Encoding Impact of Network Modification on Controllability via Edge Centrality Matrix

Prasad Vilas Chanekar*   Jorge Cortés

Abstract—This paper develops tools to quantify the importance of agent interactions and its impact on global performance metrics for networks modeled as linear time-invariant systems. We consider Gramian-based performance metrics and propose a novel notion of edge centrality that encodes the first-order variation in the metric with respect to the modification of the corresponding edge weight, including for those edges not present in the network. The proposed edge centrality matrix (ECM) is additive over the set of inputs, i.e., it captures the specific contribution to each edge's centrality of the presence of any given actuator. We provide a full characterization of the ECM structure for the class of directed stem-bud networks, showing that non-zero entries are only possible at specific sub/super-diagonals determined by the network size and the length of its bud. We also provide bounds on the value of the trace, trace inverse, and log-det of the Gramian before and after single-edge modifications, and on the edge-modification weight to ensure the modified network retains stability. Simulations show the utility of the proposed edge centrality notion and validate our results.

I. INTRODUCTION

Network control systems find application in a wide range of domains and activities, including social dynamics, energy systems, intelligent transportation, and robotics. In such scenarios, the network must respond efficiently to the inputs of its authorized users while at the same time remaining resilient against external interference or malicious attacks. As inputs are applied at nodes and propagated through the interconnections, understanding the role of each node and each edge on driving the network behavior is key. The complexity of this problem is higher in the edge case than in the nodal case since the number of edges scales quadratically with the number of nodes. To break down this complexity for agent interactions, this paper focuses on providing energy-based edge centrality notions that, given a set of inputs, allow us to quantify the relative impact of individual edges on the network controllability properties.

Literature review

Centrality notions aim to provide a way to quantify the relative importance of nodes and edges in a complex network with respect to a given performance metric, see c.f. [2]–[4]. The predominant focus on the role of nodes and the computational easiness of node-based centrality measures makes these particularly popular in the characterization of network properties. Based on the topological properties of the network,

some commonly used nodal centrality measures include degree [3], [5], closeness [3], betweeness [6], eigenvector [7], Katz [8], PageRank (Google) [9], percolation [10], cross-clique [11], Freeman [5], topological [12], Markov [13], hub and authority [14], routing [15], subgraph [16], and total communicability [17] centralities. In the case of edge centrality, notions include betweeness centrality [18], edge HITS centrality, and edge total communicability centrality [19]. These centrality measures are based on topological considerations and connectivity properties of the network, and in general overlook the role of the dynamics of individual nodes and edges in driving network behavior. As an example, one might argue that a densely connected node with a very slow timescale for its dynamics might play a lesser role than a less densely connected node with a faster dynamics.

Dynamics-based centrality measures encompass fewer notions, mostly limited to node centrality [20]–[23], and are based on performance metrics [24]–[26] based on the spectral properties of the controllability Gramian [27]. These energy-based metrics include the trace of the Gramian [21], [26], the trace of its inverse [26], its determinant [28], [29], and its minimum eigenvalue [25]. While controllability only captures the ability to steer the network between any pair of states, such metrics quantify the optimal energy required to do so, which allows for a more nuanced accounting of the interplay between topology and dynamics in determining centrality. In the cases of edges, [30] proposes an edge centrality measure with respect to the $\mathscr{H}_2$−norm for networks with continuous-time consensus dynamics having time delays and structured uncertainties. Works [31], [32] characterize networks with diagonal controllability Gramian and also propose pathways to design them for prescribed controllability properties. Our previous work [33] proposes a notion of Gramian-based edge centrality for directed topologies with non-negative weights which encodes their role in energy transmission throughout the network, irrespective of the input location. Finally, the work [34] studies conditions under which edge modifications to networks with positive edge weights do not compromise its stability and derives an upper bound on the allowable perturbation weight. This work also studies the analytical characterization of performance metrics such as coherence ($\mathscr{H}_2$−norm) and robustness ($\mathscr{H}_\infty$−norm) after edge perturbation.

Statement of contributions

We consider networks described by linear discrete-time systems, where the agent-to-agent connectivity is encoded by the system matrix. In our treatment, the network adjacency matrix is not required to be symmetric and the edge weights

can have arbitrary sign. Our first contribution is the explicit computation of the first-order variation of a generalized version of the Gramian matrix with respect to the elements of the adjacency matrix. This result allows us to express the gradients of various Gramian-based performance metrics in a unified, computationally efficient, way, which in turn is the basis for the introduction of the edge centrality matrix (ECM). The notion of ECM is tightly coupled with physically realizable energy-based system properties and provides a measure of the relative importance of each edge, including those not present in the network, on the performance metrics. ECM is additive in the input space and therefore allows to precisely identify the impact of individual network inputs in determining the importance of each edge. Our second contribution is the characterization of the structure of ECM for the family of directed stem-bud networks. Each of these networks is a combination of a line network and ring network, and possesses a diagonal controllability Gramian. We show that non-zero entries of the ECM are only possible at specific sub/super- diagonals determined by the network size and the length of its bud. Such entries correspond to edges not originally present in the stem-bud network whose addition will have the greatest impact on performance. We also establish that edge-weight modifications in the stem do not affect the stability of the resulting network. In our third contribution, we consider networks modified at a single edge with a given weight and provide bounds on the value of the trace, trace inverse, and log-det of the Gramian before and after modification. These bounds allow us to estimate the global optima of these metrics under single-edge modification, something we use in our numerical examples to verify the efficacy of ECM in capturing the most relevant network edges. We also determine a sufficient condition on the amount of change in the edge weight that ensure the network remains stable after modification. This condition is valid for arbitrary stable networks and weights. Finally, we illustrate our results in simulation on a family of 6-node stem-bud networks and 1000 random Erdős-Rényi networks.

Notation

We let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. For $x \in \mathbb{R}$ (resp. $x \in \mathbb{C}$), $|x|$ denotes its absolute value (resp. magnitude). When applied to a vector or matrix, the operation $|\cdot|$ is taken elementwise. For $j \in \{1,\dots,n\}$, $e_j \in \mathbb{R}^n$ is the jth canonical unit vector. By $(\cdot)^\top$ we denote the transpose of a vector or matrix, and by $\|\cdot\|$ its Frobenius norm. Given a square matrix A, we denote its trace, determinant, and spectral radius by $\mathrm{tr}(A)$, $\det(A)$, and $\rho(A)$ resp., and its $(i,j)^{th}$ element by a_{ij}. We use $A \succeq (\succ)0$ to denote that A is a positive semi-definite (definite) matrix, and $A_1 \succeq A_2$ to denote that $A_1 - A_2 \succeq 0$. For a symmetric matrix A, $\lambda_i(A)$ denotes its ith largest eigenvalue and $\lambda_{\min}(A)$ its smallest one. We use I to denote the identity matrix of appropriate dimensions.

II. PROBLEM STATEMENT

Consider a network of n nodes represented by the triplet $\mathscr{G}_A = (\mathscr{V}, \mathscr{E}_A, w_A)$, where $\mathscr{V} = \{1, 2, \dots, n\}$ is the node set,

$\mathscr{E}_A = \{ (i,j) \mid i \in \mathscr{V}, j \in \mathscr{V} \}$ is the edge set, and $w_A : \mathscr{E}_A \mapsto \mathbb{R}$ is a weight function. The pair (i,j) denotes an edge directed from node i to node j, i.e., $i \longrightarrow j$. The weighted adjacency matrix $A = (a_{ji}) \in \mathbb{R}^{n \times n}$ is defined by $a_{ji} = w_A(i,j) \neq 0$ if $(i,j) \in \mathscr{E}_A$, else $a_{ji} = 0$. The network follows the discrete linear time-invariant dynamics,

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \{0, \dots, T-1\}, \quad (1)$$

where $T > 0$ is a finite time horizon, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input vectors respectively. $B = \begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_m \end{pmatrix} \in \mathbb{R}^{n \times m}$ denotes the input matrix. We assume B is known and the pair (A, B) is controllable for $T = n$. The control input u might correspond to a known input specified by the designer, an unknown disturbance, or a malicious input. Note that A is stable if $\rho(A) < 1$.

The controllability of (1) refers to the ability to steer the state from an initial condition $x(0) = x_0$ to any arbitrary final condition $x(T) = x_T$ in T steps by appropriately selecting the control input sequence $\{u(0), u(1), \dots u(T-1)\}$. The controllability of (1) can be assessed in a number of ways [27]. Here, we employ the controllability Gramian,

$$\mathscr{W} = \sum_{t=0}^{T-1} A^t B B^\top A^{t^\top}. \quad (2)$$

The system (A, B) is controllable in T steps if the Gramian $\mathscr{W}$ is positive definite.

Controllability is a qualitative property that, per se, does not capture the input's energy effort required to actually steer the system state. To address this point, one can employ controllability metrics, cf. [25], [26], [31], [35], based on the spectral properties of the Gramian such as $\mathrm{tr}(\mathscr{W})$, $-\mathrm{tr}(\mathscr{W}^{-1})$, $\det(\mathscr{W})$, $\log \det(\mathscr{W})$, and $\lambda_{\min}(\mathscr{W})$ to measure the system performance. The interpretation of all these metrics stems from the fact that the minimum energy required to take a dynamical system from the $0-$state to a desired final state x_f in time T is $x_f^\top \mathscr{W}^{-1} x_f$. For instance, $\mathrm{tr}(\mathscr{W}^{-1})$ has the interpretation of the average control energy when x_f is a zero-mean, unit-variance random target state. If v_i is the eigenvector corresponding to the ith eigenvalue λ_i of $\mathscr{W}$ then, $v_i^\top \mathscr{W}^{-1} v_i = \frac{1}{\lambda_i}$ is the *eigen-energy*, i.e., the minimum energy required to move the system in the direction v_i. For the smallest eigenvalue, $\frac{1}{\lambda_{\min}}$ represents the energy required to steer the system in the most difficult direction. If one considers $y = x$ as the output, then $\mathrm{tr}(\mathscr{W})$ is the square of the $\mathscr{H}_2$-norm of the system and is related to the average controllability in all directions of the state space. This also corresponds to the energy in the output response to an unit impulse input and it is also the expected root mean square value of the output response to a white noise excitation input [36]. The metrics $\det(\mathscr{W})$ and $\log \det(\mathscr{W})$ are related to the volume of the ellipsoid containing the set of states that can be reached by one unit or less energy [26].

