Average dwell-time minimization of switched systems via sequential convex programming

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***Abstract*—This work finds a lower bound on the average dwell-time (ADT) of switching signals such that a continuous-time, graph-based, switched system is globally asymptotically stable, input-to-state stable, or integral input-to-state stable. We first formulate the lower bound on the ADT as a nonconvex optimization problem with bilinear matrix inequality constraints. Because this formulation is independent of the choice of Lyapunov functions, its solution gives a less conservative lower bound than previous Lyapunov-function-based approaches. We then design a numerical iterative algorithm to solve the optimization based on sequential convex programming with a convex-concave decomposition of the constraints. We analyze the convergence properties of the proposed algorithm, establishing the monotonic evolution of the estimates of the average dwell-time lower bound. Finally, we demonstrate the benefits of the proposed approach in two examples and compare it against other baseline methods.**

***Index Terms*—Stability of switched systems, average dwell-time, sequential convex programming**

I. Introduction

SWITCHED systems are a class of hybrid systems which play an important role in modeling real-world processes [1]. A switched system is defined by a collection of dynamical subsystems and a switching signal that governs the transitions between them. In general, switched systems do not inherit the stability properties of their subsystems under arbitrary switching, see e.g., [2]. Many works aim to find sufficient conditions on the switching signals to ensure desirable stability properties. These include *dwell-time* (DT) (resp. *average dwell-time* (ADT)) conditions, which bound the number (resp. the average number) of allowed switches over an arbitrary time interval, guaranteeing *global asymptotic stability* (GAS) of the switched system [3], [4]. For systems with inputs, *input-to-state stability* (ISS), *integral input-to-state stability* (iISS) can be guaranteed for switched systems with switching signals satisfying similar DT [5], [6] or ADT conditions [7], [8].

In practice, switching signals are often designed independently of the subsystems. Hence, a lower bound on the DT (resp. the ADT) solely based on the information of the dynamics of the subsystems can provide an important design criteria to prevent de-stabilization by the switching action of the signal. To broaden the class of admissible switching signals, one would like such lower bounds to be as small as possible. However, the DT or ADT lower bounds proposed in the literature [3]–[9] depend heavily on the Lyapunov functions chosen for stability analysis and may therefore be conservative. In particular when the switched system is graph-based (that is, when the mode changing during a switch is restricted to be an edge of a graph), the DT or ADT lower bounds computed using a naive choice of Lyapunov functions may be far from satisfactory. To address this, the work [10] proposes an optimization formulation whose constraints include matrix exponentials, which make the problem not directly solvable. [11] proposes an alternative optimization problem formulation where the decision variables are matrix functions. By assuming such matrix functions are polynomials, the problem is solved with the aid of sum-of-squares programming. On the other hand, the work [12] leverages the eigenvectors of each subsystem to propose a different formula for a DT lower bound. The research of finding DT lower bounds has also covered discrete-time systems [13], [14]. To guarantee stability of the overall system, the aforementioned works utilize the idea of pairing and neutralizing the destabilizing effects of switches with the stabilization effect provided by mode dwelling. This simple idea is inapplicable for finding ADT lower bounds, as switching signals satisfying ADT condition are more flexible, which might explain the relative low number of works on this topic. To the best of our knowledge, the work [15], extending [12], aims to use a cycle ratio of the graph-based switched system for computing an ADT lower bound. Finally, the work [16] proposes a way to verify whether a positive number is a valid ADT lower bound on the switching signals so that the switched system is stable.

The contributions of this paper address the gaps identified above by (i) formulating the problem of finding an ADT lower bound as a nonlinear optimization with *bilinear matrix inequality* (BMI) constraints; and (ii) proposing and analyzing a numerical iterative algorithm to solve the resulting optimization problem. We emphasize here that finding a lower bound on the ADT for ensuring stability using optimization techniques is novel. A key aspect of our optimization formulation is its independence of the choice of Lyapunov functions, which we accomplish building on the result in [17] establishing a common ADT lower bound guaranteeing GAS, ISS or iISS of switched systems when the unforced dynamics are linear. We tackle the nonconvexity of the optimization problem using *sequential convex programming* (SCP) with the technique of

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convex-concave decomposition of constraints. Our simulations show that our algorithm computes an ADT lower bound which is much smaller than the values produced by the baseline approaches in various sample switched systems.

II. Preliminaries

Notation: We denote by $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ the set of symmetric matrices. For any $M \in \mathbb{S}^n$, we denote $M \succ 0$ (resp. $M \succeq 0$) if M is positive definite (resp. positive semi-definite). In addition, $M_1 \succ M_2$ if $M_1 - M_2 \succ 0$. Analogous definitions hold for (semi-)negative definiteness.

