Data-Driven Optimal Control of Bilinear Systems

Zhenyi Yuan and Jorge Cortés

Abstract—**This paper develops a method to learn optimal controls from data for bilinear systems without a priori knowledge of the system dynamics. Given an unknown bilinear system, we first characterize when the available data is suitable to solve the optimal control problem. This characterization leads us to propose an online control experiment design procedure that guarantees that any input/state trajectory can be represented as a linear combination of collected input/state data matrices. Leveraging this data-based representation, we transform the original optimal control problem into an equivalent data-based optimization problem with bilinear constraints. We solve the latter by iteratively employing a convex-concave procedure to convexify it and find a locally optimal control sequence. Simulations show that the performance of the proposed data-based approach is comparable with model-based methods.**

I. Introduction

The widespread availability of data, together with increasing computational capabilities to store, process, and manipulate it, has supercharged the research activity in learning, modeling, and control of dynamical phenomena across science and engineering. Data-driven control has emerged as an appealing way of leveraging this data surge by employing solid theoretical principles to design controllers that do not require explicit a priori knowledge of the plant to be controlled. This paper contributes to this body of work by studying the data-driven synthesis of optimal control laws for bilinear systems.

Literature Review. Data-driven control approaches include indirect and direct methods [1]. Indirect methods identify system models from data prior to proceeding to the synthesis of model-based controllers, while direct approaches bypass the intermediate modeling step and construct controllers directly from data. A diverse range of factors, including the complexity of the plant, the cost and practicality of performing system identification, and the amount and quality of the available data, play a key role in the suitability and performance of each of these approaches, see e.g., [2], [3]. The direct data-driven approach has been particularly fruitful for linear systems, where tools from behavioral theory [4] have allowed to express the system trajectories in terms of sufficiently-rich data. This has resulted in the synthesis of feedback stabilizing controllers [5], [6], optimal control laws [7]–[9], predictive controllers [10], [11], network controllers [11], [12], control experiment design [13], optimization-based controllers [14], and recent extensions to nonlinear systems [15], [16]. Here we focus on direct data-driven control of bilinear systems as a building block for future work on more complex nonlinear systems. These systems are often viewed as the bridge between linear and nonlinear systems due to their special properties. Many processes in engineering, biology and ecology can be modeled as bilinear systems [17]. Moreover, [18] shows that control-affine nonlinear systems can be exactly bilinearized. The recent work [15] proposes a local stabilizing data-driven controller design for bilinear systems. Here, we focus on the synthesis of optimal controllers. Model-based approaches to optimal control of bilinear systems include [19]–[21], which treat the bilinear system as a time-varying linear system and solve the optimization problem by applying iteratively the Pontryagin's maximum principle, and [22], which gives a lower bound on the minimum control energy required to steer the bilinear system using the reachability Gramian.

Statement of Contributions. We consider[1] discrete-time bilinear control systems and study the point-to-point optimal control problem over a finite time horizon. We assume the system matrices are unknown and seek to learn the optimal control from input/state data. We introduce the notion of suitable data to characterize when it is sufficiently informative for reconstructing the optimal control. Under this hypothesis, we show that any input/state trajectory can be represented as a linear combination of the collected input/state data. Owing to the nonlinear nature of bilinear systems, the problem of ensuring that data is sufficiently informative for optimal control requires us to introduce an online control experiment design. We show our design is guaranteed to yield suitable data in a finite number of steps. Building on this, we pose the optimal control synthesis problem as a data-based optimization with bilinear constraints. We show that a local solution to this nonconvex problem can be found by iteratively solving the convexified problems that result from applying a convex-concave approximation procedure. Throughout the presentation, we draw inspiration from and make connections with existing results on data-driven control of linear systems. Simulations show that the performance of the proposed data-based approach on bilinear systems is comparable with model-based methods.

[1]Throughout the paper, we use $\mathbb{R}$ and $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}_{>0}$) to denote the sets of real and non-negative (resp. positive) integer numbers, resp. Let I and $\mathbf{0}$ (resp. $\mathbf{1}$) denote the identity matrix and zero (resp. all-ones) vector/matrix with appropriate dimensions, resp. Given a function $f:\mathbb{Z}_{\geq 0}\to\mathbb{R}^d$ and $i,j\in\mathbb{Z}_{\geq 0}$, with $i\leq j$, we denote by $f_{[i,j]}$ the restriction of f to the interval $[i,j]$ in vector form, that is, $f_{[i,j]}=[f(i)^\top\ f(i+1)^\top\ \cdots\ f(j)^\top]^\top$, and $f_{\{i,j\}}$ the sequence $\{f(i),\dots,f(j)\}$. Given a vector $\mathbf{X}=[\mathbf{x}_1^\top\ \mathbf{x}_2^\top\ \cdots\ \mathbf{x}_j^\top]^\top\in\mathbb{R}^{ij}$ with $\mathbf{x}_1,\cdots,\mathbf{x}_j\in\mathbb{R}^i$, we let $\mathcal{H}_k(\mathbf{X})$ denote the Hankel matrix of depth $k\in\mathbb{Z}_{>0}$, with $k\leq j$,

$$\mathcal{H}_k(\mathbf{X}):=\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{j-k+1}\\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{j-k+2}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_j\end{bmatrix}\in\mathbb{R}^{ik\times(j-k+1)}.$$

Given two matrices $\mathbf{Y}$ and $\mathbf{Z}$ with proper dimensions, $[\mathbf{Y}\ \mathbf{Z}]$ and $[\mathbf{Y};\mathbf{Z}]$ denote their row- and column-concatenations, resp. Moreover, we use $\mathbf{Z}^\dagger$ and $\operatorname{Im}\mathbf{Z}$ to represent the pseudo-inverse and the image space of $\mathbf{Z}$, resp. Finally, $\otimes$ denotes the Kronecker product, while $\|\cdot\|$ represents the Euclidean norm.

This work was supported by AFOSR Award FA9550-19-1-0235 and NSF Award 2044900.

The authors are with the Department of Mechanical and Aerospace Engineering, UC San Diego. {z7yuan,cortes}@ucsd.edu.