Our goal is to study the effect of changes in the network structure on its controllability properties while maintaining the input structure intact. By changes in network structure, we mean modifications of the weights of existing edges or addition of new edges of suitable weight. This has applications in practical problems such as mitigating the effect of malicious

attacks at input nodes or network edges, or suppressing output response at particular nodes caused by malicious inputs. Our analysis is motivated by two complementary situations:

(i) scenarios where we are interested in making the network more easily controllable with respect to the control input nodes,
(ii) scenarios where we seek to make a network more difficult to control with respect to malicious input nodes.

It is possible that both the scenarios occur concurrently, where the set of input nodes is a combination of known control input nodes and malicious ones. Mathematically, these scenarios can be formalized as optimization problems where the objective function corresponds to one of the Gramian-based performance metrics described above and the decision variables correspond to network edge selection along with the corresponding weight allocation, with suitable budget constraints. Such optimization problems are nonlinear and non-convex, and hence computationally challenging. As the size of the network increases, exhaustive search becomes impractical. Our aim here is to develop formal tools to characterize the importance of each network edge and its impact on the system performance metrics. Such tools could potentially be paired with known optimization procedures (e.g., by reducing the search space to the most significant edges) to address the scenarios described above.

III. EDGE CENTRALITY AS A MEASURE OF FIRST-ORDER IMPACT ON NETWORK PERFORMANCE

Here we study the effect of edge weight perturbation on the performance metrics described in Section II. We start by deriving an expression for the gradient of these metrics with respect to edge weights. We then build on this result to introduce a novel notion of edge centrality that aims to capture the importance of individual edges in determining network behavior.

A. Network First-Order Perturbation Analysis

To analyze the effect of perturbing edge weights on the performance metrics, we start by studying the first-order variation, i.e., the gradient of a generalized version of the Gramian matrix with respect to the elements of the adjacency matrix.

Theorem III.1. *(Gradient of scalar function of generalized Gramian with respect to edge weights). Consider $A \in \mathbb{R}^{n \times n}$, a symmetric matrix $P \in \mathbb{R}^{n \times n}$ and $H = (h_1 \ldots h_k \ldots h_m) \in \mathbb{R}^{n \times m}$. For $\phi(A) = \sum_{t=0}^{T-1} A^t H H^\top A^{t\top}$ then*

$$\frac{\partial}{\partial a_{ji}} \operatorname{tr}(P\phi(A)) = 2 \sum_{k=1}^{m} \sum_{t=1}^{T-1} \operatorname{tr}\left(\overline{C}_k^{(t)} \overline{O}_k^{(t)} e_i e_j^\top\right)$$
$$= 2 \sum_{t=1}^{T-1} \operatorname{tr}\left(\overline{C}_H^{(t)} \overline{O}_H^{(t)} e_i e_j^\top\right), \tag{3}$$

where

$$\overline{C}_k^{(t)} = \left(A^{t-1\top} P A^t h_k \quad \cdots \quad A^\top P A^t h_k \quad P A^t h_k\right), \tag{4a}$$

$$\overline{O}_k^{(t)} = \left(h_k \quad A h_k \quad \cdots \quad A^{t-2} h_k \quad A^{t-1} h_k\right)^\top, \tag{4b}$$

$$\overline{C}_H^{(t)} = \left(A^{t-1\top} P A^t H \quad \cdots \quad A^\top P A^t H \quad P A^t H\right), \tag{4c}$$

$$\overline{O}_H^{(t)} = \left(H \quad A H \quad \cdots \quad A^{t-2} H \quad A^{t-1} H\right)^\top. \tag{4d}$$

Proof: The derivative of ϕ with respect to an edge weight can be expressed as

$$\frac{\partial \phi}{\partial a_{ji}} = \sum_{t=1}^{T-1} \left[C_j^{(t)} O_i^{(t)} H H^\top A^{t\top} + A^t H H^\top \left(C_j^{(t)} O_i^{(t)}\right)^\top \right],$$

where we have used [33, Theorem 3.1] and the notation

$$C_j^{(t)} = \left(A^{t-1} e_j \quad A^{t-2} e_j \quad \cdots \quad A e_j \quad e_j\right), \tag{5}$$

$$O_i^{(t)} = \left(e_i \quad A^\top e_i \quad \cdots \quad A^{t-2\top} e_i \quad A^{t-1\top} e_i\right)^\top. \tag{6}$$

Now using $H H^\top = \sum_{k=1}^{m} h_k h_k^\top$ and $\frac{\partial}{\partial a_{ji}} \operatorname{tr}(P\phi(A)) = \operatorname{tr}\left(P \frac{\partial \phi(A)}{\partial a_{ji}}\right)$ gives,

$$\frac{\partial}{\partial a_{ji}} \operatorname{tr}(P\phi(A)) = \operatorname{tr}\left\{ \sum_{k=1}^{m} \sum_{t=1}^{T-1} \left[P C_j^{(t)} O_i^{(t)} h_k h_k^\top A^{t\top} \right. \right.$$
$$\left. \left. + P A^t h_k h_k^\top \left(C_j^{(t)} O_i^{(t)}\right)^\top \right] \right\}.$$

Using cyclic permutations of matrices and $\operatorname{tr}(XY^\top) = \operatorname{tr}(X^\top Y)$,

$$\frac{\partial}{\partial a_{ji}} \operatorname{tr}(P\phi(A)) = 2 \sum_{k=1}^{m} \sum_{t=1}^{T-1} \operatorname{tr}\left(O_i^{(t)} h_k h_k^\top A^{t\top} P C_j^{(t)}\right). \tag{7}$$

Consider

$$O_i^{(t)} h_k = \left(h_k^\top e_i \quad h_k^\top A^\top e_i \quad \cdots \quad h_k^\top A^{t-1\top} e_i\right)^\top \tag{8}$$
$$= \left(e_i^\top h_k \quad e_i^\top A h_k \quad \cdots \quad e_i^\top A h_k\right)^\top = \overline{O}_k^{(t)} e_i.$$

Similarly using $P = P^\top$,

$$h_k^\top A^{t\top} P C_j^{(t)} = \left(h_k^\top A^{t\top} P A^{t-1} e_j \quad \cdots \quad h_k^\top A^{t\top} P e_j\right) \tag{9}$$
$$= \left(e_j^\top A^{t-1\top} P A^t h_k \quad \cdots \quad e_j^\top P A^t h_k\right) = e_j^\top \overline{C}_k^{(t)}.$$

Using (8) and (9) in (7) yields

$$\frac{\partial}{\partial a_{ji}} \operatorname{tr}(P\phi(A)) = 2 \sum_{k=1}^{m} \sum_{t=1}^{T-1} \operatorname{tr}\left(\overline{O}_k^{(t)} e_i e_j^\top \overline{C}_k^{(t)}\right),$$
$$= 2 \sum_{k=1}^{m} \sum_{t=1}^{T-1} \operatorname{tr}\left(\overline{C}_k^{(t)} \overline{O}_k^{(t)} e_i e_j^\top\right).$$

The result now follows from the expressions in (4). ∎

One can interpret Theorem III.1 as an extension of the result in [33, Theorem 3.1] for the gradient of the Gramian with respect to an edge weight. The presence of the arbitrary symmetric matrix P in (3) provides greater versatility, and in fact leads to a unified way of computing the gradients of the various Gramian-based performance metrics, as we show next.

Corollary III.2. *(Gradient of performance metrics). Consider the network dynamics* (1). *For $P \in \mathbb{R}^{n \times n}$ symmetric, let*

$$\Theta_P = \sum_{k=1}^{m} \sum_{t=1}^{T-1} \overline{C}_k^{(t)} \overline{O}_k^{(t)} \tag{10}$$

with $\overline{C}_k^{(t)}$, $\overline{O}_k^{(t)}$ defined in Theorem III.1 with $H = B$. Then,

(i) $\frac{\partial}{\partial a_{ji}} \operatorname{tr}(\mathscr{W}) = 2\Theta_I(j,i)$;

(ii) $\frac{\partial}{\partial a_{ji}} \log \det(\mathscr{W}) = 2\Theta_{\mathscr{W}^{-1}}(j,i)$;

(iii) $\frac{\partial}{\partial a_{ji}} \{-\operatorname{tr}(\mathscr{W}^{-1})\} = 2\Theta_{\mathscr{W}^{-2}}(j,i)$;

(iv) $\frac{\partial}{\partial a_{ji}} \lambda_i(\mathscr{W}) = 2\Theta_{V_i}(j,i)$, where $V_i = v_i v_i^\top$ and v_i is the eigenvector of $\mathscr{W}$ corresponding to the ith largest eigenvalue $\lambda_i(\mathscr{W})$.