Switched systems and switching signals: Consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \cdots, p\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For each $i \in \mathcal{V}$ there is a locally Lipschitz vector field $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. According to [2], a switched system is referred by the differential equations

$$\dot{x} = f_\sigma(x, \omega), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $\omega \in \mathbb{R}^m$ is the input and $\sigma \in \mathcal{V}$ is the mode. Let $\overline{\Sigma}$ be the set of all right-continuous, piecewise constant mappings from $[0, \infty)$ to $\mathcal{V}$ with a locally finite number of discontinuities, called *switching signals*. For each switching signal $\sigma \in \overline{\Sigma}$, define the switch instants $\mathcal{T}(\sigma) := \{t > 0 : \sigma(t) \neq \sigma(t^-)\}$ where t^- denotes the left limit of the function at t. With this data, the dynamics of the switched system is described by

$$\dot{x}(t) = f_{\sigma(t)}(x(t), \omega(t)), \qquad \text{if } t \notin \mathcal{T}(\sigma), \tag{2a}$$
$$x(t) = x(t^-), \qquad \text{if } t \in \mathcal{T}(\sigma). \tag{2b}$$

We now specify the class of switching signals for which we study the stability of system (2). We say a switching signal σ has an underlying switching graph $\mathcal{G}$ if $(\sigma(t^-), \sigma(t)) \in \mathcal{E}$ for all $t \in \mathcal{T}(\sigma)$; in other words, the system is only allowed to switch from mode i to j if (i, j) is an edge of $\mathcal{G}$. According to [4], a switching signal σ has an *average dwell-time* (ADT) of τ_a (equivalently, σ satisfies the ADT constraint) if there exist $\tau_a > 0$ and $N_0 \geq 1$ such that

$$\forall t_2 \geq t_1 \geq 0 : \quad N_\sigma(t_1, t_2) \leq N_0 + \frac{t_2 - t_1}{\tau_a}, \tag{3}$$

where $N_\sigma(t_1, t_2) := |(t_1, t_2] \cap \mathcal{T}(\sigma)|$. In other words, on average there can be at most one switch per τ_a units of time.

III. A uniform ADT lower bound for stable switched systems

In this work, we study two well-known stability properties, *input-to-state stability* (ISS) and *integral input-to-state stability* (iISS), for the switched system (1). Due to space constraints, we refer the readers to [18], [19] for their definitions. It is seen that when (1) has no input ($\omega = 0$), both ISS and iISS reduce to *global asymptotic stability* (GAS). Interestingly, although GAS, ISS and iISS are different stability notions, they all can be guaranteed with the same ADT condition on the switching signal for some switched systems. Formally, consider the switched linear system without input

$$\dot{x} = A_\sigma x, \tag{4}$$

the switched linear system with linear input

$$\dot{x} = A_\sigma x + B_\sigma \omega \tag{5}$$

and the switched system with linear and bilinear inputs

$$\dot{x} = A_\sigma x + B_\sigma \omega + \sum_{j=1}^{m_c} C_{\sigma,j} x \omega_j. \tag{6}$$

The following result establishes that an appropriate lower bound on τ_a ensures that these systems are, respectively, GAS, ISS, and iISS.

Theorem 1 ([7, Theorem 3.1], [17, Proposition 12]): Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let the matrix $A_i \in \mathbb{R}^{n\times n}$ be Hurwitz for all $i \in \mathcal{V}$. Consider the switched systems (4), (5) or (6) and assume the switching signal σ has an underlying switching graph $\mathcal{G}$ with ADT τ_a. Let $P_i \in \mathbb{S}^n, P_i \succ 0$ and suppose that the inequalities

$$A_i^\top P_i + P_i A_i + \lambda P_i \preceq 0, \quad \forall i \in \mathcal{V}, \tag{7a}$$
$$P_j - \mu P_i \preceq 0, \quad \forall (i, j) \in \mathcal{E}, \tag{7b}$$

hold for some $\mu \geq 1, \lambda > 0$. If $\tau_a > \frac{\ln \mu}{\lambda}$, then the systems (4), (5) and (6) are GAS, ISS and iISS, respectively.