II. Problem formulation

Consider the following discrete-time bilinear time-invariant control system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \Big[\sum_{j=1}^{n} \mathbf{x}_j(t)\mathbf{N}_j\Big]\mathbf{u}(t), \qquad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{u}(t) \in \mathbb{R}^m$ are the system state and input, respectively, and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{N}_j \in \mathbb{R}^{n\times m}, j = 1, \ldots, n$ are system matrices. Denoting $\mathbf{N} = [\mathbf{N}_1\ \mathbf{N}_2\ \cdots\ \mathbf{N}_n] \in \mathbb{R}^{n\times mn}$, the dynamics (1) can be alternatively written as

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}(\mathbf{x}(t) \otimes \mathbf{u}(t)). \qquad (2)$$

We make the following assumption.

Assumption II.1. *The pair* $(\mathbf{A}, [\mathbf{B}\ \mathbf{N}])$ *is controllable.*

Note that Assumption II.1 is weaker than asking for the bilinear system (2) to be controllable. Given initial $\mathbf{x}_0$ and target $\mathbf{x_f}$ states, we consider the following (point-to-point) optimal control problem over the time horizon T,

$$\begin{aligned} \min_{\mathbf{u}_{[0,T-1]}} \quad & \sum_{t=0}^{T-1} \mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t) \\ \text{s.t.} \quad & \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \Big[\sum_{j=1}^{n} \mathbf{x}_j(t)\mathbf{N}_j\Big]\mathbf{u}(t), \\ & \mathbf{x}(0) = \mathbf{x_0},\ \mathbf{x}(T) = \mathbf{x_f}. \end{aligned} \qquad \text{(P1)}$$

Here, $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and $\mathbf{R} \in \mathbb{R}^{m\times m}$ are positive semi-definite. The minimum-energy control problem corresponds to $\mathbf{Q} = \mathbf{0}$ and $\mathbf{R} = I$. This optimization is nonconvex and its closed-form solution is not known in general. The optimality conditions of (P1) lead to a nonlinear two-point boundary-value problem, for which there is no analytical solution available.

The problem we address is as follows. We assume the system matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{N}_j, j = 1, \ldots, n$ are unknown. Instead, we have access to input/state data of a control experiment of the system (2), that is, a control input sequence $\mathbf{u}_{\{0,L-1\}}$ along with the corresponding state sequence $\mathbf{x}_{\{0,L\}}$ of (2). Our objective is to develop an algorithmic procedure that is able to learn from the data the optimal control sequence $\mathbf{u}^\star_{\{0,T-1\}}$ that solves (P1).

III. Data suitability for optimal control

In this section, we characterize when the available data is suitable to solve the optimal control problem and discuss a procedure to design the control experiment. To motivate our discussion, we start by considering the linear system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (3)$$

(corresponding to $\mathbf{N} = \mathbf{0}$ in (2)). Let $\mathbf{x}_{\{0,L\}}$ be a state sequence generated by (3) with input sequence $\mathbf{u}_{\{0,L-1\}}$. According to Willems' fundamental lemma [4], [23], and assuming the pair $(\mathbf{A}, \mathbf{B})$ is controllable, if $\mathbf{u}_{\{0,L-1\}}$ is persistently exciting[2] of order $n + T$, then $\tilde{\mathcal{G}}_T(L) := \big[\mathbf{x}_{[0,L-T]}; \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\big] \in \mathbb{R}^{(n+mT)\times(L-T+1)}$ is full-row rank. This ensures that any input/state trajectory $(\bar{\mathbf{u}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T-1]})$ of length T of the linear system (3) can be expressed as

$$\begin{bmatrix} \bar{\mathbf{x}}_{[0,T-1]} \\ \hline \bar{\mathbf{u}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_T(\mathbf{x}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \end{bmatrix} \tilde{\alpha},$$

for some $\tilde{\alpha} \in \mathbb{R}^{L-T+1}$. This property plays a key role in the data-driven control of linear systems, including stabilization and system identification [5], predictive control [10], and point-to-point optimal control [9].

Now consider the original bilinear system (2). If, for a moment, we regard $\mathbf{x}(t) \otimes \mathbf{u}(t)$ as an independent input, then the dynamics corresponds to a linear system with input matrix $[\mathbf{B}\ \mathbf{N}]$ and control input $\mathbf{v}(t) = [\mathbf{u}(t); \mathbf{x}(t) \otimes \mathbf{u}(t)]$. Willems' fundamental lemma applied to this linear system implies that, under the Assumption II.1, if $\mathbf{v}_{\{0,L-1\}}$ is persistently exciting of order $n+T$, then $\big[\mathbf{x}_{[0,L-T]}; \mathcal{H}_T(\mathbf{v}_{[0,L-1]})\big]$ is full-row rank, which then ensures that any input/state trajectory $(\bar{\mathbf{v}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T-1]})$ of length T of system (2) can be represented by

$$\begin{bmatrix} \bar{\mathbf{x}}_{[0,T-1]} \\ \hline \bar{\mathbf{v}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_T(\mathbf{x}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{v}_{[0,L-1]}) \end{bmatrix} \alpha, \qquad (4)$$

for some $\alpha \in \mathbb{R}^{L-T+1}$. However, as we know, the input $\mathbf{v}$ is not independent, and ensuring it is persistently exciting is not guaranteed by simply asking for $\mathbf{u}$ to be so. These observations motivate our ensuing definitions and technical treatment.

A. Data suitability and parametrization of state trajectories

We next introduce the notion of data suitability for optimal control of bilinear systems.

Definition III.1. *(Data suitability for optimal control of bilinear systems): Let* $\mathbf{x}_{\{0,L\}}$ *be a state sequence generated by* (2) *with input sequence* $\mathbf{u}_{\{0,L-1\}}$. *The data* $\mathbf{x}_{\{0,L\}}$, $\mathbf{u}_{\{0,L-1\}}$ *is* suitable (for learning optimal controls) *if*

$$\mathcal{G}_T(L) := \begin{bmatrix} \mathcal{H}_1(\mathbf{x}_{[0,L-T]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \\ \hline \mathcal{H}_T(\mathbf{x}\otimes\mathbf{u}_{[0,L-1]}) \end{bmatrix} \in \mathbb{R}^{(n+mT+mnT)\times(L-T+1)}.$$

is full-row rank.