Proof: We prove the result by making repeated use of Theorem III.1 with ϕ defined by the choice $H = B$ and considering different P as necessary. We make use of the following matrix properties. For X, Y, Z,

$$\operatorname{tr}(Xe_ie_j^\top) = X(j,i)$$
$$\operatorname{tr}(XYZ) = \operatorname{tr}(ZXY) = \operatorname{tr}(YZX),$$
$$\frac{\partial}{\partial y_{ji}} \operatorname{tr}(XY) = \operatorname{tr}\left(X \frac{\partial Y}{\partial y_{ji}}\right).$$

Now, the proof of each item follows by combining these facts with the following choices. Case *(i)* follows readily by considering $P = I$. For case *(ii)*, from [37, Appendix A],

$$\frac{\partial}{\partial a_{ji}} \log \det(\mathscr{W}) = \operatorname{tr}\left(\mathscr{W}^{-1} \frac{\partial \mathscr{W}}{\partial a_{ji}}\right).$$

This case then follows by looking at $P = \mathscr{W}^{-1}$ as a constant matrix that does not change with A. Similarly, for case *(iii)*, from [37, Appendix A],

$$\frac{\partial}{\partial a_{ji}} \{-\operatorname{tr}(\mathscr{W}^{-1})\} = \operatorname{tr}\left(\mathscr{W}^{-1} \frac{\partial \mathscr{W}}{\partial a_{ji}} \mathscr{W}^{-1}\right) = \operatorname{tr}\left(\mathscr{W}^{-2} \frac{\partial \mathscr{W}}{\partial a_{ji}}\right).$$

This case then follows by looking at $P = \mathscr{W}^{-2}$ as a constant matrix that does not change with A. Finally, for case *(iv)*, the eigenvalue equation, $\mathscr{W}v_i = \lambda_i(\mathscr{W})v_i$. As $\mathscr{W}$ is real and symmetric, we use the fact that $v_i^\top v_i = 1$ repeatedly to first express $\lambda_i(\mathscr{W}) = v_i^\top \mathscr{W} v_i$ and then

$$\frac{\partial}{\partial a_{ji}} \lambda_i(\mathscr{W}) = v_i^\top \frac{\partial \mathscr{W}}{\partial a_{ji}} v_i = \operatorname{tr}\left(v_i v_i^\top \frac{\partial \mathscr{W}}{\partial a_{ji}}\right).$$

This case then follows by looking at $P = v_i v_i^\top = V_i$ as a constant matrix that does not change with A. ∎

Note that Corollary III.2(iv) with $i = n$ corresponds to the smallest eigenvalue of the Gramian $\mathscr{W}$.

Remark III.3. *(Computational effort in gradient computation).* Corollary III.2 (i)-(iii) extend to graphs with arbitrary edge weights the result in our previous work [33, Corollary 3.2], which provides different, equivalent expressions for digraphs with positive edge weights. There is an additional key difference in the computation requirements of each expression, which are significantly lighter here. To compute the gradient with respect to an edge weight, we need to calculate for $t \in \{1, \ldots, T-1\}$ the matrices $\overline{C}_k^{(t)}$, $\overline{O}_k^{(t)}$ given in (4) according to Corollary III.2, and the matrices $C_j^{(t)}$, $O_i^{(t)}$ in (5) according to [33, Corollary 3.2]. Both sets of computations require approximately the same effort. However, with the approach here, the summation over k from 1 to m (which is independent

of n) yields the gradient of the desired performance matrix with respect to all n^2 edge weights. Instead, to obtain the latter with the approach in [33], one has to perform a computation like this n^2 times (by considering the combinations across $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$), resulting in a significantly higher computational effort as n increases. ●

Finally, we note here that the expression (2) of the Gramian, along with Theorem III.1 and Corollary III.2, are valid regardless of the stability of A. When considering stable networks, in the limit $T \to \infty$, the controllability Gramian $\mathscr{W}^\infty$ becomes the solution of the discrete-time Lyapunov equation,

$$A\mathscr{W}^\infty A^\top - \mathscr{W}^\infty + BB^\top = 0. \tag{11}$$

In fact, this equation has a valid solution only if A is stable, cf. [27]. As the time horizon T grows to infinity, one has $\|\mathscr{W}^\infty - \mathscr{W}\| \longrightarrow 0$. For such cases, one could compute the gradient of the Gramian with respect to edge weights by differentiating (11) to obtain

$$A\frac{\partial \mathscr{W}^\infty}{\partial a_{ji}}A^\top - \frac{\partial \mathscr{W}^\infty}{\partial a_{ji}} + \frac{\partial A}{\partial a_{ji}}\mathscr{W}^\infty A^\top + A\mathscr{W}^\infty \left(\frac{\partial A}{\partial a_{ji}}\right)^\top = 0. \tag{12}$$

To compute the gradient of any performance metric, we need to solve (11) and (12) for n^2 number of edges. In contrast, Corollary III.2 offers a computationally efficient way to compute the gradient of various performance metrics and given input set using simple matrix multiplications irrespective of the stability of A.

B. The Edge Centrality Matrix

Here we introduce the edge centrality matrix as a way of capturing the importance of the network connections in driving its behavior and performance. Corollary III.2 shows that, with an appropriate choice of the symmetric matrix P, the gradient of various performance metrics can be expressed by means of Θ_P in (10). Given a performance metric, we refer to

$$\Theta_P^k = \sum_{t=1}^{T-1} \overline{C}_k^{(t)} \overline{O}_k^{(t)},$$

as the *edge centrality matrix associated to the kth-input* and to

$$\Theta_P = \sum_{k=1}^{m} \Theta_P^k, \tag{13}$$

as the *edge centrality matrix* (ECM) for the network dynamics (1). Given the energy interpretations associated to these notions, the ECMs exactly encode the first-order changes in physically realizable quantities whenever a network structure is perturbed. The additivity property reflected in (13) is particularly noteworthy, because it captures the specific contribution to each edge's centrality of the presence of actuator $k \in \{1, \ldots, m\}$ in the network. This offers the system designer flexibility to examine the effects of edge perturbation due to individual inputs, a subset of inputs, or the complete set of inputs. Both properties, the one-to-one correspondence with first-order changes in the performance metric and the ability to

pinpoint the impact of each actuator, are significant advantages of the ECM concept over the edge centrality metric proposed in our previous work [33], whose construction is also based on the controllability Gramian.

IV. EDGE CENTRALITY MATRIX OF STEM-BUD NETWORKS

We are interested in characterizing the structure of the ECM. The complexity of this goal is daunting for general networks, so here we focus our attention on the particular class of directed stem-bud networks, cf. [31]. These networks possess a diagonal controllability Gramian, which significantly facilitates the study of their ECM.
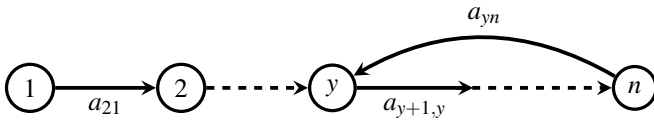

Fig. 1: A directed stem-bud network with n nodes and y as the junction.

A directed stem-bud network is a combination of a directed line network, the *stem* from node 1 to node y, with a directed ring network, the *bud* starting and ending at node y, see Fig. 1. Hence, the stem contains the node sequence $1 \rightarrow 2 \rightarrow \ldots \rightarrow y$ and the bud contains the node sequence $y \rightarrow y+1 \rightarrow \ldots \rightarrow n \rightarrow y$. The node y is called the *junction*. Consequently, the possible non-zero entries in the weighted adjacency matrix of a stem-bud network are $a_{i,i-1}$, with $i = 2,3,\ldots,n$, and a_{yn}. Note that a stem-bud network is simply a directed line network if $a_{yn} = 0$ and a directed ring network if $y = 1$. The network is controllable with only one actuator if it is placed at node 1. When multiple actuators are used, node 1 should be actuated to achieve system controllability. For $1 \leq y \leq n-1$, we let $L_b = n-y+1$ denote the length of the bud and $\Lambda_b = a_{yn}\prod_{j=y+1}^{n} a_{j,j-1}$ its contribution. For convenience, in the directed line network case, we make the convention that $y = 0$ and $L_b = \infty$.

The next result establishes that directed stem-bud networks have diagonal controllability Gramians and is a generalization to arbitrary time horizons T of the result in [31, Section 3] for the case with $T = n$.

Proposition IV.1. *(Controllability Gramian of directed stem-bud networks). Consider a directed stem-bud network without self-loops and input at node $i_b \in \{1,\ldots,n\}$. Let $T > 0$ be the time horizon. Then, the controllability Gramian $\mathscr{W}_{i_b}$ is diagonal.*

Proof: Let $b = e_{i_b}$ and $v_t = A^{t-1}b \in \mathbb{R}^n$ for $1 \leq t \leq T$. Note that, at any given time t, the input reaches only one node, say p. Consequently v_t is a vector with only one non-zero element, denoted $v_t(p)$, at the pth component. We next provide expressions for it, distinguishing between two cases, $1 \leq T \leq n-i_b+1$ and $T > n-i_b+1$,

(i) For $1 \leq T \leq n-i_b+1$, one has $p = i_b+t-1$ and

$$v_t(p) = \prod_{j=k+1}^{p} a_{j,j-1} \text{ for } t \in \{1,2,\ldots,T\}. \quad (14)$$

(ii) For $T > n-i_b+1$, if $1 \leq t \leq n-i_b+1$, then p and $v_t(p)$ are as in (i). If $t > n-i_b+1$, let $t-(n-i_b+2) = \xi L_b+\zeta$, i.e., ξ and ζ are the quotient and remainder, respectively, of dividing $t-(n-i_b+2)$ by L_b. To determine the value of p, note that it takes $n-i_b$ hops to reach node n from node i_b, and then one more hop to reach node y. Of the remaining $t-(n-i_b+2)$ hops left, after traversing the loop ξ times, we reach the ζth node on the loop, and consequently $p = y+\zeta$. The expression for $v_t(p)$ is then

$$v_t(p) = \begin{cases} a_{yn}\Lambda_b^{\xi}\prod_{j=i_b+1}^{n} a_{j,j-1}\prod_{i=y+1}^{p} a_{i,i-1} & \text{for } \zeta > 0, \\ a_{yn}\Lambda_b^{\xi}\prod_{j=i_b+1}^{n} a_{j,j-1} & \text{for } \zeta = 0. \end{cases}$$
$$(15)$$

Consequently, $\mathscr{W}_{i_b} = \sum_{t=0}^{T-1} A^t bb^{\top} A^{t^{\top}} = \sum_{t=1}^{T} v_t v_t^{\top}$ is a diagonal matrix. ∎

For undirected stem-bud networks, the controllability Gramian might in general not be diagonal. As the Gramian is an additive function in the input location space, cf. [26], in case of multiple inputs, one readily deduces from Proposition IV.1 that the Gramian is also diagonal. Building on this result, we next characterize the structure of the ECM of a directed stem-bud network. In our next result, an element q_{ji} of matrix $Q \in \mathbb{R}^{n \times n}$ belongs to the $(i-j)$th super-diagonal if $j < i$, to the $(j-i)$th sub-diagonal if $j > i$, and to the main diagonal if $j = i$.

