Informativity for centralized design of distributed controllers for networked systems

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Abstract—**Recent work in data-driven control has led to methods that find stabilizing controllers directly from measurements of an unknown system. However, for multi-agent systems we are often interested in finding controllers that take their distributed nature into account. For instance, the full state might not be available for feedback at every agent. In order to deal with such information, we consider the problem of finding a feedback controller with a given block structure based on measured data. Moreover, we provide an algorithm that, if it converges, leads to a maximally sparse controller.**

I. Introduction

In this paper, we consider the problem of finding distributed controllers on the basis of measurements of an unknown system. Such data-driven control problems have garnered a lot of attention recently, both from the viewpoints of control theory and learning. A particularly recent development is based on the works by Willems et al. in [1] and Markovsky and Rapisarda in [2]. These works have shifted the focus from the two-step approach of system identification combined with model based control towards designing controllers directly from the data.

To be precise, we are interested in finding controllers for multi-agent systems in the situation where the state matrix is completely unknown. To compensate for this lack of knowledge, we assume that we have access to measurements of the input and the corresponding state collected over a finite time window. In this paper, we take the viewpoint of the informativity framework, introduced in [3]. This means that we find a controller for the measured system by finding a controller that works for the entire set of systems consistent with the data. In contrast to [3], we do not assume that the measurements are exact, but assume that the noise on this time window satisfies bounds of the form considered in the recent paper [4]. Among the results of [4] are conditions that are necessary and sufficient for the problem of finding a stabilizing controller. These conditions are given in the form of the feasibility of linear matrix inequalities (LMI's), and therefore it is straightforward to check whether they hold.

However, the controllers found by the aforementioned methods are not necessarily *distributed*. That is, each agent might require knowledge of the state of each other agent in order to stabilize the system. As this might be undesirable or even impossible, we develop results that take into account the networked structure of the system. For this, we focus on two different types of problems. First we consider the problem of designing distributed controllers according to a given communication graph. That is, controllers such that agent number i only requires state measurements from specific other agents. In essence, this requires us to find state feedback matrices with a given block structure. After this, we move to the problem of finding controllers with maximal sparsity. Here we assume that the aforementioned communication graph is also available for design, and want to find a controller that guarantees the control objective, yet uses as little communication as possible.

Our contributions are the following:

1) We formulate necessary and sufficient conditions, in the form of linear matrix inequalities in terms of the data, under which the measured system admits a quadratically stabilizing controller. These differ from previously known results in the fact that we assume B is known.
2) Under certain specific assumptions, we show that the existence of such a controller with a given block structure can be checked using linear matrix inequalities as well.
3) We state an algorithm consisting of a repeated convex programming problem. If this algorithm converges, we show that it finds a controller with maximal sparsity.

Proofs are omitted for space reasons and will appear elsewhere.

Literature overview

As mentioned above, data-driven control has garnered a lot of attention recently. Given that it is impossible to give a complete overview of the field, we refer to the survey paper [5] and the references therein. Some additional work that needs to be highlighted combines data-driven control and networks. Specifically, the paper [6] resolves a number of data-driven problems regarding complex networks. In [7], the output synchronization problem is resolved for leader-follower multi-agent systems. Virtual reference feedback tuning and $\mathcal{H}_\infty$ are the topics of [8] and [9] respectively. Lastly, [10] provides conditions on noiseless data for specific analysis problems.

Of course, data-driven control is not only relevant in a context of networked systems. Many results from more general settings can also be applied to networks. Some such more recent developments regard the design of different types of controllers. Specifically, we note the work on data-driven predictive control [11]–[13], optimal control [14], [15] and optimization-based control [16].

Apart from data-driven methods, we should also mention the work on model-based design of distributed or decentralized

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controllers. First, we note the survey papers [17], [18] and the book [19] and the references therein. A particularly useful method for resolving distributed design problems is provided in the work on quadratic invariance [20]–[22].