This definition requires as a necessary condition that $L \geq (mn+m+1)T+n-1$. We point out that $\mathcal{G}_T(L)$ being full-row rank is equivalent to $\big[\mathcal{H}_1(\mathbf{x}_{[0,L-T]}); \mathcal{H}_T(\mathbf{v}_{[0,L-1]})\big]$ being full-row rank, as both matrices can be obtained from each other by row permutation. Using (2), one can obtain the relation (5), where $\mathcal{O}_T \in \mathbb{R}^{nT\times n}$, $\mathcal{P}_T \in \mathbb{R}^{nT\times mT}$, $\mathcal{Q}_T \in \mathbb{R}^{nT\times mnT}$. If $\mathcal{G}_T(L)$ is full-row rank, it immediately follows that

$$[\mathcal{O}_T\ \mathcal{P}_T\ \mathcal{Q}_T] = \mathcal{H}_T(\mathbf{x}_{[1,L]})\mathcal{G}_T(L)^\dagger.$$

Remark III.2. *(Suitable data for optimal control versus for identification and stabilization):* When $T = 1$, we have $[\mathcal{O}_1\ \mathcal{P}_1\ \mathcal{Q}_1] = [\mathbf{A}\ \mathbf{B}\ \mathbf{N}]$, which corresponds to system identification. Moreover, if $\mathcal{H}_1(\mathbf{x}_{[0,L-1]})$ is full-row rank, then under knowledge of an upper bound on $\|\mathbf{N}\|$, one can construct locally stabilizing controllers directly from data, cf. [15]. Notice $\mathcal{G}_T(L)$ is full-row rank $\Rightarrow$ $\mathcal{G}_1(L)$ is full-row rank

[2] The signal $f_{\{0,L-1\}}$ is persistently exciting of order k if the matrix $\mathcal{H}_k(f_{[0,L-1]})$ is full-row rank.

$$\mathcal{H}_T(\mathbf{x}_{[1,L]}) = \Big[\underbrace{\begin{matrix}\mathbf{A}\\ \mathbf{A}^2\\ \vdots\\ \mathbf{A}^T\end{matrix}}_{\mathcal{O}_T} \Big| \underbrace{\begin{matrix}\mathbf{B} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{AB} & \mathbf{B} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}^{T-1}\mathbf{B} & \mathbf{A}^{T-2}\mathbf{B} & \cdots & \mathbf{B}\end{matrix}}_{\mathcal{P}_T} \Big| \underbrace{\begin{matrix}\mathbf{N} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{AN} & \mathbf{N} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A}^{T-1}\mathbf{N} & \mathbf{A}^{T-2}\mathbf{N} & \cdots & \mathbf{N}\end{matrix}}_{\mathcal{Q}_T}\Big]\mathcal{G}_T(L) \tag{5}$$

$\Rightarrow \mathcal{H}_1(\mathbf{x}_{[0,L-1]})$ is full-row rank. We deduce that suitable data comprises data needed for system identification and local stabilization. Also, although $\mathcal{O}_T, \mathcal{P}_T$ and $\mathcal{Q}_T$ can be constructed only using $\mathbf{A}, \mathbf{B}$ and $\mathbf{N}$ when $\mathcal{G}_1(L)$ is full-row rank, this is not enough to express any input/state trajectory of length T as a linear combination of the collected input/state data, and thus $\mathcal{G}_1(L)$ being full-row rank is not sufficient to recover optimal controls. Another important observation is that, in the linear case ($\mathbf{N} = \mathbf{0}$), according to [9], optimal controls over the time horizon T can be learned if $\tilde{\mathcal{G}}_T(L)$ is full-row rank. Moreover, one can identify and globally stabilize linear systems directly using data if $\tilde{\mathcal{G}}_1(L)$ is full-row rank, cf. [5]. Since $\tilde{\mathcal{G}}_T(L)$ is full-row rank $\Rightarrow \tilde{\mathcal{G}}_1(L)$ is full-row rank, this further reinforces the parallelism between the bilinear and linear cases regarding data conditions for different control problems. □

We have established that any input/state trajectory of (2) admits a data-based representation of the form (4). The next result establishes when the converse is also true, i.e., when a trajectory of the form (4) corresponds to a trajectory of (2).

Lemma III.3. *(Data-based representation of input/state trajectory in terms of suitable data): Let $\mathbf{x}_{\{0,L\}}$ and $\mathbf{u}_{\{0,L-1\}}$ be a suitable data set. Then*

(i) Any input/state trajectory $(\bar{\mathbf{u}}_{[0,T-1]}, \bar{\mathbf{x}}_{[0,T]})$ of system (2) can be represented as

$$\left[\frac{\bar{\mathbf{x}}_{[0,T]}}{\bar{\mathbf{u}}_{[0,T-1]}}\right] = \left[\frac{\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]})}{\mathcal{H}_T(\mathbf{u}_{[0,L-1]})}\right]\alpha$$

for some $\alpha \in \mathbb{R}^{L-T+1}$;

(ii) Conversely, let $\alpha \in \mathbb{R}^{L-T+1}$ such that

$$\bar{\mathbf{x}} \otimes \bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{x} \otimes \mathbf{u}_{[0,L-1]})\alpha, \tag{6}$$

where $\bar{\mathbf{x}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{x}_{[0,L-1]})\alpha$ and $\bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\alpha$. Then, $\left[\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}); \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\right]\alpha$ is an input/state trajectory of (2) over the time horizon T.