Theorem IV.2. *(Structure of ECM of stem-bud networks). Consider a controllable $n-$node directed stem-bud network with dynamics* (1) *and m inputs. For the choices of matrix $P \in \mathbb{R}^{n \times n}$ specified in Corollary III.2, the ECM Θ_P may have non-zero elements in*

(i) *the set of sub-diagonals $N_{\text{sub}} = \{1 + iL_b : i \in \{0,\ldots,k_{\text{sub}}\}\}$, with $k_{\text{sub}} = \lfloor \frac{n-2}{L_b} \rfloor$;*

(ii) *the set of super-diagonals $N_{\text{sup}} = \{iL_b - 1 : i \in \{1,\ldots,k_{\text{sup}}\}\}$, with $k_{\text{sup}} = \lfloor \frac{n}{L_b} \rfloor$.*

Proof: Note that since Θ_P is additive in the input space, $\Theta_P = \sum_{i_b=1}^{m} \Theta_P^{i_b}$, cf. (13), it is enough to reason for the single-input case. Consider then the case of an input located at node i_b, i.e., with $b_{i_b} = e_{i_b}$. Note that $A^t e_{i_b}$ is a column vector with only one non-zero value at the $t+i_b$ coordinate. From (10), $\Theta_P^{i_b}$ consists of additive terms of $\overline{C}_{i_b}^{(t)}\overline{O}_{i_b}^{(t)}$ running in $t \in \{1,\ldots,T-1\}$. We consider three cases as follows.

Case 1 ($t < y-i_b$): Relying on (4), consider one general term in the product $\overline{C}_{i_b}^{(t)}\overline{O}_{i_b}^{(t)}$ as $A^{s^{\top}}PA^t e_{i_b}e_{i_b}^{\top}A^{t-1-s^{\top}}$, where s is an integer and $0 \leq s \leq t-1$. In this case, the only non-zero entry of $A^t e_{i_b}$ corresponds to a node $t+i_b$ at the stem since $t < y-i_b$. From Proposition IV.1, the Gramian $\mathscr{W}$ is diagonal, and hence the matrix P in Corollary III.2 is diagonal, and may be singular or non-singular. If $PA^t e_{i_b}$ is non-zero, then it retains the same structure as $A^t e_{i_b}$. In that case, the quantity $A^{s^{\top}}PA^t e_{i_b}$ is a column vector with only one non-zero element at the

$t + i_b - s$ coordinate. Now, $e_{i_b}{}^\top A^{t-1-s^\top}$ results in a row vector with only non-zero element at the $t + i_b - s - 1$ coordinate. So, $A^{s^\top} P A^t e_{i_b} e_{i_b}{}^\top A^{t-1-s^\top}$ is a matrix with only one non-zero element at the $(t + i_b - s, t + i_b - s - 1)$ position, which is part of the first sub-diagonal.

Case 2 $(y - i_b \leq t \leq n - i_b)$: In this case, $A^{s^\top} P A^t e_{i_b}$ results in the input reaching nodes $t + i_b - s$, $t + i_b - s + \zeta L_b$, where $\zeta \in \{1, 2, \ldots, \zeta_{max}\}$ on the stem and node $(n - s + t + i_b - y + \zeta_{max} L_b + 1)$ on bud (here, $\zeta_{max} = \left\lfloor \frac{s - t - i_b + y}{L_b} \right\rfloor$). Now, $e_{i_b}{}^\top A^{t-1-s^\top}$ results in a row vector with only non-zero coordinate at $t + i_b - s - 1$ place. So, $A^{s^\top} P A^t e_{i_b} e_{i_b}{}^\top A^{t-1-s^\top}$ is a matrix with non-zero elements at positions,

(i) $(t + i_b - s, t + i_b - s - 1)$ i.e., first sub-diagonal.
(ii) $(t + i_b - s + \zeta L_b, t + i_b - s - 1)$ i.e., $(\zeta L_b + 1)$ sub-diagonal for $\zeta \in \{1, 2, \ldots, \zeta_{max}\}$.
(iii) $(n - s + t + i_b - y + \zeta_{max} L_b + 1, t + i_b - s - 1)$ i.e., $n - y + \zeta_{max} L_b + 2 = ((\zeta_{max} + 1) L_b + 1)$ sub-diagonal.

Case 3 $(t > n - i_b)$: Proceeding as in Case 2, $A^t e_{i_b}$ will make the input reach node $y + r_1$ where $r_1 = t - y + i_b - \delta_1 L_b$ and $\delta_1 = \left\lfloor \frac{t - y + i_b}{L_b} \right\rfloor$. $A^{s^\top} P A^t e_{i_b}$ will result in nodes $y - s + r_1 + \delta_2 L_b$ where $\delta_2 = \{0, 1, 2 \ldots, \delta_2^{max}\}$, $\delta_2^{max} = \left\lceil \frac{1 + s - y - r}{L_b} \right\rceil$ on the stem. On the bud, the node is $n - r_2$ with $r_2 = s - r_1 - 1 - \delta_3 L_b$ and $\delta_3 = \left\lfloor \frac{s - r - 1}{L_b} \right\rfloor$. $e_{i_b}{}^\top A^{t-1-s^\top}$ is the node $y + r_3$ where $r_3 = t - 1 - s - y + i_b - \delta_4 L_b$ and $\delta_4 = \left\lfloor \frac{t - 1 - s - y + i_b}{L_b} \right\rfloor$. So, $A^{s^\top} P A^t e_{i_b} e_{i_b}{}^\top A^{t-1-s^\top}$ is a matrix with non-zero elements at positions,

(i) $(t + i_b - s - \delta_1 L_b + \delta_2 L_b, t - 1 + i_b - s - \delta_4 L_b)$ i.e., $((\delta_2 + \delta_4 - \delta_1) L_b + 1)$ sub-diagonal or $((\delta_1 - \delta_2 - \delta_4) L_b - 1)$ super-diagonal.
(ii) $(n + t + i_b - y - s - \delta_1 L_b + 1 + \delta_3 L_b, t - 1 + i_b - s - \delta_4 L_b)$ i.e., $((\delta_3 + \delta_4 + 1 - \delta_1) L_b + 1)$ sub-diagonal or $((\delta_1 - \delta_3 - 1 - \delta_4) L_b - 1)$ super-diagonal.

In all the three cases, we find the non-zero elements belong to the sub-diagonals $1, L_b + 1, 2L_b + 1, \ldots, kL_b + 1$, with $kL_b + 1 \leq n - 1$, or the super-diagonals $L_b - 1, 2L_b - 1, \ldots, kL_b - 1$, with $kL_b - 1 \leq n - 1$, as stated. ∎

Theorem IV.2 states that for stem-bud networks, the structure of ECM depends only on the size n of the network and the length L_b of its bud. In general, the performance of the network may be improved by suitably modifying its edges and their corresponding weights. However, determining such modifications is a computationally intensive process. The edge centrality matrix offers a criterium to guide the selection of edges to modify by identifying those with higher first-order impact on network performance. All edges corresponding to sub-diagonal elements (except those in the first sub-diagonal) and all super-diagonal elements (except the element at (y, n)) of ECM do not exist in the stem-bud network and are therefore new. Edges corresponding to an element in a sub-diagonal run along the stem. Edges corresponding an element in a super-diagonal form a new cycle in the network.

When a stem-bud network is either a directed line network ($y = 0$) or a directed ring network ($y = 1$), the next result shows that the structure of their respective ECMs is the same as that of the weighted adjacency matrix A.

Corollary IV.3. *(Structure of ECM of directed line and ring networks). For directed line and ring networks, the edge centrality matrix Θ_P has the same structure as the weighted adjacency matrix A.*

Proof: A directed line network corresponds to a stem-bud network with $L_b = \infty$. From Theorem IV.2, $k_{sup} = 0$, meaning that Θ_P has non-zero elements in the first sub-diagonal only. This structure is the same as that of A. A directed ring network corresponds to a stem-bud network with $L_b = n$, which yields $k_{sub} = 0$ and $k_{sup} = 1$. Thus Θ_P has non-zero elements in the first sub-diagonal and the $n - 1$ super-diagonal only, i.e., Θ_P has the same structure as that of A. ∎

V. NETWORK PERFORMANCE AND STABILITY BOUNDS

Here, we examine two complementary aspects regarding network performance and stability motivated by our observations in Section IV. On one hand, we seek to bound the impact on network performance that edge modification might have and to understand to what extent ECM is a good indicator of it. Given the challenges in addressing this question, here we focus on quantifying the impact over an infinite time horizon caused by the modification of a single edge. On the other hand, to ensure that edge modifications do not result in network instability, we characterize bounds on the weight changes for each edge such that the network retains its stability properties.