More specifically, finding controllers that are as sparse as possible, while still guaranteeing certain design goals, is also a topic of significant interest. For this topic, a good overview can be found in [23]. Special mention is made of the paper [24], which, like this paper employs LMI's and [25] which deals with an efficient method for resolving these problems. An important ingredient of most methods noted above is the idea of reweighted ℓ_1 minimization of Candes et al. [26] (see also [27]). Specific applications of sparse controllers can be found within the field of power networks [28], [29] and security [30].

Organization

The paper is organized as follows. We start with a problem formulation in Section II. After this, we introduce the formalities regarding informativity in Section III. In particular, that section focuses on the quadratic stabilizability problem, and provides conditions for finding a centralized controller for each. In Section IV we consider the problem of finding a controller corresponding to a specific communication graph, which we resolve for two special cases. We develop an algorithm for finding a controller that is as sparse as possible in Section V. After this, Section VI illustrates the proposed algorithm using a simulation example. Lastly, we end the paper with conclusions.

II. Problem formulation

Suppose we have a heterogeneous networked system given by r agents of the form:

$$x_i(t+1) = \sum_{j=1}^{r} A_{ij}x_j(t) + B_i u_i(t) + w_i(t). \tag{1}$$

Denote the state and input dimensions of agent i by n_i and m_i. We can represent the entire system by

$$x(t+1) = A_s x(t) + Bu(t) + w(t), \tag{2}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_r(t) \end{bmatrix}, \ u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix}, \ w(t) = \begin{bmatrix} w_1(t) \\ \vdots \\ w_r(t) \end{bmatrix},$$

and

$$A_s = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rr} \end{bmatrix}, \ B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_r \end{bmatrix}.$$

We assume that the input matrix B is known, but that A_s is unknown. In lieu of this, we assume that we have access to data, consisting of a finite time window of input and state measurements. Based on these data we are interested in finding distributed controllers. In order to formalize this notion, we introduce some additional notation.

Suppose that we have a state-feedback controller K that guarantees some control objective for the system (A_s, B). We can partition K in the same fashion as A_s and B, and obtain

$$K = \begin{bmatrix} K_{11} & \cdots & K_{1r} \\ \vdots & & \vdots \\ K_{r1} & \cdots & K_{rr} \end{bmatrix},$$

with $K_{ij} \in \mathbb{R}^{p_i \times n_j}$. Note that, if we close the loop, we get $u(t) = Kx(t)$. In other words, for each agent i we have that

$$u_i(t) = \sum_{j=1}^{r} K_{ij}x_j(t).$$

An essential observation is the following: If $K_{ij} = 0$, then agent i does not require knowledge of the state of agent j in order to compute the feedback. As such, we can guarantee the absence of such dependencies by imposing that certain blocks K_{ij} are equal to zero. A number of interesting problems now arise.

First of all, there is the problem of **centralized control**, that is controller that stabilizes the system based on measured data. For this problem we make use the *informativity framework* of [3]. This means that we make the observation that we can only guarantee that a controller attains the objective for the true system, if it does so for *all systems that could have generated the data*.

Following the standard centralized problem, we consider a number of variants. For the problem of **control with a given communication graph** we suppose that the controller is allowed a given *communication graph*, that is, for each agent i, a set of 'neighboring' agents $\mathcal{N}_i$ are available for feedback. In line with the previous discussion, this is equivalent to finding a controller K such that certain blocks K_{ij} are equal to zero.

Alternatively, we might be tasked with controlling the system as *efficiently as possible* in a number of ways. The problem of **data-driven control with minimal actuation** consists of finding a controller with the least number of nonzero block-rows. This means that the controller acts on the minimal number of agents. Similarly, we can consider **data-driven control with minimal observation**. By finding a controller with the least number of nonzero block-columns, the controller is required to measure the state of the least number of agents.

Lastly, we look at minimizing the number of nonzero blocks in K. We refer to this problem as **data-driven control with maximal sparsity**.