We refer to the parameter $\frac{\ln \mu}{\lambda}$ as the *ADT lower bound*. The smaller this bound is, the larger the set of switching signals to which Theorem 1 applies, and hence the greater the design flexibility for the switched system is. Notice that Theorem 1 only provides a sufficient condition, and hence the ADT lower bound might be conservative. This bound depends on the choices of P_i's, μ, and λ. We observe that if P_i's are fixed, μ and λ can be optimized to give a minimal ADT lower bound while preserving the inequalities (7a) and (7b). However, the matrices P_i's are related to the Lyapunov functions chosen for the subsystems and they are not unique. In order to maximize λ, each P_i needs to be tailored to A_i, in which case the P_i's might be very different from each other, causing μ to become large. On the other hand, in order for μ to be as close to 1 as possible, the P_i's need to be close to each other. In that case the Lyapunov functions of some subsystems may dissipate slowly and thus λ may become small. These observations point out to the trade-offs in the selection of the matrices P_i's when minimizing the ADT lower bound. Instead, one can formulate the following optimization problem to find the best choice of μ, λ and P_i's which give the minimal ADT lower bound:

(P1)
$$\underset{\{P_i\}_{i\in\mathcal{V}}, \mu, \lambda}{\text{minimize}} \quad \frac{\ln \mu}{\lambda} \tag{8a}$$
$$\text{subject to} \quad \mu \geq 1, \tag{8b}$$
$$\lambda > 0, \tag{8c}$$
$$P_i \succ 0 \quad \forall i \in \mathcal{V}, \tag{8d}$$
$$\text{and (7a), (7b).}$$

The next result is a direct consequence of Theorem 1.

Corollary 1: Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let the matrices $A_i \in \mathbb{R}^{n\times n}$ be Hurwitz for all $i \in \mathcal{V}$. Denote the optimal value of (P1) by $\tau^* = \frac{\ln \mu^*}{\lambda^*}$. Then if a switching signal σ has underlying switching graph $\mathcal{G}$ with ADT satisfying $\tau_a > \tau^*$, then the switched system (4) is GAS, (5) is ISS and (6) is iISS.

Comparing Corollary 1 with Theorem 1, we see that the ADT lower bound τ^* in Corollary 1 only depends on the system matrices A_i's, and it is less conservative than the one given in Theorem 1.

We remark here that (P1) is NP-hard in general and cannot be solved directly since 1) the objective function (8a) is nonlinear and nonconvex, and 2) the constraints (7a) and (7b) are BMI constraints, which are nonconvex. The problem we address in the rest of this paper is how to tackle the challenges involved in solving (P1).

IV. Solving the nonlinear optimization problem

In this section, we first establish the feasibility of problem (P1) in Section IV-A and then design a SCP to solve it.

A. Feasibility of the problem

We first establish the feasibility of problem (P1); that is, whether there exist $P_i \in \mathbb{S}^n$ for all $i \in \mathcal{V}$ and $\mu, \lambda \in \mathbb{R}$ satisfying the constraints (7a), (7b), (8b), (8c) and (8d).

Lemma 1: The optimization problem (P1) is feasible when all A_i's are Hurwitz matrices.

Proof: Since all A_i's are Hurwitz, there exist $P_i \in \mathbb{S}^n, P_i \succ 0$ that solve the Lyapunov equations

$$A_i^\top P_i + P_i A_i + I = 0 \quad \forall i \in \mathcal{V}. \tag{9}$$

Set $\lambda := \frac{1}{\bar{\sigma}}$, $\mu := \frac{\bar{\sigma}}{\underline{\sigma}}$ where $\bar{\sigma} := \max_{i\in\mathcal{V}} \sigma_{\max}(P_i)$, $\underline{\sigma} := \min_{i\in\mathcal{V}} \sigma_{\min}(P_i)$ and $\sigma_{\max}(Q), \sigma_{\min}(Q)$ denotes the largest/smallest singular values of a matrix Q, respectively. Note that all constraints in (P1) are satisfied since $\underline{\sigma} I \preceq P_i \preceq \bar{\sigma} I$, for all $i \in \mathcal{V}$. ■

B. Approximation of the objective and constraint functions

Here we design an SCP to solve problem (P1). The pseudocode is summarized in Algorithm 1. Our ensuing discussion elaborates on each of the steps in Algorithm 1. In the k-th iteration, we convexify the problem (P1) around $\{P_i^{(k-1)}\}_{i\in\mathcal{V}}, \mu^{(k-1)}, \lambda^{(k-1)}$. This convexification is done by linearizing the objective function (8a), followed by adding a quadratic regularization term, and convex-concave decomposition of the BMI constraints (7a), (7b) using techniques similar to the ones in [20]. We then compute an ADT lower bound $\tau^{(k)}$ using the data $\mu^{(k)}, \lambda^{(k)}$ and repeat this process. The convergence analysis is presented in Section IV-C.