Proof. For statement (i), note that any input/state trajectory $(\bar{\mathbf{x}}_{[0,T]}, \bar{\mathbf{u}}_{[0,T-1]})$ of (2) is uniquely determined by $\bar{\mathbf{x}}(0) \in \operatorname{Im} \mathcal{H}_1(\mathbf{x}_{[0,L-T]})$ and $\bar{\mathbf{u}}_{[0,T-1]} \in \operatorname{Im} \mathcal{H}_T(\mathbf{u}_{[0,L-1]})$. Recalling that $\mathcal{G}_T(L)$ is full-row rank, statement (i) follows. For statement (ii), let α satisfy (6) and consider the initial state $\bar{\mathbf{x}}(0) = \mathcal{H}_1(\mathbf{x}_{[0,L-T]})\alpha$ and input sequence $\bar{\mathbf{u}}_{[0,T-1]} = \mathcal{H}_T(\mathbf{u}_{[0,L-1]})\alpha$. Then,

$$\begin{aligned}\bar{\mathbf{x}}_{[1,T]} &= [\mathcal{O}_T \ \mathcal{P}_T \ \mathcal{Q}_T]\left[\frac{\bar{\mathbf{x}}(0)}{\frac{\mathcal{H}_T(\bar{\mathbf{x}} \otimes \bar{\mathbf{u}}_{[0,T-1]})}{\bar{\mathbf{u}}_{[0,T-1]}}}\right] \\ &= [\mathcal{O}_T \ \mathcal{P}_T \ \mathcal{Q}_T]\, \mathcal{G}_T(L)\alpha = \mathcal{H}_T(\mathbf{x}_{[1,L]})\alpha,\end{aligned}$$

where we have employed (5). The conclusion follows by noting $\mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}) = [\mathcal{H}_1(\mathbf{x}_{[0,L-T]}); \mathcal{H}_T(\mathbf{x}_{[1,L]})]$. □

B. Online design of control experiment for data suitability

We discuss next how to ensure that the available data is suitable. Based on our discussion above, $\mathbf{v}_{\{0,L-1\}}$ being persistently exciting of order $n+T$ is enough to ensure the suitability of the data for bilinear systems. In contrast to the linear case, where $\mathbf{u}_{\{0,L-1\}}$ can be designed to be persistently exciting of any order by selecting control inputs offline, the persistence of excitation of $\mathbf{v}_{\{0,L-1\}}$ depends on both the control input $\mathbf{u}(t)$ and the system state $\mathbf{x}(t)$. Due to the unknown nonlinear dynamics, there is no available closed-form expression of $\mathbf{x}(t)$ in terms of $\mathbf{u}(t)$. Hence, selecting control inputs offline may not guarantee $\mathbf{v}_{\{0,L-1\}}$ to be persistently exciting of order $n+T$, which motivates an online approach to design $\mathbf{u}$ to ensure data suitability. The next results state some useful facts for our experiment design.

Proposition III.4. *(Scaled persistently exciting input returns a full-row rank Hankel matrix of state data): Consider system (2) and further assume that the pair $(\mathbf{A}, \mathbf{B})$ is controllable. Then, for any input sequence $\mathbf{u}_{\{0,L-1\}}$ that is persistently exciting of order $n+k$, there exists $\bar{\varepsilon}$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, the input sequence $\varepsilon\mathbf{u}_{\{0,L-1\}}$ with initial state $\mathbf{x}(0) = \mathbf{0}$ ensures $\mathcal{H}_k(\mathbf{x}_{[1,L]})$ is full-row rank.*

We omit the proof for space reasons, but note that the result follows by using for the higher-order case of $k \geq 1$ the same arguments employed in [16] for the case of $k = 1$.

Proposition III.5. *(Property on the left kernel of $\mathcal{G}_T(t)$ when it is not full-row rank): Suppose $\mathcal{G}_T(t)$ is not full-row rank for some $t \geq T$. If*

$$\left[\begin{array}{c}\mathbf{x}(t-T+1)\\ \hline \mathbf{u}_{[t-T+1,t-1]}\\ \hline \mathbf{x} \otimes \mathbf{u}_{[t-T+1,t-1]}\end{array}\right] \in \operatorname{Im}\left[\begin{array}{c}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-2]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-2]})\end{array}\right], \tag{7}$$

then there must exist $\xi \in \mathbb{R}^n$, $\eta_1, \ldots, \eta_T \in \mathbb{R}^m$, and $\chi_1, \ldots, \chi_T \in \mathbb{R}^{mn}$ such that the following holds

$$[\xi^\top \ \eta_1^\top \ \cdots \ \eta_T^\top \ \chi_1^\top \ \cdots \ \chi_T^\top]\mathcal{G}_T(t) = \mathbf{0}, \tag{8}$$

with at least one in $\{\eta_T, \chi_T\}$ not equal to $\mathbf{0}$.

Proof. We reason by contradiction. Suppose all vectors of the form $\left[\xi^\top \ \eta_1^\top \ \cdots \ \eta_T^\top \ \chi_1^\top \ \cdots \ \chi_T^\top\right]$ in the left kernel of $\mathcal{G}_T(t)$ satisfy that both η_T and χ_T are equal to $\mathbf{0}$. Then,

$$[\xi^\top \ \eta_1^\top \cdots \eta_{T-1}^\top \ \chi_1^\top \cdots \chi_{T-1}^\top]\left[\begin{array}{c}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-2]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-2]})\end{array}\right] = \mathbf{0}.$$

Combining this with (7), we deduce that

$$[\xi^\top \ \eta_1^\top \cdots \eta_{T-1}^\top \ \chi_1^\top \cdots \chi_{T-1}^\top]\left[\begin{array}{c}\mathcal{H}_1(\mathbf{x}_{[0,t-T+1]})\\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[0,t-1]})\\ \hline \mathcal{H}_{T-1}(\mathbf{x} \otimes \mathbf{u}_{[0,t-1]})\end{array}\right] = \mathbf{0}.$$

Combining this with the fact that

$$\begin{bmatrix} \mathcal{H}_1(\mathbf{x}_{[1,t-T+1]}) \\ \hline \mathcal{H}_{T-1}(\mathbf{u}_{[1,t-1]}) \\ \hline \mathcal{H}_{T-1}(\mathbf{x}\otimes\mathbf{u}_{[1,t-1]}) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix} \mathcal{G}_T(t),$$

we obtain

$$\left[\xi^\top \mathbf{A}\ \xi^\top \mathbf{B}\ \eta_1^\top\ \cdots\ \eta_{T-1}^\top\ \xi^\top \mathbf{N}\ \chi_1^\top\ \cdots\ \chi_{T-1}^\top\right] \mathcal{G}_T(t) = \mathbf{0}.$$