III. Preliminaries on informativity

Before we return to the question of distributed controller design, we first formulate results regarding the non-distributed case. In this section, we use the rather general noise model that was introduced in [4]. Note that the results presented here differ from those in the latter paper due to the fact that we assume B is known.

Suppose that we collect data from system (2) in the form of state and input trajectories $x(t)$ and $u(t)$. We capture these measurements in the matrices:

$$X := \begin{bmatrix} x(0) & \cdots & x(T) \end{bmatrix},$$
$$U_- := \begin{bmatrix} u(0) & \cdots & u(T-1) \end{bmatrix},$$

and subsequently write

$$X_+ := \begin{bmatrix} x(1) & \cdots & x(T) \end{bmatrix},$$
$$X_- := \begin{bmatrix} x(0) & \cdots & x(T-1) \end{bmatrix}.$$

We assume that the noise w is unknown, that is, the samples of $w(0), w(1), \dots, w(T-1)$ are not measured. However, we do assume that the noise samples collected in the matrix

$$W_- := \begin{bmatrix} w(0) \; w(1) \; \cdots \; w(T-1) \end{bmatrix}$$

satisfy a given *noise model*. Let

$$\Phi := \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix}$$

be such that $\Phi_{11} = \Phi_{11}^\top \in \mathbb{R}^{n \times n}$, $\Phi_{12} \in \mathbb{R}^{n \times T}$ and $\Phi_{22} = \Phi_{22}^\top \in \mathbb{R}^{T \times T}$ and $\Phi_{22} < 0$. We now assume that the noise satisfies

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0. \tag{3}$$

Remark III.1 (Special cases of the noise model)**.** Note that this noise model encompasses, among others, energy bounds of the form $W_- W_-^\top \leqslant Q$, where $Q \in \mathbb{R}^{n \times n}$. For a further discussion on the special cases of this noise model, we refer to [4]. □

Clearly, the true state matrix A_s satisfies

$$W_- = X_+ - A_s X_- - BU_-,$$

where W_- satisfies (3). As such, it is clear that we can define the set of all state-matrices compatible with the data as

$$\Sigma = \{A \in \mathbb{R}^n \mid W_- = X_+ - AX_- - BU_- \text{ satisfies (3)}\}.$$

Let $N \in \mathbb{R}^{2n \times 2n}$ be given by:

$$N := \begin{bmatrix} I & X_+ - BU_- \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ - BU_- \\ 0 & -X_- \end{bmatrix}^\top. \tag{4}$$

Then it is straightforward to show that $A \in \Sigma$ if and only if

$$\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top N \begin{bmatrix} I \\ A^\top \end{bmatrix} \geqslant 0. \tag{5}$$

As noted, we are interested in determining properties on the true system, based on the measurements (U_-, X) as described above. Note that we can only conclude that the true system (A_s, B) has a given property if (A, B) has that property for all $A \in \Sigma$. This observation leads to the following definition.

Definition III.2. *Let B be given. We say that the data (U_-, X) are informative for quadratic stabilization if there exists a feedback gain K and a matrix $P > 0$ such that for each $A \in \Sigma$:*

$$(A + BK)P(A + BK)^\top < P. \tag{6}$$

Remark III.3 (Exact measurements)**.** Suppose that B is known, and that the measurements are exact, that is, each $w(t) = 0$. Then it is straightforward to show that if the data are informative for quadratic stabilization only if there is precisely one $A \in \Sigma$, or equivalently, X_- has full row rank. □

Note that informativity for quadratic stabilization not only requires all systems in Σ to admit the same feedback gain K. We also require all systems in Σ to admit the same Lyapunov function P. In particular a shared Lyapunov function is given by $V(x) = x^\top P^{-1} x$.