Algorithm 1 Computation of minimum ADT lower bound

Input: $(\mathcal{V}, \mathcal{E}), \{A_i\}_{i\in\mathcal{V}}, \{P_i^{(0)}\}_{i\in\mathcal{V}}, \mu^{(0)}, \lambda^{(0)}, \epsilon$

1: $\tau^{(0)} \leftarrow \frac{\ln \mu^{(0)}}{\lambda^{(0)}}$
2: **for** $k = 1, 2, \cdots$ **do**
3: Convexify (P1) around $\{P_i^{(k-1)}\}_{i\in\mathcal{V}}$, $\mu^{(k-1)}$, $\lambda^{(k-1)}$
4: Solve the convexified problem, set $\{P_i^{(k)}\}_{i\in\mathcal{V}}$, $\mu^{(k)}$, $\lambda^{(k)}$ equal to the obtained minimizer
5: $\tau^{(k)} \leftarrow \frac{\ln \mu^{(k)}}{\lambda^{(k)}}$

1) Approximation of the objective function: We approximate f in (P1) linearly around $(\mu^\dagger, \lambda^\dagger)$ by

$$\mathbf{L}\, f_{\mu^\dagger,\lambda^\dagger}(\mu, \lambda) := \frac{\ln \mu^\dagger}{\lambda^\dagger} + \begin{pmatrix} \frac{1}{\mu^\dagger \lambda^\dagger} & -\frac{\ln \mu^\dagger}{(\lambda^\dagger)^2} \end{pmatrix} \begin{pmatrix} \mu - \mu^\dagger \\ \lambda - \lambda^\dagger \end{pmatrix}. \tag{10}$$

Note that the second term is the directional derivative of f at $(\mu^\dagger, \lambda^\dagger)$, in the direction of $(\mu - \mu^\dagger, \lambda - \lambda^\dagger)$. The objective function f can also be locally approximated by a quadratic function by taking into account its Hessian. However, since f is nonconvex, its Hessian is sign indefinite and the quadratic optimization problem cannot be solved efficiently.

2) Approximation of the BMI constraints: We first approximate the BMI constraints (7a) with *linear matrix inequality* (LMI) constraints. For each $i \in \mathcal{V}$, (7a) can be rewritten in quadratic form as

$$\begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top \Sigma \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix} \preceq 0, \quad \text{with } \Sigma := \begin{pmatrix} 0 & I & 0 \\ I & 0 & I \\ 0 & I & 0 \end{pmatrix}. \tag{11}$$

Notice that Σ is a $3n \times 3n$ symmetric matrix, with eigenvalues $-\sqrt{2}$, 0, and $\sqrt{2}$, all of multiplicity n. Let V denote an orthogonal matrix whose columns are eigenvectors of Σ. Divide both sides of (11) by $\sqrt{2}$ and define

$$\hat{R}(P_i, \lambda) := \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top V \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \begin{pmatrix} 0 & 0 & I \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix},$$

$$\check{R}(P_i, \lambda) := \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix}^\top V \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix},$$

the inequality in (11) becomes

$$\hat{R}(P_i, \lambda) - \check{R}(P_i, \lambda) \preceq 0. \tag{12}$$

Notice that by definition, both $\hat{R}$ and $\check{R}$ are positive semidefinite and convex. Using this fact, we can write, for any $P_i, P_i^\dagger \in \mathbb{S}^n$ and $\lambda, \lambda^\dagger > 0$,

$$\begin{aligned} \check{R}(P_i, \lambda) &\succeq \mathbf{L}\, \check{R}_{P_i^\dagger,\lambda^\dagger}(P_i, \lambda) \\ &:= \check{R}(P_i^\dagger, \lambda^\dagger) + D\check{R}(P_i, \lambda)(P_i - P_i^\dagger, \lambda - \lambda^\dagger), \end{aligned} \tag{13}$$

where $\mathbf{L}\, \check{R}_{P_i^\dagger,\lambda^\dagger}$ is the linearization of $\check{R}$ around $(P_i^\dagger, \lambda^\dagger)$ and $D\check{R}(\mathbf{x})(\mathbf{v})$ is the directional derivative of $\check{R}$ evaluated at $\mathbf{x}$ in the direction of $\mathbf{v}$. The latter can be explicitly computed as

$$\begin{aligned} &D\check{R}(P_i, \lambda)(P_i - P_i^\dagger, \lambda - \lambda^\dagger) \\ &= \begin{pmatrix} 0 \\ P_i - P_i^\dagger \\ \frac{\lambda - \lambda^\dagger}{2} I \end{pmatrix}^\top V \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i^\dagger \\ \frac{\lambda^\dagger}{2} I \end{pmatrix} \\ &+ \begin{pmatrix} A_i \\ P_i^\dagger \\ \frac{\lambda^\dagger}{2} I \end{pmatrix}^\top V \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \end{pmatrix} V^\top \begin{pmatrix} 0 \\ P_i - P_i^\dagger \\ \frac{\lambda - \lambda^\dagger}{2} I \end{pmatrix}. \end{aligned}$$