A. Bounding the Change in Network Performance of Single-Edge Modification

Our interest here lies in quantifying the effect of edge perturbations in stable network systems. We consider the trace, log-det and, trace inverse of the Gramian as performance metrics and provide bounds for each of them for the original (unmodified) network as well as for the network modified at one edge. Consistent with the interpretation of these metrics regarding network controllability, we derive upper bounds for the trace and log-det and a lower bound for the trace inverse. The following notation is useful in our forthcoming discussion. For $w \in \mathbb{R}$, let

$$\mathcal{X} = (I - |A|)^{-1}, \quad \alpha_{ij} = \frac{|w|}{1 - |w| e_i^\top \mathcal{X} e_j}, \quad \alpha = \max_{i \neq j} \alpha_{ij},$$

$$\beta = \max \left\{ \max_{i \neq j} 2 e_i^\top \mathcal{X} e_j, \max_{i \neq j} e_i^\top \mathcal{X} e_j + \max_j \|\mathcal{X} e_j\| \right\}, \quad (16)$$

$$\gamma = \max_k e_k^\top \mathcal{X}^\top \mathcal{X} e_k, \quad \bar{\gamma} = \max_k e_k^\top \mathcal{X} |B| |B|^\top \mathcal{X}^\top e_k.$$

We are ready to state the first result of this section.

Theorem V.1. *(Upper bound on trace of Gramian). Given a network with adjacency matrix A and $\rho(|A|) < 1$, consider the modified network resulting from adding the weight $w \in \mathbb{R}$ to an edge $i \longrightarrow j$, $i \neq j$, such that $\rho(|A| + |w| e_j e_i^\top) < 1$. Then,*

$$\mathrm{tr}(\mathscr{W}_A) \leq \mathrm{tr}(\mathscr{W}_A^\infty) \leq \mathrm{tr}(\mathscr{H}_{\mathcal{X}}), \tag{17a}$$

$$\mathrm{tr}(\mathscr{W}_{A + w e_j e_i^\top}) \leq (1 + \alpha \beta) \mathrm{tr}(\mathscr{H}_{\mathcal{X}}) + \alpha^2 \gamma \bar{\gamma}, \tag{17b}$$

where $\mathscr{H}_{\mathscr{X}} = \mathscr{X}|B||B|^\top\mathscr{X}^\top$.

Proof: To prove (17a), note that since $\mathscr{W}_A^\infty = \sum_{t=0}^\infty A^t BB^\top A^{t^\top}$, one has $\mathscr{W}_A^\infty \succeq \mathscr{W}_A$. As $|A| \geq A$ and $|B| \geq B$,

$$\mathrm{tr}\left(\sum_{t=0}^\infty |A|^t|B||B|^\top|A|^{t^\top}\right) \geq \mathrm{tr}(\mathscr{W}_A^\infty) \geq \mathrm{tr}(\mathscr{W}_A). \quad (18)$$

As $|A| \geq 0$ and $\rho(|A|) < 1$, from [38, Lemma 2.3.3], $\mathscr{X}$ is non-singular and

$$\mathscr{X} = \sum_{t=0}^\infty |A|^t. \quad (19)$$

Now consider,

$$\mathscr{X}|B||B|^\top\mathscr{X}^\top = \sum_{t=0}^\infty |A|^t|B||B|^\top|A|^{t^\top} + \Psi(|A|,|B|), \quad (20)$$

where $\Psi(|A|,|B|)$ is a function of cross-terms involving $|A|$ and $|B|$. Taking trace throughout in (20), we obtain $\mathrm{tr}(\mathscr{X}|B||B|^\top\mathscr{X}^\top) \geq \mathrm{tr}\left(\sum_{t=0}^\infty |A|^t|B||B|^\top|A|^{t^\top}\right)$, which combined with (18) gives us (17a).

Next, we show (17b). Define $\delta A = we_je_i^\top$, $\mathscr{Y} = \left(I - |A| - |w|e_je_i^\top\right)^{-1}$, and $\mathscr{H}_{\mathscr{Y}} = \mathscr{Y}|B||B|^\top\mathscr{Y}^\top$. From (17a),

$$\mathrm{tr}(\mathscr{W}_{A+\delta A}) \leq \mathrm{tr}(\mathscr{W}_{A+\delta A}^\infty) \leq \mathrm{tr}(\mathscr{H}_{\mathscr{Y}}).$$

In addition, from [39, Section 0.7.4], we have

$$\mathscr{Y} = (I-|A|)^{-1} + \frac{|w|(I-|A|)^{-1}e_je_i^\top(I-|A|)^{-1}}{1 - |w|e_i^\top(I-|A|)^{-1}e_j}$$

Therefore, $\mathscr{Y} = \mathscr{X} + \alpha_{ij}\mathscr{X}e_je_i^\top\mathscr{X}$, and hence,

$$\begin{aligned}\mathscr{H}_{\mathscr{Y}} &= \mathscr{X}|B||B|^\top\mathscr{X}^\top + \alpha_{ij}\mathscr{X}|B||B|^\top\mathscr{X}^\top e_ie_j^\top\mathscr{X}^\top \\ &+ \alpha_{ij}\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top \\ &+ \alpha_{ij}^2\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top e_ie_j^\top\mathscr{X}^\top.\end{aligned} \quad (21)$$

Taking trace throughout, using $\mathrm{tr}(U_1U_2) = \mathrm{tr}(U_2U_1)$ repeatedly, and the fact that $\alpha_{ij} \leq \alpha$ from (16),

$$\begin{aligned}\mathrm{tr}(\mathscr{H}_{\mathscr{Y}}) &\leq \mathrm{tr}(\mathscr{X}|B||B|^\top\mathscr{X}^\top) \\ &+ \alpha\,\mathrm{tr}\left(\mathscr{X}|B||B|^\top\mathscr{X}^\top(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top)\right) \\ &+ \alpha^2\,\mathrm{tr}(e_j^\top\mathscr{X}^\top\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top e_i).\end{aligned} \quad (22)$$

Using [40, Proposition 8.4.13], $\mathrm{tr}\left(\mathscr{X}|B||B|^\top\mathscr{X}^\top(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top)\right) \leq \lambda_1(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top)\mathrm{tr}(\mathscr{X}|B||B|^\top\mathscr{X}^\top) = \lambda_1(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top)\mathrm{tr}(\mathscr{H}_{\mathscr{X}})$. From Lemma A.1, $\lambda_1(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top) \leq \max\{2e_i^\top\mathscr{X}e_j, e_i^\top\mathscr{X}e_j + \|\mathscr{X}e_j\|\}$. Noting that $\mathrm{tr}(e_j^\top\mathscr{X}^\top\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top e_i) = e_j^\top\mathscr{X}^\top\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top e_i$, we get from (16) that

$$\begin{aligned}\mathrm{tr}\left(\mathscr{X}|B||B|^\top\mathscr{X}^\top(e_ie_j^\top\mathscr{X}^\top + \mathscr{X}e_je_i^\top)\right) &\leq \beta\,\mathrm{tr}(\mathscr{H}_{\mathscr{X}}), \\ \mathrm{tr}(e_j^\top\mathscr{X}^\top\mathscr{X}e_je_i^\top\mathscr{X}|B||B|^\top\mathscr{X}^\top e_i) &\leq \gamma\bar{\gamma},\end{aligned} \quad (23)$$

which together with (22) yields the result. ∎

Remark V.2. *(Comparison of upper bound of trace of Gramian with the literature).* The work [41] derives a different upper bound for the trace of Gramian as

$$\mathrm{tr}(\mathscr{W}_A) \leq \frac{\mathrm{tr}(BB^\top)}{1 - \lambda_1(AA^\top)}, \quad (24)$$

which is valid under the assumption that $\sqrt{\lambda_1(AA^\top)} < 1$. This restricts its utility as this condition might be violated even if $\rho(|A|) < 1$. In contrast, the upper bound (17a) in Theorem V.1 needs only $\rho(|A|) < 1$ and hence is more generally applicable. The 1000 random Erdős-Rényi networks (used in our numerical simulations, cf. Section VI-B), all have $\sqrt{\lambda_1(AA^\top)} < 1$ and $\rho(|A|) < 1$, and we observe that the upper bound (17a) is tighter than the upper bound (24). ●

Theorem V.1 can be used to lower bound the trace of the inverse of the Gramian. The following result follows by using $\mathrm{tr}(U^{-1}) \geq \frac{n^2}{\mathrm{tr}(U)}$, for $U \succeq 0$, cf. [40, Fact 8.12.2].

Theorem V.3. *(Lower bound on trace inverse of Gramian). Given a network with adjacency matrix A and $\rho(|A|) < 1$, consider the modified network resulting from adding the weight $w \in \mathbb{R}$ to its edge $i \longrightarrow j$, $i \neq j$, such that $\rho(|A| + |w|e_je_i^\top) < 1$. Then,*

$$\mathrm{tr}(\mathscr{W}_A^{-1}) \geq \frac{n^2}{\mathrm{tr}(\mathscr{H}_{\mathscr{X}})},$$

$$\mathrm{tr}(\mathscr{W}_{A+we_je_i^\top}^{-1}) \geq \frac{n^2}{(1+\alpha\beta)\mathrm{tr}(\mathscr{H}_{\mathscr{X}}) + \alpha^2\gamma\bar{\gamma}}.$$

Next, we derive an upper bound for the log-det of Gramian.

Theorem V.4. *(Upper bound on log-det of Gramian). Given a network with adjacency matrix A and $\rho(|A|) < 1$, consider the modified network resulting from adding the weight $w \in \mathbb{R}$ to an arbitrary edge $i \longrightarrow j$, $i \neq j$, such that $\rho(|A| + |w|e_je_i^\top) < 1$. Then,*

$$\log\det(\mathscr{W}_A) \leq \sigma n\log\left(\frac{\mathrm{tr}(\mathscr{W}_A)}{n^{1/\sigma}}\right) \leq \sigma n\log\left(\frac{\mathrm{tr}(\mathscr{H}_{\mathscr{X}})}{n^{1/\sigma}}\right), \quad (25a)$$

with $\sigma = 1$ if $\mathrm{tr}(\mathscr{W}_A) \leq 1$ and $\sigma = 2$ if $1 < \mathrm{tr}(\mathscr{W}_A)$ and

$$\log\det(\mathscr{W}_{A+we_je_i^\top}) \leq \sigma n\log\left(\frac{\tau}{n^{1/\sigma}}\right), \quad (25b)$$

with $\tau := (1+\alpha\beta)\mathrm{tr}(\mathscr{H}_{\mathscr{X}}) + \alpha^2\gamma\bar{\gamma}$ and $\sigma = 1$ if $\tau \leq 1$ and $\sigma = 2$ for $1 < \tau$.