Consequently, given our hypothesis of contradiction, η_{T-1} and χ_{T-1} must both be equal to $\mathbf{0}$. Following a similar procedure iteratively, we conclude that $\eta_{T-1} = \cdots = \eta_1 = \mathbf{0}$ and $\chi_{T-1} = \cdots = \chi_1 = \mathbf{0}$. This implies that $\operatorname{Im}\mathcal{G}_T(t) = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) \times \mathbb{R}^{(m+mn)T}$. Left multiplying by $[\mathbf{A}\ \mathbf{B}\ \mathbf{0}\ \mathbf{N}\ \mathbf{0}]$ on both sides, we obtain $\mathbf{A}\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) + \operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{N} = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[1,t-T+1]})$. Since $\mathbf{x}(t-T+1) \in \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$, then $\mathbf{A}\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]}) + \operatorname{Im}\mathbf{B} + \operatorname{Im}\mathbf{N} = \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$. This implies $\operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$ is an $\mathbf{A}$-invariant subspace containing $\operatorname{Im}[\mathbf{B}\ \mathbf{N}]$. Since the reachable subspace of the pair $(\mathbf{A},[\mathbf{B}\ \mathbf{N}])$ is $\mathbb{R}^n$ by Assumption II.1, and the fact that it is also the smallest $\mathbf{A}$-invariant subspace containing $\operatorname{Im}[\mathbf{B}\ \mathbf{N}]$, we deduce that $\mathbb{R}^n \subseteq \operatorname{Im}\mathcal{H}_1(\mathbf{x}_{[0,t-T]})$. Therefore $\mathbb{R}^{(m+mn)T+n} \subseteq \operatorname{Im}\mathbf{x}_{[0,t-T]} \times \mathbb{R}^{(m+mn)T} = \operatorname{Im}\mathcal{G}_T(t)$, which contradicts the fact that $\mathcal{G}_T(t)$ is not full-row rank. □

Based on Propositions III.4 and III.5, we introduce the online control experiment procedure in Algorithm 1 to ensure data suitability. The underlying idea of the strategy is to increase the row rank of $\mathcal{G}_T(t)$ at each step.

Algorithm 1 Online control experiment design

Input: $\mathbf{x}(0) = \mathbf{0}$, $\|\mathbf{u}(i)\| < \epsilon$ for $i = 0,\ldots,T-1$ s.t. $\mathcal{G}_T(T) \neq \mathbf{0}$, ϵ sufficient close to 0, $t := T$, $k := 1$
repeat
  while $\mathcal{H}_{n+k}(\mathbf{u}_{[0,t-1]})$ is full-row rank **do**
    $k \leftarrow k+1$ ▷ Increase order
  end while
  if (7) holds **then**
    select $\xi \in \mathbb{R}^n$, $\eta = \left[\eta_1^\top \ldots \eta_T^\top\right]^\top \in \mathbb{R}^{mT}$, and $\chi = \left[\chi_1^\top \ldots \chi_T^\top\right]^\top \in \mathbb{R}^{mnT}$ s.t. (8) holds, with $\left[\eta_T^\top\ \chi_T^\top\right] \neq \mathbf{0}$
    if $\eta_T^\top + \chi_T^\top(\mathbf{x}(t)\otimes I) \neq \mathbf{0}$ **then**
      choose $\|\mathbf{u}(t)\| < \epsilon$ s.t. $\xi^\top\mathbf{x}(t-T+1) + \eta^\top\mathbf{u}_{[t-T+1,t]} + \chi^\top\mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]} \neq \mathbf{0}$ holds
    else
      choose $\|\mathbf{u}(t)\| < \epsilon$ s.t. rowrk $(\mathcal{H}_{n+k}(\mathbf{u}_{[0,t]}))$ increases
    end if
  else
    choose $\|\mathbf{u}(t)\| < \epsilon$ arbitrarily
  end if
  $t \leftarrow t+1$ ▷ Update iteration
until $\mathcal{G}_T(t)$ is full-row rank
$L \leftarrow t$
Output: Full-row rank $\mathcal{G}_T(L)$

Theorem III.6. *(Online control experiment design to ensure data suitability): Let* $(\mathbf{A},\mathbf{B})$ *be controllable and design the control experiment for system* (2) *according to Algorithm 1. Then the output* $\mathcal{G}_T(L)$ *is full-row rank.*