We can equivalently write (6) in the form of:

$$\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top \begin{bmatrix} P - BKPK^\top B^\top & -BKP \\ -PK^\top B^\top & -P \end{bmatrix} \begin{bmatrix} I \\ A^\top \end{bmatrix} > 0. \tag{7}$$

This means that characterizing informativity for quadratic stabilization is equivalent to characterizing when the quadratic matrix inequality (5) implies (7).

We say that the *Slater condition* holds if

$$\exists \hat{A} \text{ s.t. } \begin{bmatrix} I \\ \hat{A}^\top \end{bmatrix}^\top N \begin{bmatrix} I \\ \hat{A}^\top \end{bmatrix} > 0. \tag{8}$$

Theorem III.4 (LMI conditions for stabilization)**.** *Suppose that the Slater condition* (8) *holds. Then the data (U_-, X) are informative for quadratic stabilization if and only if there exist matrices $P > 0$, L and scalars $\alpha \geqslant 0, \beta > 0$ such that*

$$\left[\begin{array}{cc|c} P - \beta I & 0 & BL \\ 0 & 0 & P \\ \hline L^\top B^\top & P & P \end{array}\right] - \alpha \left[\begin{array}{c|c} N & 0 \\ \hline 0 & 0 \end{array}\right] \geqslant 0, \tag{9}$$

where N is as defined in (4)*, holds. Moreover, in this case the gain $K := LP^{-1}$ stabilizes all systems in Σ.*

IV. Control with a given sparsity structure

Having resolved the centralized control problems, we move our attention to distributed controllers. For this, we introduce some notation.

Let $p \in \mathbb{N}^k$ and $q \in \mathbb{N}^\ell$ such that $m = \sum_{i=1}^k p_i$ and $n = \sum_{j=1}^\ell q_j$. Given $M \in \mathbb{R}^{m \times n}$ we can partition it according to the vectors p and q by

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1\ell} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{k\ell} \end{bmatrix}, \tag{10}$$

with $M_{ij} \in \mathbb{R}^{p_i \times q_j}$.

We call $\sigma \in \{0,1\}^{k \times \ell}$ a *block sparsity structure*, and define the space of matrices corresponding to σ by:

$$\mathcal{M}_{p,q}^\sigma := \{M \in \mathbb{R}^{m \times n} \mid M_{ij} = 0 \text{ if } \sigma_{ij} = 0\}.$$

As such, it is clear that the problem of control with a given sparsity structure is a special case of the following.

Problem 1 (Control with a given sparsity structure)**.** *Given vectors $p \in \mathbb{N}^k$, $q \in \mathbb{N}^\ell$ such that $m = \sum_{i=1}^k p_i$ and $n = \sum_{j=1}^\ell q_j$, and block sparsity structure $\sigma \in \{0,1\}^{k \times \ell}$. Provide necessary and sufficient conditions for the data (U_-, X) to be informative for quadratic stabilization with feedback gain $K \in \mathcal{M}_{p,q}^\sigma$.*

Remark IV.1 (Block partitions and network systems)**.** Recall the problem formulation of Section II. There we decompose K according to the state and input dimensions of the specific subsystems. Clearly, this corresponds to the choice of $r = k = \ell$, and the partition $p_i = m_i$ and $q_i = n_i$ for each $i = 1, \dots, r$. As such, finding a controller with a given communication graph is a special case of Problem 1. However, it is important to stress that for Problem 1, this is not required. An interesting alternative case we consider is the case where $r = k$ and $p_i = m_i$ for each $i = 1, \dots, r$, but where $\ell = 1$. In terms of networked systems, this corresponds to actuating only the agents i for which $\sigma_{i1} = 1$. Similarly, we can look at $k = 1$, which, in terms of the set-up of Section II, would correspond to measuring only agent i for $\sigma_{1i} = 1$. □

Recall that in Theorem III.4 we formulate conditions for quadratic stabilization in the form of LMI (9) in the variables $P > 0$, L, $\alpha \geqslant 0$ and $\beta > 0$. If (9) is feasible, we can find a suitable feedback gain by taking $K = LP^{-1}$. However, note that the latter is not linear in the variables. This means that testing feasibility of the subspace constraint $LP^{-1} \in \mathcal{M}^{\sigma}_{p,q}$ together with the LMI (9) is no longer linear. However, certain special cases can be resolved in an efficient manner.