Consider the following inequality

$$\hat{R}(P_i, \lambda) - \mathbf{L}\, \check{R}_{P_i^\dagger,\lambda^\dagger}(P_i, \lambda) \preceq 0. \tag{14}$$

From (13), we see that (12) holds whenever (14) holds. This means the feasible set of P_i, λ given by the constraint (14) is a subset of the feasible set given by the constraint (12). Thus, if we fix $P_i^\dagger, \lambda^\dagger$ and replace the constraint (7a) by (14) in (P1), the optimal value of the objective function will in general be larger. Note that by using the definition of $\hat{R}$ and applying the Schur complement [21], (14) is equivalent to the LMI

$$\begin{pmatrix} I & \begin{pmatrix} 0 & 0 & I \end{pmatrix} V^\top \begin{pmatrix} A_i \\ P_i \\ \frac{\lambda}{2} I \end{pmatrix} \\ \begin{pmatrix} A_i^\top & P_i & \frac{\lambda}{2} I \end{pmatrix} V \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} & \mathbf{L}\, \check{R}_{P_i^\dagger, \lambda^\dagger}(P_i, \lambda) \end{pmatrix} \succeq 0. \tag{15}$$

Using similar convex-concave decomposition and linearization, the constraints (7b) can also be approximated by LMI constraints

$$\begin{pmatrix} I & \begin{pmatrix} I & 0 \end{pmatrix} U^\top \begin{pmatrix} P_i \\ \frac{\mu}{2} I \end{pmatrix} \\ \begin{pmatrix} P_i & \frac{\mu}{2} I \end{pmatrix} U \begin{pmatrix} I \\ 0 \end{pmatrix} & \mathbf{L}\, \hat{S}_{P_i^\dagger, \mu^\dagger}(P_i, \mu) - P_j \end{pmatrix} \succeq 0, \tag{16}$$

where the columns of U are the eigenvectors of $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and

$$\mathbf{L}\, \hat{S}_{P_i^\dagger, \mu^\dagger}(P_i, \mu) = \begin{pmatrix} P_i - P_i^\dagger \\ \frac{\mu}{2} I \end{pmatrix}^\top U \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} 0 & I \end{pmatrix} U^\top \begin{pmatrix} P_i^\dagger \\ \frac{\mu^\dagger}{2} I \end{pmatrix} + \begin{pmatrix} P_i^\dagger \\ \frac{\mu^\dagger}{2} I \end{pmatrix}^\top U \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} 0 & I \end{pmatrix} U^\top \begin{pmatrix} P_i \\ \frac{\mu}{2} I \end{pmatrix}$$

C. The convex subproblem

In this section we combine the aforementioned approximations for the objective function and constraints and summarize the convex subproblem formulated and solved in Steps 3 and 4 of Algorithm 1. To this end, for $P_i^\dagger \in \mathbb{S}^n, \mu^\dagger, \lambda^\dagger \in \mathbb{R}$ and parameters $c_\lambda, c_\mu, c_P \geq 0$, define the regularization function

$$r_{P_i^\dagger, \mu^\dagger, \lambda^\dagger}(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) := c_P \sum_{i \in \mathcal{V}} \|P_i - P_i^\dagger\|_F^2 + c_\mu (\mu - \mu^\dagger)^2 + c_\lambda (\lambda - \lambda^\dagger)^2. \tag{17}$$

This function is a weighted sum of the squared distance between P_i, μ, λ and $P_i^\dagger, \mu^\dagger, \lambda^\dagger$, and is convex in P_i, μ, λ. Consider the following problem

$$\text{(P2)} \quad \underset{\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda}{\text{minimize}}\ \mathbf{L}\, f_{\mu^\dagger, \lambda^\dagger}(\mu, \lambda) + r_{P_i^\dagger, \mu^\dagger, \lambda^\dagger}(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda)$$
$$\text{subject to} \quad \text{(8b) – (8d)}, \quad \text{and (15) } \forall i \in \mathcal{V}, \text{(16) } \forall (i,j) \in \mathcal{E},$$

where $\mathbf{L}\, f_{\mu^\dagger, \lambda^\dagger}$ is defined in (10) and $r_{P_i^\dagger, \mu^\dagger, \lambda^\dagger}$ is defined in (17). If the point $(\{P_i^\dagger\}_{i \in \mathcal{V}}, \mu^\dagger, \lambda^\dagger)$ is itself a solution to (P2), then we call it *optimal*. The problem (P2) has a quadratic and convex objective function with LMI constraints, and it is a standard *semi-definite programming* (SDP) problem which can be efficiently solved.