Proof. Given $t \geq T$, assume $\mathcal{G}_T(t)$ is not full-row rank. If (7) does not hold, it is easy to see that any choice of $\mathbf{u}(t)$ leads to rowrk $(\mathcal{G}_T(t+1)) >$ rowrk $(\mathcal{G}_T(t))$. Hence, we concentrate on the case when (7) holds. In this case, from Proposition III.5, we know there exist $\xi \in \mathbb{R}^n$, $\eta_1,\ldots,\eta_T \in \mathbb{R}^m$, and $\chi_1,\ldots,\chi_T \in \mathbb{R}^{mn}$, with at least one in $\{\eta_T,\chi_T\}$ not equal to $\mathbf{0}$ making (8) hold. We aim to design $\mathbf{u}(t)$ to satisfy $\xi^\top\mathbf{x}(t-T+1) + \eta^\top\mathbf{u}_{[t-T+1,t]} + \chi^\top\mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]} \neq \mathbf{0}$ so that $[\mathbf{x}(t-T+1);\mathbf{u}_{[t-T+1,t]};\mathbf{x}\otimes\mathbf{u}_{[t-T+1,t]}]$ does not belong to $\operatorname{Im}\mathcal{G}_T(t)$, which ensures rowrk $(\mathcal{G}_T(t+1)) >$ rowrk $(\mathcal{G}_T(t))$. Such $u(t)$ can be found as long as $\eta_T^\top + \chi_T^\top(\mathbf{x}(t)\otimes I) \neq \mathbf{0}$. If this is not the case, any selection of $\mathbf{u}(t)$ will not affect whether the row rank of $\mathcal{G}_T(t)$ will increase or not at this time step. We prove by contradiction that this situation will not occur indefinitely under Algorithm 1. Suppose $\eta_T^\top + \chi_T^\top(\mathbf{x}(\ell)\otimes I) = \mathbf{0}$ holds for all $\ell \geq t$, it then follows that $\eta_T^\top + \chi_T^\top(\mathcal{H}_1(\mathbf{x}_{[t,\ell]})\alpha\otimes I) = \mathbf{0}$ for any $\alpha \in \mathbb{R}^{\ell-t+1}$ with $\mathbf{1}^\top\alpha = 1$. According to Algorithm 1, the order k is increased, followed by an input selection that makes $\mathcal{H}_{n+k}(\mathbf{u}_{[0,\ell]})$ full-row rank. Let ℓ sufficiently large so that $k > t$. In this case, $\mathcal{H}_{n+t}(\mathbf{u}_{[0,\ell]})$ is full-row rank and, using Proposition III.4, $\mathcal{H}_t(\mathbf{x}_{[1,\ell]})$ is full-row rank too. The latter implies that $\mathcal{H}_1(\mathbf{x}_{[t,\ell]})$ is full-row rank. Together with the fact that at least one in $\{\eta_T,\chi_T\}$ is not equal to $\mathbf{0}$, there must exist $\alpha \in \mathbb{R}^{\ell-t+1}$ with $\mathbf{1}^\top\alpha = 1$ such that $\eta_T^\top + \chi_T^\top(\mathcal{H}_1(\mathbf{x}_{[t,\ell]})\alpha\otimes I) \neq \mathbf{0}$ holds, which is a contradiction. This argument shows that Algorithm 1 increases the row rank of $\mathcal{G}_T(t)$ by one after finitely many steps. Hence, in a finite number of steps, the algorithm terminates with a full-row rank $\mathcal{G}_T(L)$. □

Note that the controllability assumption on the pair $(\mathbf{A},\mathbf{B})$ is only necessary to ensure Algorithm 1 is successful, cf. Theorem III.6. Our design methodology below is still valid as long as a full row-rank matrix $\mathcal{G}_T(L)$ can be obtained.

IV. Data-driven control design

Here, we lay out our algorithmic procedure to find a local solution of the optimal control problem (P1) using suitable data. Our first step is to provide an equivalent data-based representation of the optimization problem. We then iteratively apply a convex-concave procedure to solve it efficiently. The next result provides a data-based formulation of the optimal control problem (P1), provided the available data is suitable.

Theorem IV.1. *(Data-based reformulation of optimal control problem): Given a data set* $\mathbf{x}_{\{0,L\}}$ *and* $\mathbf{u}_{\{0,L-1\}}$ *suitable for optimal control, problem* (P1) *is equivalent to the following data-based optimization:*

$$\begin{aligned} \min_{\alpha} \quad & \sum_{t=0}^{T-1} \bar{\mathbf{x}}^\top(t)\mathbf{Q}\bar{\mathbf{x}}(t) + \bar{\mathbf{u}}^\top(t)\mathbf{R}\bar{\mathbf{u}}(t) \\ \text{s.t.} \quad & \begin{bmatrix} \bar{\mathbf{x}}_{[0,T]} \\ \hline \bar{\mathbf{u}}_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{T+1}(\mathbf{x}_{[0,L]}) \\ \hline \mathcal{H}_T(\mathbf{u}_{[0,L-1]}) \end{bmatrix}\alpha, \\ & \bar{\mathbf{x}}(0) = \mathbf{x_0}, \bar{\mathbf{x}}(T) = \mathbf{x_f}, (6) \text{ holds.} \end{aligned} \qquad \text{(P2)}$$

The proof of this result readily follows from Lemma III.3. Notice that the optimization problem (P2) is nonconvex because of the presence of bilinear terms $\alpha_i\alpha_j$, $i,j \in \{1,\ldots,L-$

$T+1\}$, in the constraints. Here, we describe a convex–concave procedure from [24] that can be iteratively employed to solve it[3]. We first describe the bilinear terms with new variables $r_{i,j} = \alpha_i \alpha_j$, which we employ in the constraints in (P2) to make them all become affine. We represent this set of constraints by $\mathcal{A}_1(\alpha, r) = \mathbf{0}$. Additionally, we write each equality $r_{i,j} = \alpha_i \alpha_j$ with the following equivalent representation

$$(\alpha_i + \alpha_j)^2 - (\alpha_i^2 + \alpha_j^2) - 2r_{i,j} \leq 0,$$
$$(\alpha_i^2 + \alpha_j^2) - (\alpha_i + \alpha_j)^2 + 2r_{i,j} \leq 0.$$

We gather all these new nonconvex constraints in the expression $\mathcal{C}_1(\alpha) - \mathcal{C}_2(\alpha) + \mathcal{A}_2(r) \leq 0$, where $\{\mathcal{C}_i\}_{i=1}^2$, and $\mathcal{A}_2$ are vector-valued convex function and affine functions, resp. Using $\mathcal{C}_0$ to denote the convex cost function, (P2) reads

$$\begin{aligned} \min_{\alpha,r} \quad & \mathcal{C}_0(\alpha) \\ \text{s.t.} \quad & \mathcal{C}_1(\alpha) - \mathcal{C}_2(\alpha) + \mathcal{A}_2(r) \leq \mathbf{0}, \\ & \mathcal{A}_1(\alpha, r) = \mathbf{0}. \end{aligned} \tag{P3}$$

The nonconvex constraint in (P3) can be convexified by linearizing the concave function $-\mathcal{C}_2$. We perform such convexification iteratively to yield Algorithm 2. The next result follows from [24, Section 1.3].

Lemma IV.2. *(Convergence to critical point of* (P2)*): Given a feasible initial point α^0, all iterates of Algorithm 2 are feasible, $\{\mathcal{C}_0(\alpha^k)\}_{k=1}^\infty$ decreases monotonically, and $\{\alpha^k\}_{k=1}^\infty$ converges to a critical point $\alpha^\star$ of* (P2).