First of all, it is straightforward to show that L and $K = LP^{-1}$ have exactly the same (non-)zero rows, regardless of P. As such, we have the following result.

Corollary IV.2 (Control with given block-rows)**.** *Suppose that $\ell = 1$ and that the Slater condition holds. Then the data (U_-, X) are informative for quadratic stabilization with feedback gain $K \in \mathcal{M}^{\sigma}_{p,q}$ if and only if there exists $P > 0$, $L \in \mathcal{M}^{\sigma}_{p,q}$, $\alpha \geqslant 0$ and $\beta > 0$ such that* (9)*, where N is defined as in* (4)*, holds.*

Let $\bar{\sigma} := I_\ell \in \{0,1\}^{\ell\times\ell}$. Note that if $P \in \mathcal{M}^{\bar{\sigma}}_{q,q}$, then P is a block diagonal $n \times n$ matrix. Moreover, if the matrix $P \in \mathcal{M}^{\bar{\sigma}}_{q,q}$ is (block) diagonal, then so is P^{-1}. Furthermore, it is straightforward to prove that if this is the case, then $L \in \mathcal{M}^{\sigma}_{p,q}$ if and only if $K = LP^{-1} \in \mathcal{M}^{\sigma}_{p,q}$.

Remark IV.3 (Block diagonal P and networks)**.** Consider the case of networked systems, that is, $\ell = r$ and $q_i = n_i$. Then, the assumption that P is block diagonal corresponds to the case where

$$x^\top P^{-1} x = \sum_{i=1}^{r} x_i^\top P_{ii}^{-1} x_i.$$

That is, the Lyapunov function is decoupled. □

As such, we can resolve Problem 1 efficiently under the additional assumption that P is block diagonal.

Corollary IV.4 (Control with diagonal Lyapunov function)**.** *Suppose that the Slater condition holds. The data (U_-, X) are informative for quadratic stabilization with feedback gain $K \in \mathcal{M}^{\sigma}_{p,q}$ and Lyapunov matrix $0 < P \in \mathcal{M}^{\bar{\sigma}}_{q,q}$ if and only if there exists $0 < P \in \mathcal{M}^{\bar{\sigma}}_{q,q}$, $L \in \mathcal{M}^{\sigma}_{p,q}$, $\alpha \geqslant 0$ and $\beta > 0$ such that* (9)*, where N is defined as in* (4)*, holds.*

V. Sparse control

After considering finding controllers with a given block structure, we now move to the problem of finding controllers that are as sparse as possible.

Let $p \in \mathbb{N}^k$ and $q \in \mathbb{N}^\ell$ and let M be a matrix that is partitioned as in (10). We define the *block cardinality* of a matrix M, denoted $\mathrm{bcard}_{p,q}(M)$ as the number of non-zero blocks in M.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$\phi(x) := \begin{cases} 0 & x = 0, \\ 1 & x \neq 0. \end{cases}$$

Note that the number of nonzero elements of $\sigma \in \{0,1\}^{k\times\ell}$ is equal to $\sum_{i=1}^{k}\sum_{j=1}^{\ell} \sigma_{ij}$. As such, we have the following equivalent statements

$$\mathrm{bcard}_{p,q}(M) = \min_{\sigma \text{ s.t. } M \in \mathcal{M}^{\sigma}_{p,q}} \sum_{i=1}^{k}\sum_{j=1}^{\ell} \sigma_{ij} = \sum_{i=1}^{k}\sum_{j=1}^{\ell} \phi(\|M_{ij}\|_F),$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

In the case where $p_i = q_j = 1$ for all i and j, the block cardinality is equal to the number of non-zero elements in M. This is often referred to as the ℓ_0-pseudo norm or simply the *cardinality* of M. It should be stressed, however, that the (block) cardinality is not a norm, nor a convex function.