The convex subproblem solved at the k-th iteration of Algorithm 1 is precisely (P2) with the choice $(P_i^\dagger, \mu^\dagger, \lambda^\dagger) = (P_i^{(k-1)}, \mu^{(k-1)}, \lambda^{(k-1)})$. A fixed point of Algorithm 1 is therefore an optimal point in the sense defined above. Under some mild assumptions, we show next that any solution of (P1) is a fixed point. We also show that $P_i^{(k)}, \mu^{(k)}, \lambda^{(k)}$ generated by Algorithm 1 converge to a fixed point.

Proposition 1 (Convergence of Algorithm 1): Suppose that there is a compact subset D of the feasible set of (P1) such that $c_\lambda \geq \frac{1 + 2\ln\mu}{\lambda^3}$ for all $(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) \in D$. Let $(\{P^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*) \in D$ be a solution of (P1), then it is a fixed point. In addition, if Algorithm 1 generates the sequence $(P_i^{(k)}, \mu^{(k)}, \lambda^{(k)}) \in D$ for all $k \in \mathbb{N}$, then the associated $\tau^{(k)}$ monotonically decrease and the sequence $(\{P_i^{(k)}\}_{i \in \mathcal{V}}, \mu^{(k)}, \lambda^{(k)})$ converges to a fixed point when k approaches infinity.

Proof: For each $k \in \mathbb{N}$, denote $g^*(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) := \mathbf{L}\, f_{\mu^*, \lambda^*}(\mu, \lambda) + r_{P_i^*, \mu^*, \lambda^*}(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda)$ and $e^*(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) := g^*(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) - \frac{\ln\mu}{\lambda}$. Computing the Hessian of $e_{P_i^*, \mu^*, \lambda^*}$, it is found that

$$\mathbf{H}\, e^* = \begin{pmatrix} c_\mu + \frac{1}{\mu^2 \lambda} & \frac{1}{\mu \lambda^2} & 0 \\ \frac{1}{\mu \lambda^2} & c_\lambda - \frac{2\ln\mu}{\lambda^3} & 0 \\ 0 & 0 & c_P I \end{pmatrix}.$$

Since $c_\mu, c_P \geq 0$ and $c_\lambda \geq \frac{1 + 2\ln\mu}{\lambda^3}$, $\mathbf{H}\, e^* \succeq 0$ so the function e^* is convex in D. In addition, the value of e^* and the gradient of e^* at $(\{P_i^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*)$ are found to be 0, which implies that the minimum of e^* in D is 0 and the optimizer is $(\{P_i^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*)$. In other words, $\frac{\ln\mu^*}{\lambda^*} \leq \frac{\ln\mu}{\lambda} \leq g^*(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda)$, where the first inequality comes from the fact that $(\{P^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*)$ is a solution of (P1). Because both equalities above hold iff $(\{P\}_{i \in \mathcal{V}}, \mu, \lambda) = (\{P^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*)$, we conclude that $(\{P^*\}_{i \in \mathcal{V}}, \mu^*, \lambda^*)$ is a fixed point.

To show that the sequence $\tau^{(k)}$ is monotonically decreasing, we use the shorthand notation that $g^{(k)}(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda) := \mathbf{L}\, f_{\mu^{(k)}, \lambda^{(k)}}(\mu, \lambda) + r_{P_i^{(k)}, \mu^{(k)}, \lambda^{(k)}}(\{P_i\}_{i \in \mathcal{V}}, \mu, \lambda)$. We have

$$\tau^{(k+1)} = \frac{\ln\mu^{(k+1)}}{\lambda^{(k+1)}} \leq g^{(k)}(\{P_i^{(k+1)}\}_{i \in \mathcal{V}}, \mu^{(k+1)}, \lambda^{(k+1)}) \leq g^{(k)}(\{P_i^{(k)}\}_{i \in \mathcal{V}}, \mu^{(k)}, \lambda^{(k)}) = \frac{\ln\mu^{(k)}}{\lambda^{(k)}} = \tau^{(k)}$$

where the first inequality comes from similar reasoning as above and the second inequality comes from the fact that $(\{P_i^{(k+1)}\}_{i \in \mathcal{V}}, \mu^{(k+1)}, \lambda^{(k+1)})$ is the minimizer of $g^{(k)}$. Hence the sequence $\tau^{(k)}$ is monotonically decreasing. Lastly, the convergence to a fixed point is concluded by appealing to [22, Theorem 3.1] and realizing the fact that $\tau^{(k)}$ is monotonic. ■

We conclude here with some remarks regarding Proposition 1. Firstly, the assumption that $c_\lambda \geq \frac{1 + 2\ln\mu}{\lambda^3}$ over D imposes constraints on D and hence only local convergence is guaranteed for Algorithm 1. In practice, we use a sufficiently large parameter c_λ in order for $(P_i^{(k)}, \mu^{(k)}, \lambda^{(k)}) \in D$ for all $k \in \mathbb{N}$. The limitation of local convergence is caused by the nonconvexity of (P1). Meanwhile, Proposition 1 guarantees the convergence of Algorithm 1 to a fixed point which is only a necessary condition of being an optimizer of (P1). In other words, the initial guesses $P_i^{(0)}, \mu^{(0)}, \lambda^{(0)}$ affect the output of Algorithm 1. Nevertheless, monotonicity implies that the ADT lower bound improves by applying Algorithm 1.