Algorithm 2 Convex-concave procedure to solve ((P3))

1: **Given** Initial feasible point α^0, $k := 0$.
2: **repeat**
3: Let $\bar{\mathcal{C}}_2(\alpha, \alpha^k) \triangleq \mathcal{C}_2(\alpha^k) + \nabla_\alpha \mathcal{C}_2(\alpha^k)^\top (\alpha - \alpha^k)$ ▷ Convexifying the constraint
4: Set α^{k+1} to be the solution of the convex problem ▷ Convex optimization

$$\begin{aligned} \min_{\alpha,r} \quad & \mathcal{C}_0(\alpha) \\ \text{s.t.} \quad & \mathcal{C}_1(\alpha) - \bar{\mathcal{C}}_2(\alpha, \alpha^k) + \mathcal{A}_1(r) \leq 0 \\ & \mathcal{A}_2(\alpha, r) = 0 \end{aligned}$$

5: $k \leftarrow k+1$ ▷ Update iteration
6: **until** convergence

V. Simulation examples

Here we illustrate the effectiveness of the proposed data-based approach to solve the optimal control problem (P1). For comparison, we use the model-based approaches taken in [20], [22] designed specifically for bilinear systems.

Example V.1. *(Population control):* We consider a population control problem introduced in [20, Example 1] evolving in continuous time. For the horizon $T = 20$, we use a first-order Euler discretization with stepsize 0.1. The resulting discrete-time bilinear system is

$$\mathbf{x}(t+1) = \mathbf{x}(t) + 0.1\mathbf{x}(t)\mathbf{u}(t).$$

We take $\mathbf{Q} = \mathbf{R} = 1$ and consider $\mathbf{x_0} = 1$, $\mathbf{x_f} = \frac{1}{3}$. We perform a control experiment with $L = 60$ using randomly generated inputs, and verify that the resulting $\mathcal{G}_{20}(60)$ is full-row rank. Algorithm 2 obtains a local optimum $\alpha^\star$ of (P2). Fig. 1(a) shows the trajectories, both displaying similar performance, obtained from the data-based solution in Theorem IV.1 with that of the model-based iterative method [20]. □

Example V.2. *(Minimum-energy control problem):* Consider the following bilinear system from [22, Example 4.5]:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t),$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.024 & 0 & 0 \\ 1 & 0 & -0.26 & 0 & 0 \\ 0 & 1 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & -0.06 \\ 0 & 0 & 0.15 & 1 & 0.5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.5 \end{bmatrix},$$
$$\mathbf{N} = \text{diag}(0.1, 0.2, 0.3, 0.4, 0.5).$$

We consider the minimum-energy control problem ($\mathbf{Q} = \mathbf{0}$, $\mathbf{R} = I$) with $T = 10$. Let $\mathbf{x_0} = \mathbf{0}$ and $\mathbf{x_f} = [0.0004\ -0.00038\ 0.00318\ 0.00062\ 0.00219]^\top$. We perform a control experiment with $L = 74$ using Algorithm 1. We solve (P2) using Algorithm 2. For comparison, we use the Gramian-based lower bound of the optimal cost value obtained in [22],

$$\sum_{t=0}^{T-1} \mathbf{u}^{\star\top}(t)\mathbf{u}^\star(t) \geq \mathbf{x}^\top(T)\mathcal{W}^{-1}\mathbf{x}(T),$$

where $\mathcal{W}$ is the reachability Gramian of the bilinear system. Fig. 1(b) compares this lower bound with the values obtained with the trajectories from the data-based solution in Theorem IV.1, showing a close agreement between the two. □

Example V.3. *(Minimum-energy control problem):* We consider another minimum-energy control example from [20, Example 2], for which we use a first-order Euler discretization with stepsize 0.02. The discrete-time bilinear system is:

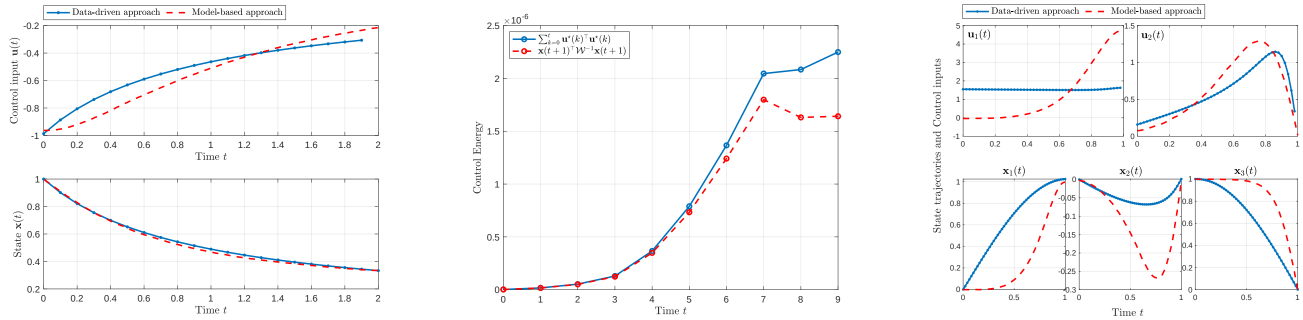
$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \Big[\sum_{j=1}^{3} \mathbf{x}_j(t)\mathbf{N}_j\Big]\mathbf{u}(t),$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & -0.01 & 0 \\ 0.01 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{B} = \mathbf{0}, \mathbf{N}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.02 & 0 \end{bmatrix},$$
$$\mathbf{N}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.02 \end{bmatrix}, \mathbf{N}_3 = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.02 \\ 0 & 0 \end{bmatrix}.$$

We consider $T = 50$ and perform a control experiment with $L = 452$ randomly generated inputs, and verify $\mathcal{G}_{50}(452)$ is full-row rank. We let $\mathbf{x_0} = [0\ 0\ 1]^\top$, $\mathbf{x_f} = [1\ 0\ 0]^\top$. We solve (P2) using Algorithm 2 to obtain $\alpha^\star$ and compare, cf. Fig. 1(c), the trajectories obtained from the data-based

[3]Local optimal solutions to bilinear programs can also be found using the OPTI Toolbox in MATLAB.



