_4, and g_5, see Figure 2(b).

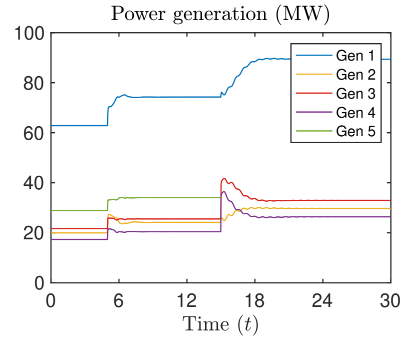
At $t = 15\,$s, generator g_5 stops producing power and exits the market mechanism. This reflects the case of a generator failure. We observe in Figure 2(c) that all the other generators start producing more power to meet the load mismatch. Further, the system converges to the optimal generation $(P_{g1}, P_{g2}, P_{g3}, P_{g4}, P_{g5}) =$



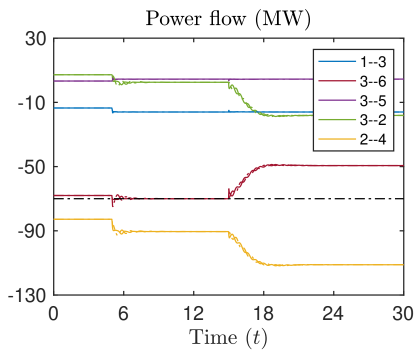
(a) Evolution of the frequency deviation. After each change of the load or generator failure, the frequency is restored to its nominal value.



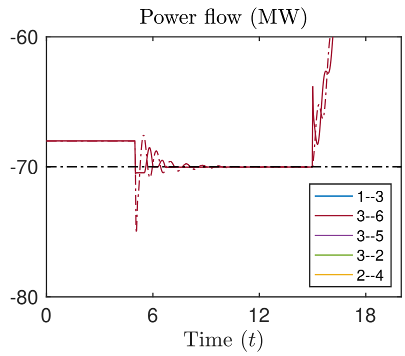
(b) Evolution of bids and Lagrange multipliers (dashed lines). At the second steady-state (14 s) the Lagrange multipliers differ at bus 4 and 6 due to congestion.



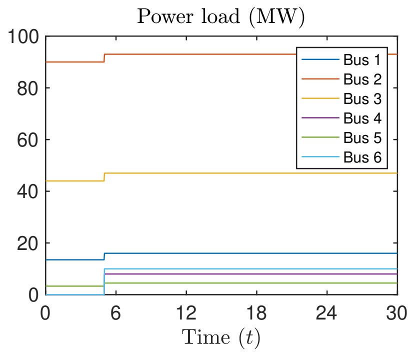
(c) Evolution of the power generation at each node. After the incraese in the load and the generator failure, all other generators increase their production.



(d) Evolution of physical (solid lines) and virtual power flows (dashed lines). The physical power flow tracks the virtual power flow, and the latter is bounded for line $(3, 6)$ at 70 MW.



(e) Evolution of physical (solid lines) and virtual power flows (dashed lines) near the maximal power flow. The virtual power flow remains within the bounds while the physical power may exceed it for a limited amount of time.



(f) Evolution of the piecewise constant power demand. At time 1 s the demand is increased, resulting in an initial frequency drop as observed in Figure 2(a).

Figure 2. Simulations of the interconnection (12) between the ISO-generation bidding mechanism and the power network dynamics. At $t = 5$ s the overall load increases. At $t = 15$ s, generator g_5 fails and stops contributing power to the network. The line colors of plots (b) and (c) match the ones assigned to generator nodes in Figure 1.

$(89.36, 29.76, 32.98, 26.38, 0)$ where no line is congested, as opposed to before where line $(3, 6)$ has the maximum allowable flow of 70 MW, see Figure 2(d). Since no line is congested, the bids of all generators become equal at the equilibrium as seen in Figure 2(b). We note that the generator drop-out does not cause major oscillations in the frequency, as seen in Figure 2(a).

The proposed closed-loop dynamics (12) consists of several tunable design parameters $(\tau_b, \tau_g, \tau_v, \tau_\lambda, \rho, \sigma)$ that might affect the convergence rate of the system. In our example, we have observed that the time constants have a great impact on convergence. We do not report these comparisons in detail due to space constraints.

7 Conclusions

We have studied a market-based power dispatch scheme and its interconnection with the swing dynamic of the physical network. From the market perspective, we have considered a continuous-time bidding scheme that describes the negotiation process between the independent system operator and a group of competitive generators. Using the frequency as a feedback signal in the bidding dynamics, we have shown that the interconnected system provably converges to an efficient Nash equilibrium (where generation levels minimize the total cost) and to zero frequency deviation. This way, competitive generators are enabled to participate in the real-time electricity market without compromising efficiency and stability of the power system. Future work will investigate the effect of the design parameters on the convergence and transient behavior of the system, finite-horizon scenarios incorporating ramp rates and storage assets, more general bidding mechanisms, including scenarios where generators interact simultaneously with multiple ISOs, the incorporation of demand response elements, and objective functions that consider the cost of changing operational set points at the frequency regulation scale. In general, the addition of capacity constraints on generator's production might lead to the lack of existence of efficient Nash equilibrium. However, if this is not the case, we believe the convergence results presented here still hold, and plan to investigate it as part of our future work. We also would like to drop the requirement of edge-disjoint cycles to deal with general undirected networks. Finally, we are interested in scenarios where generators share information among each other to maximize their profits. In such a case, we aim to determine conditions under which collusion can be prevented. Lastly, our swing equations assume positive inertia for all generators and we would like to extend our analysis to inertialess resources, where one would study the interconnection of a differential algebraic system defining the physics with the bidding mechanisms.