V. Examples and comparison

Here, we show two examples of continuous-time, graph-based switched systems and compute the ADT lower bounds which guarantee GAS. We use Algorithm 1 for the computation and compare the result with other alternative approaches.

A. Baseline approaches and an approach from literature

We start with an introduction to three baseline approaches for computing ADT lower bounds. These approaches are all based on Theorem 1 with some particular choices of P_i's.

a) *Naive choice of P_i's:* We choose the matrices P_i's solving the Lyapunov equations (9), and then find the maximal λ and minimal μ satisfying the constraints (7a) and (7b).

b) *Greedy choice of maximizing λ:* We first choose P_i's maximizing λ subject to (7a). This is a *generalized eigenvalue problem* (GEVP) and can be solved using the techniques in [23], [24]. We then find the minimal μ satisfying the constraints (7b).

c) *Greedy choice of minimizing μ:* We first choose P_i's minimizing μ subject to (7b). This is again a GEVP. We then find the maximal λ satisfying the constraints (7a).

Algorithm 1 is implemented via the platform `YALMIP` in Matlab, and for each iteration the SDP (P2) is solved using `SeDuMi`. We run the algorithm until the difference $|\tau^{(k)} - \tau^{(k-1)}|$ between consecutive estimates of the ADT lower bound get below a tolerance $\epsilon = 0.001$. In each example, we use the results generated by the three baseline approaches – which are feasible points to (P1) – as the initial guesses for our algorithm, and then take the best output.

As an additional comparison, we also use the approach in [15] for computing ADT lower bounds for both examples. This approach requires the matrices A_i's to be diagonalizable and examines the relation between their matrices of eigenvectors. With this information, one finds the maximal ratio between the total destabilising effects and total stabilizing effects in a cycle (termed as *cycle ratio*) over all possible cycles in $\mathcal{G}$, and then computes an ADT lower bound.

	Naive	Max. λ	Min. μ	Alg. 1	[15]
μ	10.47	5489	1.056	1.171	-
λ	0.2046	1.52	0.0053	0.5568	-
ADT lb.	11.48	5.664	10.39	0.2844	2.899
time (ms)	9	12	14	518	2

TABLE I: ADT lower bounds for Section V-B.

B. Two-mode, 4-dimensional switched system

Consider a two-mode, 4-dimensional switched linear system of form (4) with matrices

$$A_1 = \begin{pmatrix} -15 & 9 & -12 & -1 \\ -2 & 2 & -5 & -7 \\ 13 & -5 & -17 & 23 \\ 2 & 2 & -15 & 10 \end{pmatrix}, \; A_2 = \begin{pmatrix} -14 & 11 & -19 & 6 \\ -10 & 7 & -15 & 5 \\ 3 & -1 & -7 & 9 \\ -6 & 5 & -15 & 8 \end{pmatrix}.$$

Both switches, from mode 1 to 2 and from mode 2 to 1, are allowed. Table I shows the computed values of the ADT lower bounds, λ, and μ using the approaches in Section V-A. From the table, we observe that the approaches which greedily choose P_i's so that either λ is maximized or μ is

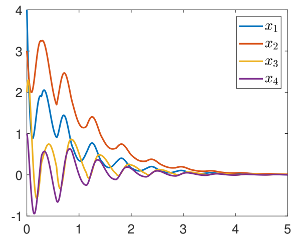

Fig. 1: Trajectories for the example in Section V-B starting from $x(0) = (4, 3, 2, 1)$ with dwell time 0.285.