This leads us to the following general problem formulation.

Problem 2 (Control with maximal sparsity)**.** *Given vectors $p \in \mathbb{N}^k$, $q \in \mathbb{N}^\ell$ such that $m = \sum_{i=1}^{k} p_i$ and $n = \sum_{j=1}^{\ell} q_j$ and data (U_-, X) that are informative for quadratic stabilization, resolve the following problem:*

$$\begin{aligned} &\textit{minimize} && \mathrm{bcard}_{p,q}(K), \\ &\textit{subject to} && \exists P > 0 \text{ s.t. } (6) \ \ \forall A \in \Sigma. \end{aligned} \tag{11}$$

Remark V.1 (Interpretation in terms of networked systems)**.** It follows immediately from the reasoning in Remark IV.1 that this statement can be used for the problems of control with minimal actuation/observation and for control with maximal sparsity. □

Note that in Problem 2 the objective function is not a convex function of K, and the constraint set is linear in P and KP, but not necessarily in K. As such, the problem above is not a convex problem. This means that the problem can not be resolved by many standard methods.

An approach that can work for networked systems with a relatively low number of agents is a simple exhaustive search. In cases where we can efficiently solve Problem 1, we can simply test feasibility for different block sparsity patterns σ with increasing number of nonzero elements. This method is guaranteed to provide the correct answer, but scales in a combinatorial way with $k\ell$.

As a first step towards resolving the minimization problem (11), we formulate the following corollary of Theorem III.4.

Corollary V.2 (Equivalent formulation of control with maximal sparsity)**.** *Suppose that the Slater condition holds. Then resolving* (11) *in Problem 2 is equivalent to:*

$$\begin{aligned} &\text{minimize} && \mathrm{bcard}_{p,q}(LP^{-1}), \\ &\text{subject to} && P > 0, \exists \alpha \geqslant 0, \beta > 0. \text{ s.t. } (9). \end{aligned} \tag{12}$$

In the following we take an approach based on the method of reweighted ℓ_1 minimization, as introduced in [26]. As such, we propose a strategy consisting of repeating a weighted optimization problem and updating the weights, as shown in Algorithm 1.

Algorithm 1 Reweighted optimization

Inputs: Vectors $p \in \mathbb{N}^k$, $q \in \mathbb{N}^\ell$ with $m = \sum_{i=1}^k p_i$ and $n = \sum_{j=1}^\ell q_j$, matrix N as in (4).
Outputs: $\{(L_{\hat{t}}, P_{\hat{t}})\}_{\hat{t}=0}^{t}$ for some $t \geqslant 1$.
Initialize: Set $t = 0$ and find L_0 and $P_0 > 0$ for which there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (9) holds
while $(L_{t-1}, P_{t-1}) \neq (L_t, P_t)$ **do**
  for $i = 1, \dots k$, $j = 1, \dots, \ell$ **do**
    Update the weights by:
    if $(L_t P_t^{-1})_{ij} \neq 0$ **then**
      Let $w_{ij}(t) := \dfrac{1}{\|(L_t P_t^{-1})_{ij}\|_F}$
    else
      Let $w_{ij}(t) := \infty$
    end if
  end for
  Set $f_t(L) := \sum_{i=1}^{k} \sum_{j=1}^{\ell} w_{ij}(t) \|(L P_t^{-1})_{ij}\|_F$
  Update the estimates by solving:

$$\begin{aligned} (L_{t+1}, P_{t+1}) &:= \operatorname*{arg\,min}_{(L,P)} f_t(L), \\ &\text{subject to } P > 0, \exists \alpha \geqslant 0, \beta > 0 \text{ s.t. (9)} \end{aligned} \tag{13}$$

  Update $t \leftarrow t + 1$
end while

Note that the objective function of the optimization problem (13) is not dependent on P, but on P_t. As such, it is straightforward to show that the objective function is a convex function of L. Furthermore, the constraint set is given by an LMI, making it straightforward to resolve (13).