A Appendix

Theorem A.1 *(Invariance principle for complementarity systems [Brogliato and Goeleven, 2005]): Consider*

$$\dot{x} = F(x) + C^\top \Lambda, \tag{A.1a}$$
$$0 \leq Cx + d \perp \Lambda \geq 0, \tag{A.1b}$$

with Lipschitz continuous F and let K be the polyhedron

$$K = \{x : Cx + d \geq 0\}. \tag{A.2}$$

Let $\Psi \subset \mathbb{R}^n$ be a compact set and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function such that

$$\textbf{(I)} \quad x - \nabla V(x) \in K, \quad \text{for all } x \in \partial K \cap \Psi,$$
$$\textbf{(II)} \quad \langle \nabla V(x), F(x) \rangle \leq 0, \quad \text{for all } x \in K \cap \Psi.$$

Let $\mathbb{R}^n \supset E := \{x \in K \cap \Psi : \langle F(x), \nabla V(x) \rangle = 0\}$ and denote the largest invariant subset of E by $\mathcal{M}$. Then, for each $x_0 \in K$ such that its orbit satisfies $\gamma(x_0) \subset \Psi$,

$$\lim_{t\to\infty} d(x(t; t_0, x_0), \mathcal{M}) = 0.$$

References

F. L. Alvarado, J. Meng, C. L. DeMarco, and W. S. Mota. Stability analysis of interconnected power systems coupled with market dynamics. *IEEE Transactions on Power Systems*, 16(4):695–701, 2001.

T. Başar and G. J. Oldser. *Dynamic Noncooperative Game Theory*. Academic Press, 1982.

B. Brogliato and D. Goeleven. The Krakovskii-LaSalle invariance principle for a class of unilateral dynamical systems. *Mathematics of Control, Signals and Systems*, 17(1):57–76, 2005.

B. Brogliato, A. Daniilidis, C. Lemaréchal, and V. Acary. On the equivalence between complementarity systems, projected systems and differential inclusions. *Systems & Control Letters*, 55(1):45–51, 2006.

S. T. Cady, A. D. Domínguez-García, and C. N. Hadjicostis. A distributed generation control architecture for islanded AC microgrids. *IEEE Transactions on Control Systems Technology*, 23(5):1717–1735, 2015.

A. Cherukuri and J. Cortés. Iterative bidding in electricity markets: rationality and robustness. *IEEE Transactions on Network Science and Engineering*, 7(3):1265–1281, 2020.

A. Cherukuri, B. Gharesifard, and J. Cortés. Saddle-point dynamics: conditions for asymptotic stability of saddle points. *SIAM Journal on Control and Optimization*, 55(1):486–511, 2017.

F. Dörfler, J. W. Simpson-Porco, and F. Bullo. Breaking the hierarchy: Distributed control and economic optimality in microgrids. *IEEE Transactions on Control of Network Systems*, 3(3):241–253, 2016.

D. Fudenberg and J. Tirole. *Game Theory*. MIT Press, Cambridge, MA, 1991.

R. Goebel. Stability and robustness for saddle-point dynamics through monotone mappings. *Systems & Control Letters*, 108:16–22, 2017.

J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex analysis and minimization algorithms I: Fundamentals*, volume 305. Springer, 2013.

N. Li, C. Zhao, and L. Chen. Connecting automatic generation control and economic dispatch from an optimization view. *IEEE Transactions on Control of Network Systems*, 3(3): 254–264, 2016.

J. Machowski, J. W. Bialek, and J. R. Bumby. *Power System Dynamics: Stability and Control*. John Wiley & Sons, Ltd, second edition, 2008.

D. J. Shiltz, M. Cvetković, and A. M. Annaswamy. An integrated dynamic market mechanism for real-time markets and frequency regulation. *IEEE Transactions on Sustainable Energy*, 7(2):875–885, 2016.

T. Stegink, C. De Persis, and A. van der Schaft. A unifying energy-based approach to stability of power grids with market dynamics. *IEEE Transactions on Automatic Control*, 62(6):2612–2622, 2017.

T. Stegink, A. Cherukuri, C. De Persis, A. van der Schaft, and J. Cortés. Stable interconnection of continuous-time price-bidding mechanisms with power network dynamics. In *Power Systems Computation Conference*, Dublin, Ireland, 2018. electronic proceedings.

T. Stegink, A. Cherukuri, C. De Persis, A. J. van der Schaft, and J. Cortés. Hybrid interconnection of iterative bidding and power network dynamics for frequency regulation and optimal dispatch. *IEEE Transactions on Control of Network Systems*, 6(2):572–585, 2019.

S. Trip, M. Bürger, and C. De Persis. An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages. *Automatica*, 64:240–253, 2016.

A. J. van der Schaft and J. M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, 1998.

X. Zhang and A. Papachristodoulou. A real-time control framework for smart power networks: Design methodology and stability. *Automatica*, 58:43–50, 2015.

M. Zholbaryssov and A. D. Domínguez-García. Convex relaxations of the network flow problem under cycle constraints. *IEEE Transactions on Control of Network Systems*, 2020. To appear.