Proof: Let Z be an arbitrary adjacency matrix with Gramian $\mathscr{W}_Z$. We use [42, Theorem 2] to write $\log\det(\mathscr{W}_Z) \leq \phi_1(\phi_2 + \phi_3)$, where

$$\begin{aligned}\phi_1 &= \frac{1}{\tilde{\lambda}\tilde{\omega}^2 - \tilde{\lambda}^2\tilde{\omega}}, \quad \phi_2 = (\mu_1\tilde{\omega}^2 - \mu_2\tilde{\omega})\log(\tilde{\lambda}), \\ \phi_3 &= (\mu_2\tilde{\lambda} - \mu_1\tilde{\lambda}^2)\log(\tilde{\omega}), \\ \tilde{\omega} &= \frac{\tilde{\lambda}\mu_1 - \mu_2}{\tilde{\lambda}n - \mu_1},\end{aligned}$$

for $\mu_1 = \mathrm{tr}(\mathscr{W}_Z)$, $\lambda_1(\mathscr{W}_Z) \leq \tilde{\lambda} \leq \mu_1$ and $\mu_2 = \|\mathscr{W}_Z\|^2$. Direct substitution of the value of $\tilde{\omega}$ and some simplifications lead to

$$\phi_1\phi_2 = \frac{n\mu_2 - \mu_1^2}{\mu_2 + n\tilde{\lambda}^2 - 2\mu_1\tilde{\lambda}}\log(\tilde{\lambda}),$$

$$\phi_1\phi_3 = \frac{(n\tilde{\lambda}-\mu_1)^2}{\mu_2+n\tilde{\lambda}^2-2\tilde{\lambda}\mu_1}\log\left(\frac{\tilde{\lambda}\mu_1-\mu_2}{n\tilde{\lambda}-\mu_1}\right).$$

We look at these as functions of μ_2. From Lemma A.2, $\mu_2 \in \left[\frac{\mu_1^2}{n}, \tilde{\lambda}\mu_1\right]$. Since $\phi_1\phi_3$ is a decreasing function of μ_2, its maximum value occurs at $\mu_2 = \frac{\mu_1^2}{n}$, yielding $n\log(\mu_1) - n\log(n)$. To find the maximum value of $\phi_1\phi_2$, we compute its derivative as $\frac{d}{d\mu_2}(\phi_1\phi_2) = \left(\frac{n\tilde{\lambda}-\mu_1}{\mu_2+n\tilde{\lambda}^2-2\mu_1\tilde{\lambda}}\right)^2\log(\tilde{\lambda})$. If $\log(\tilde{\lambda}) \le 0$, then $\phi_1\phi_2$ is decreasing and has maximum value of 0 at $\mu_2 = \frac{\mu_1^2}{n}$. If $\log(\tilde{\lambda}) > 0$, then $\phi_1\phi_2$ is increasing and has maximum value of $\frac{\mu_1}{\tilde{\lambda}}\log(\tilde{\lambda})$ at $\mu_2 = \tilde{\lambda}\mu_1$. From the above discussion, and from $\tilde{\lambda} \le \mu_1 \le n\tilde{\lambda}$, we deduce

$$\log\det(\mathscr{W}_Z) \le \begin{cases} n\log(\mu_1) - n\log(n) & \text{if } \mu_1 \le 1, \\ 2n\log(\mu_1) - n\log(n) & \text{if } \mu_1 > 1. \end{cases} \quad (26)$$

For $Z = A$, (25a) follows from (26) by using $\text{tr}(\mathscr{W}_A) \le \text{tr}(\mathscr{H}_{\mathcal{X}})$, cf. Theorem V.1. For $Z = A + we_je_i^\top$, (25b) follows from (26) by using $\text{tr}(\mathscr{W}_{A+we_je_i^\top}) \le (1+\alpha\beta)\text{tr}(\mathscr{H}_{\mathcal{X}}) + \alpha^2\gamma\bar{\gamma}$, cf. Theorem V.1. ∎

Note that the upper bound on the log-det of the Gramian in Theorem V.4 depends on the trace of Gramian and the size of the network.

B. Ensuring Stability of the Modified Network

The modification of edges in a network can lead to instability. Here we provide bounds on the edge-weight addition parameter $w \in \mathbb{R}$ that ensures instability does not arise. The next result generalizes [34, Theorem 1] by considering arbitrary edge weight modifications instead of only positive ones.

Theorem V.5. *(Bounds on edge weight perturbation resulting in stable modified network). Consider the network dynamics (1) with a stable adjacency matrix A and let $w \in \mathbb{R}$ be the edge modification on the edge $i \longrightarrow j$. If $\rho(|A|) < 1$ and $\frac{-1}{\mathcal{X}_{ij}} < w < \frac{1}{\mathcal{X}_{ij}}$, then the modified network $A + we_je_i^\top$ is stable.*

Proof: We seek to prove that $\rho(A + we_je_i^\top) < 1$. We know that $A \le |A|$ and $A + we_je_i^\top \le |A| + |w|e_je_i^\top$. Consider $(I - |A| - |w|e_je_i^\top)^{-1}$. Using [39, Section 0.7.4],

$$(I - |A| - |w|e_je_i^\top)^{-1} = \mathcal{X} + \frac{|w|\mathcal{X}e_je_i^\top\mathcal{X}}{1 - |w|e_i^\top\mathcal{X}e_j}. \quad (27)$$

As $\rho(|A|) < 1$, we have $\mathcal{X} \ge 0$ from [43, Theorem 1.2]. So, if $1 - |w|e_i^\top(I - |A|)^{-1}e_j > 0$, then $(I - |A| - |w|e_je_i^\top)^{-1} \ge 0$. Note that $\frac{-1}{\mathcal{X}_{ij}} < w < \frac{1}{\mathcal{X}_{ij}}$ is equivalent to the condition $1 - |w|e_i^\top(I - |A|)^{-1}e_j > 0$. Therefore, in such case, and using [43, Theorem 1.2], we get $\rho(|A| + |w|e_je_i^\top) < 1$. From [39, Theorem 8.1.18], $\rho(A + we_je_i^\top) \le \rho(|A| + |w|e_je_i^\top)$ and the required result follows. ∎

Note that Theorem V.5 is a sufficient condition for general networks, i.e., for $A \in \mathbb{R}^{n\times n}$ and $w \in \mathbb{R}$, but is a necessary and sufficient condition for $A \ge 0$ and $w \ge 0$, cf. [34, Theorem 1]. The element (j,i) of $|A|^t$ represents the sum of the products of the (absolute values of) weights of all paths from node i

to node j of length exactly t, where t is an arbitrary positive integer. Hence, from (19), if there is a path (of any length) from node i to node j, then the (j,i) element of $\mathcal{X}$ is non-zero. From Theorem V.5, the bounds on the modification weight w of edge $i \longrightarrow j$ depend upon the (i,j) element of $\mathcal{X}$, i.e., on the existence of a path from node j to node i. Therefore, a stable network could be made unstable by forming cycles in the network through edge modifications. These observations lead us to state the following result regarding the stability of stem-bud networks after edge modification.

Corollary V.6. *(Stability of modified stem-bud networks). A stable stem-bud network with $A \ge 0$ remains stable if modifications $w \ge 0$ are performed to the weight of edges in its stem.*

Consequently, a directed line network can be made unstable only by adding a new edge (that will create a bud and hence break its pure line structure). A directed ring network can be made unstable by suitably modifying the weight of any of its edges. If all edges of a stem-bud network have equal positive weights and only edges of the bud are modified, then the bounds on the edge weight modification and upper bound on the trace of Gramian depend upon the length of the bud, as shown next.

Theorem V.7. *(Stem-bud networks with equal edge weights). Consider a stable stem-bud network with $A \ge 0$, all edge weights equal to $0 \le a < 1$, and a single input at node 1. We have the following,*

(i) Let $w > 0$ denote the weight modification performed only on the edges of the bud. Then, the modified network is stable if $0 < w < \frac{1-a^{L_b}}{a^{L_b-1}}$;

(ii) $\text{tr}(\mathscr{W}_A) \le \sum_{k=0}^{y-2} a^{2k} + \sum_{k=y-1}^{n-1} \frac{a^{2k}}{(1-a^{L_b})^2}.$

Proof: To show the result, we rely on Theorems V.5 and V.1. Before invoking them, we compute the expression for the elements of $\mathcal{X} = (\mathcal{X}_{ij})$. Since $A \ge 0$ and $\rho(A) < 1$ with $0 \le a < 1$, we have from (19) that $\mathcal{X} = \sum_{t=0}^\infty A^t$. Note that the ijth element of A^t corresponds to the product of weights in the path of length t (if it exists) from node j to node i. Any node i in the stem ($1 \le i < y$) can only be reached by the input once while any node i in the bud ($y \le i \le n$) can be reached multiple times. Therefore, for $1 \le i < y$, we have

$$\mathcal{X}_{ij} = \begin{cases} 0 & \text{for } i < j, \\ a^{i-j} & \text{for } i \ge j, \end{cases}$$

and for $y \le i \le n$, we have

$$\mathcal{X}_{ij} = \begin{cases} \sum_{k=0}^\infty a^{L_b+i-j}a^{kL_b} = \frac{a^{L_b+i-j}}{1-a^{L_b}} & \text{for } y \le i < j, \\ \sum_{k=0}^\infty a^{i-j}a^{kL_b} = \frac{a^{i-j}}{1-a^{L_b}} & \text{for } j \le i \le n, \end{cases}$$

where we have used the infinite geometric series formula $\sum_{k=0}^\infty a_1 r^k = \frac{a_1}{1-r}$, for $|r| < 1$.