Theorem V.3 (If reweighted optimization converges, its output solves the stabilization problem with maximal sparsity)**.** *Given vectors $p \in \mathbb{N}^k$, $q \in \mathbb{N}^\ell$ such that $m = \sum_{i=1}^k p_i$ and $n = \sum_{j=1}^\ell q_j$. Suppose that the Slater condition holds and that the data (U_-, X) are informative for quadratic stabilization. Then, we can initialize Algorithm 1. Moreover, if $(L_{t-1}, P_{t-1}) = (L_t, P_t)$, and we denote $L := L_t$ and $P := P_t$ then LP^{-1} is the minimizer of* (11).

It is important to realize that Theorem V.3 only gives sufficient conditions for resolving Problem 2, since we have not formulated conditions under which the algorithm converges.

VI. Simulations

Let the true system be given by 3 agents, where $n_i = 2$, $m_i = 1$ and $B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for each $i \leqslant 3$. Assume that the true state matrix is given by:

$$A_s = \frac{3}{5} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We generate measurements using Matlab, by choosing an initial condition $x(0)$, inputs $u(t)$ and noise $w(t)$ randomly for $t = 0, \dots 9$, such that $W_- W_-^\top \leqslant \frac{1}{20} I$. The precise measurements can be found in (14).

We use Yalmip [31] with Mosek as a solver in combination with Theorem III.4 to resolve the informativity problem. First, we note that the Slater condition holds. Then, the solver returns $P > 0$, L, $\alpha \geqslant 0$ and $\beta > 0$ such that LMI (9) holds. As such, the data are informative for quadratic stabilization. In addition this results in the stabilizing feedback gain $K_1 = LP^{-1}$ for all $A \in \Sigma$, given by:

$$\begin{bmatrix} -0.071335 & 0.53919 & -0.36814 & 0.23887 & -0.72051 & -0.74332 \\ 0.088392 & 0.091179 & -0.38196 & -0.37764 & -0.64738 & -0.060889 \\ -0.076069 & 0.54351 & 0.11392 & 0.10647 & -1.2478 & -0.66924 \end{bmatrix}$$

It can be easily verified that this gain indeed stabilizes the true system. However, if we decompose K_1 according to the input and state dimension of the agents, we obtain

$$u_i(t) = \sum_{j=1}^{r} K_{ij} x_j(t).$$

As such, we see that in order to compute the input u_i, we require for each $j = 1, \dots, r$ the state x_j. That is, the controller is not sparse. As such, we move our attention to Problem 2, the problem of control with maximal sparsity. Note that we are not in the situation Corollary IV.2 or Corollary IV.4. As such, we have no efficient way of resolving Problem 1. This prevents us from performing an exhaustive search for a maximally sparse controller. In addition, note that finding a feedback gain with, for example, less than or equal to 4 nonzero blocks would require us to check up to 255 different sparsity patterns. In line with Section V, we implement Algorithm 1 numerically. This requires making a number of straightforward changes regarding machine precision to the pseudo code. Again, we apply Yalmip with the solver Mosek. After 21 iterations, the algorithm has stabilized up to the required precision. The corresponding feedback gain, denoted K_{21}, is found as:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.19134 & -0.048629 & 0 & 0 \\ -0.94584 & -0.052014 & -0.11946 & 0.073348 & -0.98268 & -0.14899 \end{bmatrix}$$

As such, we have obtained a feedback gain with just 4 nonzero blocks that stabilizes all systems in Σ.