(a) Example V.1 (b) Example V.2 (c) Example V.3

Fig. 1: Performance of the proposed data-driven approach (solid blue lines) versus model-based approaches (dashed red lines). The total cost values are (a) $0.1 \times \sum_{t=0}^{19} \mathbf{x}^2(t) + \mathbf{u}^2(t) = 1.3346$ for the data-based approach and $\int_0^2 \mathbf{x}^2(\tau) + \mathbf{u}^2(\tau) d\tau = 1.3506$ for the model-based iterative method in [20]; (b) $\sum_{t=0}^{9} \mathbf{u}(t)^\top \mathbf{u}(t) = 2.25 \times 10^{-6}$ for the data-based approach and $\mathbf{x}(10)^\top \mathcal{W}^{-1} \mathbf{x}(10) = 1.64 \times 10^{-6}$ for the Gramian-based lower bound in [22]; and (c) $0.02 \times \sum_{t=0}^{49} \mathbf{u}^\top(t)\mathbf{u}(t) = 2.7999$ for the data-based approach and $\int_0^1 \mathbf{u}^\top(\tau)\mathbf{u}(\tau) d\tau = 4.7976$ for the model-based iterative method in [20].

solution in Theorem IV.1 with that of the model-based iterative method [20], showing that the data-driven approach finds a better local optimum. □

VI. Conclusions

We have presented a data-driven method to learn optimal controls of bilinear systems directly from input/state data without a priori knowledge of the system matrices. The nonlinear nature of the dynamics has led us to introduce a notion to describe when data is suitable to recover the optimal controls. We have provided an online control experiment design method to obtain data with such properties and introduced an equivalent data-based reformulation of the original optimal control problem. Given its nonconvexity, we have employed an iterative convex-concave algorithmic procedure to solve it. Simulations show that the data-based approach finds optimal control trajectories with comparable performance to those obtained by model-based methods. Future work will explore extensions to noisy data and robustness analysis, investigate weaker notions under which data is suitable to reconstruct optimal controls, and design distributed implementations for large-scale bilinear networks.

References

[1] Z. Hou and Z. Wang, "From model-based control to data-driven control: Survey, classification and perspective," *Information Sciences*, vol. 235, pp. 3–35, 2013.

[2] V. Krishnan and F. Pasqualetti, "On direct vs indirect data-driven predictive control," *arXiv preprint arXiv:2103.14936*, 2021.

[3] F. Dörfler, J. Coulson, and I. Markovsky, "Bridging direct & indirect data-driven control formulations via regularizations and relaxations," *arXiv preprint arXiv:2101.01273*, 2021.

[4] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[5] C. D. Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality and robustness," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2019.

[6] H. J. V. Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4753–4768, 2020.

[7] G. R. Gonçalves da Silva, A. S. Bazanella, C. Lorenzini, and L. Campestrini, "Data-driven LQR control design," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 180–185, 2018.

[8] G. Baggio, V. Katewa, and F. Pasqualetti, "Data-driven minimum-energy controls for linear systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 589–594, 2019.

[9] G. Baggio and F. Pasqualetti, "Learning minimum-energy controls from heterogeneous data," in *American Control Conference*, Denver, CO, Jul. 2020, pp. 3991–3996.

[10] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: in the shallows of the DeePC," in *European Control Conference*, Naples, Italy, Jun. 2019, pp. 307–312.

[11] A. Allibhoy and J. Cortés, "Data-based receding horizon control of linear network systems," *IEEE Control Systems Letters*, vol. 5, no. 4, pp. 1207–1212, 2021.

[12] G. Baggio, D. S. Bassett, and F. Pasqualetti, "Data-driven control of complex networks," *Nature Communications*, vol. 12, no. 1, pp. 1–13, 2021.

[13] H. J. van Waarde, "Beyond persistent excitation: Online experiment design for data-driven modeling and control," *IEEE Control Systems Letters*, vol. 6, pp. 319–324, 2022.

[14] G. Bianchin, M. Vaquero, J. Cortés, and E. Dall'Anese, "Online stochastic optimization of unknown linear dynamical systems: data-driven controller synthesis and analysis," *IEEE Transactions on Automatic Control*, 2021, submitted.

[15] A. Bisoffi, C. D. Persis, and P. Tesi, "Data-based stabilization of unknown bilinear systems with guaranteed basin of attraction," *Systems & Control Letters*, vol. 145, p. 104788, 2020.

[16] C. De Persis and P. Tesi, "Designing experiments for data-driven control of nonlinear systems," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 285–290, 2021.

[17] C. Bruni, G. Dipillo, and G. Koch, "Bilinear systems: An appealing class of "nearly linear" systems in theory and applications," *IEEE Transactions on Automatic Control*, vol. 19, no. 4, pp. 334–348, 1974.

[18] D. Goswami and D. A. Paley, "Bilinearization, reachability, and optimal control of control-affine nonlinear systems: A Koopman spectral approach," *IEEE Transactions on Automatic Control*, 2021, to appear.

[19] Z. Aganovic and Z. Gajic, "The successive approximation procedure for finite-time optimal control of bilinear systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1932–1935, 1994.

[20] S. Wang and J.-S. Li, "Fixed-endpoint optimal control of bilinear ensemble systems," *SIAM Journal on Control and Optimization*, vol. 55, no. 5, pp. 3039–3065, 2017.

[21] ——, "Free-endpoint optimal control of inhomogeneous bilinear ensemble systems," *Automatica*, vol. 95, pp. 306–315, 2018.

[22] Y. Zhao and J. Cortés, "Gramian-based reachability metrics for bilinear networks," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 3, pp. 620–631, 2017.

[23] H. J. van Waarde, C. D. Persis, M. K. Camlibel, and P. Tesi, "Willems' fundamental lemma for state-space systems and its extension to multiple datasets," *IEEE Control Systems Letters*, vol. 4, no. 3, pp. 602–607, 2020.

[24] T. Lipp and S. Boyd, "Variations and extension of the convex–concave procedure," *Optimization and Engineering*, vol. 17, no. 2, pp. 263–287, 2016.