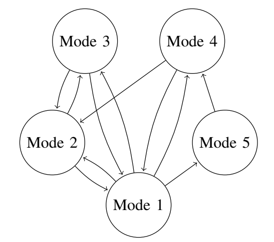

Fig. 2: Switching graph $\mathcal{G}$ of Section V-C.

minimized do not yield small values for the ADT lower bound. However, Algorithm 1 is capable of balancing μ and λ so that the ADT lower bound is further minimized. It is also significant that Algorithm 1 finds an ADT lower bound that is about 10 times smaller than the value computed with the cycle ratio approach [15]. To explain this improvement, we note that, although the cycle ratio approach is not based on Lyapunov functions, the idea of eigendecomposition can be interpreted as choosing Lyapunov functions that maximize the decay rates of all subsystems. Therefore, this approach is similar to the greedy approach of maximizing λ (albeit with better performance because the ratio computation per cycle is less conservative than employing the uniform parameter μ). In contrast, Algorithm 1 is capable of finding a smaller ADT lower bound by employing the fact that the gain at switches may be further minimized if the Lyapunov functions for the subsystems are chosen differently. As an illustration, Figure 1 shows a trajectory converging to the equilibrium implemented with a dwell time significantly smaller than the ADT lower bound computed by the cycle ratio approach but slight larger than the one computed by Algorithm 1.

The last row of Table I displays the total computation times for each approach. The approach of [15] is the fastest because it just implements eigendecomposition, followed with finding the maximal cycle ratio, and hence it has a complexity of $O(mn^3 + mp + m^2 \log m)$, where $m = |\mathcal{V}|$ and $p = |\mathcal{E}|$. Instead, each iteration of Algorithm 1 has a complexity of $O(m^{4.5}n^{6.5}+m^2n^{6.5}p^{2.5}+n^{3.5}p^{3.5})$, based on the complexity of `SeDuMi` provided in [25] and assuming no simplification is used for block diagonal LMI. This results in Algorithm 1 having the largest computation time (albeit we should also note that we have not optimized the implementation on Matlab, and currently rely on the external solver `SeDuMi` to solve the SDP optimization). As noted above, this is because our design puts the emphasis on minimizing the ADT lower bound, whereas optimality is not in the scope of the other approaches. We also point out that computation time is not an issue in the scenarios of application we envision, where the algorithm is run offline to provide a reference for the switched system designer. Interestingly, Algorithm 1 terminates after 3 iterations and, after the first iteration, already achieves $\tau^{(1)} = 0.4382$, which is much smaller than the results produced by the other approaches. This shows that while other approaches are fast, they severely underestimate the possible improvement in the ADT lower bound by optimizing over the Lyapunov functions.

C. Five-mode, 3-dimensional switched system

Consider a five-mode, 3-dimensional switched system of form (4) with matrices given by

$$A_1 = \begin{pmatrix} -5 & 1 & 2 \\ 0 & -5 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 3 \\ -2 & -1 & -3 \\ -1 & 0 & -2 \end{pmatrix}, \; A_4 = \begin{pmatrix} -4 & 0 & -3 \\ 2 & -2 & 4 \\ 1 & 0 & -1 \end{pmatrix},$$
$$A_5 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & 0 & -4 \end{pmatrix}.$$

The switching graph of this system is shown in Figure 2 and the computed ADT lower bounds through different approaches are summarized in Table II. The approach in [15] is not applicable because the matrix A_2 is not diagonalizable. Note that when the number of modes is large and the switching graph gets more complex, the greedy approaches produce lower-quality ADT lower bounds as they compute extremely conservative values of μ or λ while optimizing the other one. Algorithm 1 finds an ADT lower bound which is significantly smaller by balancing μ and λ.

	Naive	Max. λ	Min. μ	Alg. 1	[15]
μ	20.17	3964900	1.443	3.071	-
λ	0.286	1.959	0.0011	0.9178	-
ADT lb.	10.5	7.757	334.6	1.222	-
time (ms)	13	23	38	2423	-

TABLE II: ADT lower bounds for Section V-C.

Consistent with the previous example, Algorithm 1 takes a computation time to find an optimal ADT lower bound longer than the other approaches for the reasons explained above. Here, Algorithm 1 takes 7 iterations to converge, with each iteration taking about 340 ms. After the first iteration, the algorithm already yields $\tau^{(1)} = 2.597$, which is much smaller than the other approaches.

VI. Conclusions

We have studied the problem of finding ADT lower bounds for switching signals that can guarantee GAS, ISS or iISS of continuous-time, graph-based switched systems. We formulated the problem as an optimization problem, which essentially minimizes the ADT lower bound computed over the parameters given by different choices of Lyapunov functions. This optimization problem was then solved via an iterative algorithm with local convergence guarantees. From the demonstration of examples and the comparison with previous results, we found that the ADT lower bounds produced by our algorithm are relatively small and, hence, favorable for practical switching-signal design purposes. Future research will develop an analysis of computational complexity that addresses the SCP initialization and characterizes its convergence rate. We also plan to explore the combination of cycle ratios with our technique of optimizing over the matrices that define the Lyapunov functions by defining constraints on cycles, rather than on edges, to further improve the ADT lower bound.

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