To establish (i), we resort to Theorem V.5. Since only edges in the bud are modified, to ensure stability it is enough to enforce that $|w|$ is smaller than $1/\mathcal{X}_{i-1,i}$, for $y+1 \le i \le n$, and $1/\mathcal{X}_{ny}$. The minimum of these values is $\frac{1-a^{L_b}}{a^{L_b-1}}$, and the result

follows. Regarding (ii), since there is only one input at node 1, $B = e_1$, and hence $\mathscr{X}B$ corresponds to the first column of $\mathscr{X}$. Therefore, $\mathrm{tr}(\mathscr{H}_{\mathscr{X}}) = \sum_{k=0}^{y-2} a^{2k} + \sum_{k=y-1}^{n-1} \frac{a^{2k}}{(1-a^{L_b})^2}$, and the result follows from Theorem V.1. ■

VI. NUMERICAL EXAMPLES

Here, we provide two sets of simulations to illustrate the efficacy of the proposed edge centrality measure and the bounds derived for the performance metrics. All the simulations are performed using MATLAB on a desktop with Intel core-i7-8700, 3.20 GHz processor with 16 GB of RAM. We consider the objective function f as the trace and as the log-det of the Gramian and use the following notation in both examples: f_I denotes the value of f for the original network; f_{EC} denotes the value of f for the network after modification of the edge with the best edge centrality; and f_{EX} denotes the global maximum of f computed by exhaustive search of all single-edge modifications.

A. 6-Node Stem-Bud Networks

Consider a family of 6-node stem-bud networks, all with parameters $a_{21} = 0.9$, $a_{32} = 0.7$, $a_{43} = 0.8$, $a_{54} = 0.6$, and $a_{65} = 0.8$. We consider different networks depending on where the junction node y lies, i.e., $0 \leq y \leq 5$. Figure 2 shows an example with $y = 2$. When $y = 0$, the network is a directed line and when $y = 1$, the network is a directed ring. Whenever a backward edge is present, we set $a_{y6} = 0.7$. We consider actuators at nodes 1 and 3, and take $T = 2n$. In all cases, the controllability Gramian is a diagonal matrix, validating Proposition IV.1.
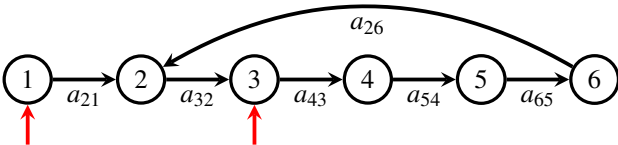
Fig. 2: Example of a 6-node stem-bud network with inputs at node 1 and node 3 with junction at node 2. The structure of ECM for this particular network is shown in Figure 3c.

Following Theorem IV.2, Table I describes the properties of the edge centrality matrix as the junction node y goes from 0 to 5, including the list of sub- and super-diagonals having non-zero elements. As y moves towards the end node n, the bud length decreases, but more edges (existing as well as non-existing) become influential. Note that the structure of the ECMs is the same irrespective of the number of inputs and the performance metric. Figure 3 shows the corresponding sparsity patterns of ECMs for each network.

We analyze the improvements in network controllability with a single edge modification (excluding self-loops). For each possible junction node y, we first compute $\mathscr{X}$ and determine its largest non-diagonal element. According to Theorem V.5, this determines the maximum allowable perturbation weight, denoted $w_{\max}$, which ensures stability of the modified

Junction node y		0	1	2	3	4	5
Bud length L_b		∞	6	5	4	3	2
Sub-diagonals	k_{sub}	0	0	0	1	1	2
	N_{sub}	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,5\}$	$\{1,4\}$	$\{1,3,5\}$
Super-diagonals	k_{sup}	0	1	1	1	2	3
	N_{sup}	$\emptyset$	$\{5\}$	$\{4\}$	$\{3\}$	$\{2,5\}$	$\{1,3,5\}$

TABLE I: Structure of ECM for a 6-node stem-bud network as a function of the junction node location. Columns correspond, in order, to plots (a)-(f) in Figure 3.
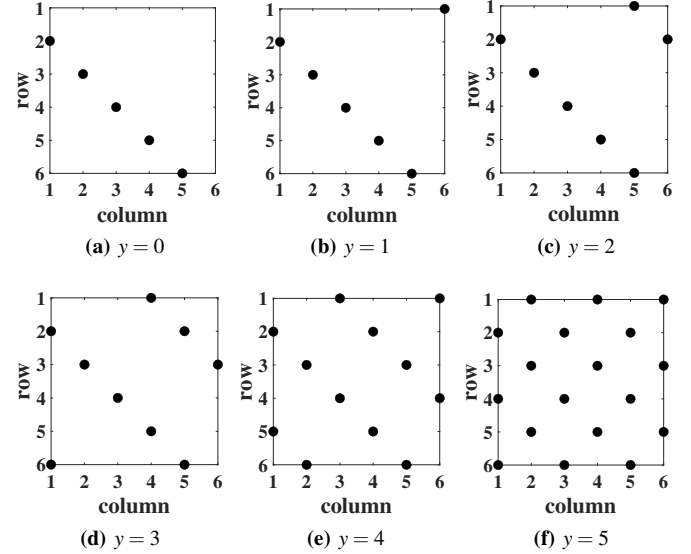
Fig. 3: Sparsity pattern of ECM for a 6-node stem-bud network as a function of junction node location with either trace or log-det as performance metric. '•' represents a non-zero element.

network. We set $w = 0.99 w_{\max}$ as the edge modification weight and exhaustively search for the global solution, including the edges not present in the network. Table II shows the results of the comparison with the solution obtained by modifying the edge with the best edge centrality. We observe that, for the trace of the Gramian, the global solution and the best edge centrality solution are the same except for the directed line and when the junction node is at 3 or 4. For the log-det case, the best edge centrality solution matches the global solution for all the junction nodes. This shows the usefulness of the proposed edge centrality notion even when the modification weight is large.

Metric		y 0	1	2	3	4	5
w		1.1	0.91	0.89	0.90	0.82	0.55
Trace	f_I	4.63	4.90	4.83	4.84	4.81	4.87
	f_{EC}	11	11.7	11.4	13	9.8	7.9
	f_{EX}	16	11.7	11.4	14.3	11.6	7.9
Log-det	f_I	-2.7	-2.39	-2.44	-2.4	-2.39	-1.96
	f_{EC}	2.5	3.2	2.9	3.2	3	1.2
	f_{EX}	2.5	3.2	2.9	3.2	3	1.2

TABLE II: Improvement in performance metrics with single-edge modification for a 6-node stem-bud network as a function of junction node location. w denotes the edge modification weight.

B. Random Erdős-Rényi Networks

Here, we show the efficacy of ECM and the utility of the performance metric bounds obtained in Section V-A on 1000 random Erdős-Rényi (ER) networks [44] without self-loops. We consider networks with $n = 100$ nodes, $m = 30$ input nodes, and edge probability of 0.35. For uniformity, the spectral radius $\rho(A)$ of each ER network belongs to the interval $(0.85, 0.90)$. The time horizon is $T = n$ and we neglect self-loops. Using Theorem V.5, we compute 3.82 as the bound on the edge modification weight beyond which stability of the random ER networks cannot be guaranteed. We compute the improvement f_{EC} in the performance metrics obtained by selecting the edge with the best centrality, and compare this improvement with that obtained using exhaustive search, f_{EX}.

Metric \ w		0.5	1	1.5	2	2.5	3	3.5
Avg $|f_I|$		32	32	32	32	32	32	32
% $\frac{f_{XC}}{|f_I|}$	Worst	0.06	0.32	0.92	2.16	4.80	10.4	23.8
	Best	0	0	0	0	0	0	0
	Avg	0.00	0.04	0.18	0.61	1.63	3.91	9.17
Avg f_{XC}		0.00	0.01	0.06	0.20	0.51	1.25	2.93
% $\frac{f_{CI}}{|f_I|}$	Worst	1.26	4.26	9.03	15.6	24.0	34.1	46.2
	Best	1.90	5.97	12.6	22.5	37.5	61.5	108
	Avg	1.57	5.08	10.6	18.4	28.6	41.9	59.1
Avg f_{CI}		0.50	1.63	3.39	5.90	9.15	13.4	18.9
Avg % $\frac{|f_{EX} - f_g|}{|f_I|}$		1.46	3.12	5.63	7.37	8.91	10.6	14.4

TABLE III: Performance analysis with trace of Gramian as objective for 1000 Random Erdős-Rényi networks. Here, $f_{XC} = |f_{EX} - f_{EC}|$ and $f_{CI} = |f_{EC} - f_I|$.

Metric \ w		0.5	1	1.5	2	2.5	3	3.5
Avg $|f_I|$		553	553	553	553	553	553	553
% $\frac{f_{XC}}{|f_I|}$	Worst	0.56	0.67	0.72	0.76	0.77	0.80	0.83
	Best	0	0	0	0	0	0	0
	Avg	0.19	0.24	0.26	0.29	0.30	0.32	0.33
Avg f_{XC}		1.05	1.33	1.44	1.60	1.66	1.77	1.83
% $\frac{f_{CI}}{|f_I|}$	Worst	1.16	1.63	1.95	2.20	2.43	2.62	2.79
	Best	2.59	3.36	3.86	4.24	4.54	4.81	5.03
	Avg	1.88	2.63	3.12	3.50	3.81	4.08	4.32
Avg f_{CI}		10.4	14.5	17.3	19.4	21.0	22.6	23.9
Avg % $\frac{|f_{EX} - f_g|}{|f_I|}$		1.04	1.36	2	1.81	1.49	1.13	0.94

TABLE IV: Performance analysis with log-det of Gramian as objective for 1000 Random Erdős-Rényi networks. Here, $f_{XC} = |f_{EX} - f_{EC}|$ and $f_{CI} = |f_{EC} - f_I|$.