VII. Conclusions

We have considered data-driven distributed and sparse control. In particular, we started with defining and resolving informativity problems regarding centralized stabilization. As such, we formulated conditions under which a controller guarantees stabilization for all systems compatible with given measurements. After this, we have considered the same problem while restricting the allowed controllers to those corresponding to a given communication graph. For two specific cases, it was shown that efficient solutions are possible. Lastly we formulated an algorithm whose steps can be calculated efficiently. If this algorithm converges, it results in the most sparse stabilizing controller for all systems compatible with the data. Future work will investigate the synthesis of stabilizing controllers with a given sparsity structure (cf. Problem 1) for

$$
\begin{aligned}
X &= \begin{bmatrix} 0.75274 & 1.2276 & 1.5028 & 1.4546 & 2.2505 & 3.2402 & 4.0554 & 4.8123 & 5.0687 & 5.8844 & 7.3989 \\ 0.48475 & 1.6001 & 2.0504 & 3.067 & 5.4602 & 8.2031 & 11.736 & 16.6432 & 22.2254 & 28.3431 & 36.5077 \\ 0.62701 & 0.28679 & 0.56613 & 1.9483 & 1.9467 & 2.3218 & 3.3144 & 3.4338 & 3.7058 & 5.0454 & 5.0957 \\ 0.80199 & 0.30132 & 0.99168 & 2.6294 & 4.0138 & 5.7936 & 8.6326 & 12.1519 & 16.2375 & 21.5731 & 27.317 \\ 0.11059 & 0.60892 & 1.6934 & 1.9273 & 2.9284 & 3.9676 & 4.7534 & 5.4348 & 6.8431 & 7.5784 & 8.6763 \\ 0.39059 & 1.0436 & 2.6886 & 4.7617 & 6.7268 & 10.4199 & 15.4993 & 21.6268 & 29.1105 & 37.9496 & 47.8538 \end{bmatrix} \\
U_- &= \begin{bmatrix} 0.39914 & 0.59328 & 0.21324 & 0.20845 & 0.72101 & 0.71757 & 0.39015 & 0.12077 & 0.61899 & 0.8402 \\ 0.22042 & 0.20061 & 0.93207 & 0.79012 & 0.56395 & 0.93289 & 0.58158 & 0.4449 & 0.9393 & 0.54806 \\ 0.090819 & 0.59133 & 0.0087293 & 0.89861 & 0.85981 & 0.42837 & 0.14863 & 0.69451 & 0.43057 & 0.59851 \end{bmatrix} \qquad (14) \\
W_- &= 10^{-4} \begin{bmatrix} 0.56402 & 0.85894 & 0.078075 & 0.30536 & 0.81527 & 0.68118 & 0.19788 & 0.30939 & 0.66536 & 0.8844 \\ 0.21199 & 0.93952 & 0.38109 & 0.63732 & 0.34066 & 0.82892 & 0.067992 & 0.74664 & 0.63701 & 0.16617 \\ 0.020618 & 0.17608 & 0.26612 & 0.25169 & 0.81665 & 0.99683 & 0.21282 & 0.0048493 & 0.20266 & 0.57528 \\ 0.61413 & 0.1923 & 0.19338 & 0.42205 & 0.42013 & 0.11501 & 0.24711 & 0.46404 & 0.91496 & 0.25192 \\ 0.10097 & 0.13537 & 0.88955 & 0.63512 & 0.39169 & 0.35093 & 0.91207 & 0.34179 & 0.69204 & 0.14824 \\ 0.35514 & 0.51728 & 0.61431 & 0.50191 & 0.33043 & 0.84755 & 0.31911 & 0.24342 & 0.93888 & 0.53028 \end{bmatrix}
\end{aligned}
$$

the general case, the application of efficient solution methods to the stabilization with maximal sparsity (cf. Problem 2), establishing convergence of Algorithm 1. A last problem of interest is the case where a block structure of the state matrix is known, in addition to one for the controller. This corresponds to knowledge of the network structure of the system. Being able to use this knowledge might lead to stronger results.

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