The results are displayed in Table III for trace of Gramian and Table IV for log-det of Gramian. Regarding the performance of the ECM-based solution versus exhaustive search for the trace of Gramian, we observe that, as the modification weight increases, the worst-case value for $\% \frac{f_{XC}}{|f_I|}$ increases. Such behavior is expected as f_{EC} is computed using the ECM according to the first-order effects of edge modification. However the average $\% \frac{f_{XC}}{|f_I|}$ remains significantly lower, less than half the worst case. Regarding improvement, we see that the average $\% \frac{f_{CI}}{|f_I|}$ increases with the modification weight and is more prominent for the trace as objective with an

average more than 50% for $w = 3.5$. In the case of log-det of Gramian as objective, even though the values of f_{XC} and f_{CI} are significant, the percentage values are small as the initial objective value f_I is large. Note that even for higher modification weights, in some cases the use of edge centrality leads to the global solution. Note that the exhaustive search procedure is weight dependent, while the ECM-based one is not. Thus, the computation of the global solution exhaustively involves a significantly larger computational cost than the ECM-based one as the network size increases. For the considered ER networks, the cost of computing the best edge using ECM is approximately one-tenth the cost of exhaustive search for a given weight.

We also illustrate the capability of the network performance bounds to capture the global maxima of the corresponding metric. For the trace of the Gramian, we employ (17b) in Theorem V.1 to compute a global upper bound h_g^{tr}. Similarly, for the log-det of the Gramian, we employ (25b) with $\tau = h_g^{\text{tr}}$ in Theorem V.4 to compute the global upper bound

$$h_g^{\log \det} = 2n \log \left(\frac{h_g^{\text{tr}}}{\sqrt{n}} \right).$$

We observe that these bounds of the true global optimum f_{EX} (obtained through exhaustive search) tend to be conservative, so we refine them by computing estimates f_g^{tr} and $f_g^{\log \det}$ of the value of f corresponding to the global edge modification solution as follows.

We take the 30 edges with the largest ECM. For each of these edges, we modify its weight by $w > 0$ and consider the resulting adjacency matrix $\overline{A}$. We then compute $\mathscr{W}_{\overline{A}}$, $\overline{\mathscr{X}} = (I - \overline{A})^{-1}$, and $\mathscr{H}_{\overline{\mathscr{X}}} = \overline{\mathscr{X}} B B^\top \overline{\mathscr{X}}^\top$. According to (17a) in Theorem V.1, we have $\overline{f} = \text{tr}(\mathscr{W}_{\overline{A}}) \leq \overline{h} = \text{tr}(\mathscr{H}_{\overline{\mathscr{X}}})$. We take all the data pairs $(\overline{f}, \overline{h})$ obtained in this way and fit a curve using the curve-fitting function '*fit*' and model type '*rat55*' of MATLAB. Using the obtained curve-fit we compute the estimate f_g^{tr} corresponding to h_g^{tr}. We run this procedure for different modification weights w, from 0.5 to 3.5 with an increment of 0.5 for all considered ER networks.

We follow the same procedure for the log-det of the Gramian, computing the appropriate bounds, now resorting to Theorem V.4, as follows. As the trace of the Gramian for all 1000 ER networks is greater than 1, we use (25a) with $\sigma = 2$. For each of the 30 edges with the largest ECM we obtain $\overline{f} = \log \det(\mathscr{W}_{\overline{A}}) \leq \overline{h} = 2n \log \left(\frac{\text{tr}(\mathscr{H}_{\overline{\mathscr{X}}})}{\sqrt{n}} \right)$. As in the case of the trace, we take all the data pairs $(\overline{f}, \overline{h})$ obtained in this way and fit a curve, which we use to compute the estimate $f_g^{\log \det}$ corresponding to $h_g^{\log \det}$.

The last rows of Tables III and IV list the average percentage values of the error, relative to the initial metric value, between the global solution computed by exhaustive search and the estimate of the global solution obtained as described above. This error is less than $\approx 15\%$ for the trace of Gramian and less than $\approx 2\%$ for the log-det of Gramian. Note that, for creating the estimate of the global solution, we have used just 30 of 9900 edges, i.e., 0.3% of the total number of edges. For the trace of Gramian, the error between the estimate and the

actual global solution increases with the modification weight. This may be due to the nonlinear dependence of h_g^{tr} on the modification weight w.

VII. CONCLUSIONS

We have studied network systems modeled as controlled linear-time invariant systems and addressed the question of characterizing the importance for controllability of individual agent connections. We have considered a suite of performance metrics based on the spectral properties of the associated controllability Gramian and formally characterized the effect on them of perturbing the weights of all possible edges, including those not present in the network. This analysis has led us to propose a novel notion of edge centrality as a way of measuring the first-order variation in network performance. The edge centrality matrix (ECM) encodes important physically realizable quantities and is additive on the set of inputs, meaning that it captures the specific contribution to each edge's centrality of the presence of any given actuator. We have fully characterized the structure of ECM for the class of directed stem-bud networks, which possess a diagonal controllability Gramian, and shown that it only depends on the network size and the length of its bud, with possible non-zero entries only at specific sub/super-diagonals. Finally, given an edge modification weight, we have developed novel bounds on the value of the trace, trace inverse, and log-det of the Gramian before and after single-edge modifications. We have also determined bounds on the weight that ensure the resulting modified network remains stable. Numerical examples illustrate the usefulness of the proposed edge centrality notion and the derived results. Future work will employ the proposed notions in the algorithmic synthesis of stable networks via edge modification with enhanced guarantees on convergence and controllability and the mitigation of the effect of malicious attacks in network strategic scenarios, explore the design of distributed schemes for the computation of the proposed edge centrality measures, and develop bounds of the impact on network performance of multiple-edge modifications.

APPENDIX

Lemma A.1. *(Upper bound on largest eigenvalue of sum of two rank-one matrices). For $v \in \mathbb{R}^n$, let $U = ve_i^\top + e_i v^\top$. Then,*

$$\lambda_1(U) = 2v^\top e_i \qquad \text{if } \operatorname{rank}(U) = 1,$$
$$\lambda_1(U) \le v^\top e_i + \|v\| \qquad \text{if } \operatorname{rank}(U) = 2.$$

Proof: Denote $\operatorname{rank}(U) = r_U$. From [45, Theorem 2.1],

$$\lambda_1(U) \le \frac{\operatorname{tr}(U)}{r_U} + \sqrt{\left(r_U - 1\right)\left(\frac{\operatorname{tr}(U^2)}{r_U} - \left(\frac{\operatorname{tr}(U)}{r_U}\right)^2\right)}. \quad (28)$$

Now,

$$\operatorname{tr}(U) = \operatorname{tr}(ve_i^\top) + \operatorname{tr}(e_i v^\top) = 2v^\top e_i,$$
$$\operatorname{tr}(U^2) = 2(v^\top e_i)^2 + 2v^\top v. \quad (29)$$

As $\operatorname{rank}(ve_i^\top) = \operatorname{rank}(e_i v^\top) = 1$, from [39, Section 0.4.5], $\operatorname{rank}(U) \le 2$. Using (29) and $\|v\| = \sqrt{v^\top v}$ in (28) for $r_U = 1$ and $r_U = 2$ gives the required result. $\blacksquare$

Lemma A.2. *(Bounds on norm of positive definite matrix). Let $\mathscr{L}$ be a $n \times n$ symmetric positive definite matrix and $\tilde{\lambda} \ge \lambda_1(\mathscr{L})$. Then,*

$$\frac{(\operatorname{tr}(\mathscr{L}))^2}{n} \le \|\mathscr{L}\|^2 \le \lambda_1(\mathscr{L})\operatorname{tr}(\mathscr{L}) \le \tilde{\lambda}\operatorname{tr}(\mathscr{L}).$$

Proof: From [40, Proposition 8.4.13], $\|\mathscr{L}\|^2 = \operatorname{tr}(\mathscr{L}^2) \le \lambda_1(\mathscr{L})\operatorname{tr}(\mathscr{L}) \le \tilde{\lambda}\operatorname{tr}(\mathscr{L})$. From [40, Fact 8.12.1], $\frac{(\operatorname{tr}(\mathscr{L}))^2}{n} \le \operatorname{tr}(\mathscr{L}^2) = \|\mathscr{L}\|^2$. $\blacksquare$

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Jorge Cortés (M'02, SM'06, F'14) received the Licenciatura degree in mathematics from Universidad de Zaragoza, Zaragoza, Spain, in 1997, and the Ph.D. degree in engineering mathematics from Universidad Carlos III de Madrid, Madrid, Spain, in 2001. He held postdoctoral positions with the University of Twente, Twente, The Netherlands, and the University of Illinois at Urbana-Champaign, Urbana, IL, USA. He was an Assistant Professor with the Department of Applied Mathematics and Statistics, University of California, Santa Cruz, CA, USA, from 2004 to 2007. He is currently a Professor in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA, USA. He is the author of Geometric, Control and Numerical Aspects of Nonholonomic Systems (Springer-Verlag, 2002) and co-author (together with F. Bullo and S. Martínez) of Distributed Control of Robotic Networks (Princeton University Press, 2009). He is a Fellow of IEEE and SIAM. His current research interests include distributed control and optimization, network science, nonsmooth analysis, reasoning and decision making under uncertainty, network neuroscience, and multi-agent coordination in robotic, power, and transportation networks.

Prasad Vilas Chanekar received the undergraduate degree in Mathematics and Computing from the Indian Institute of Technology, Kharagpur, India. He then obtained M.Sc. and Ph.D. degrees in Mechanical Engineering from the Indian Institute of Science, Bangalore, India and the University of Maryland, College Park, USA respectively. He is currently a Post-Doctoral researcher in the Department of Mechanical and Aerospace Engineering at the University of California, San Diego, USA. His research interests include dynamics, control and optimization of large-scale engineering systems.