Continuity and Boundedness of Minimum-Norm CBF-Safe Controllers

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Abstract— **The existence of a Control Barrier Function (CBF) for a control-affine system provides a powerful design tool to ensure safety. Any controller that satisfies the CBF condition and ensures that the trajectories of the closed-loop system are well defined makes the zero superlevel set forward invariant. Such a controller is referred to as *safe*. This paper studies the regularity properties of the minimum-norm safe controller as a stepping stone towards the design of general continuous safe feedback controllers. We characterize the set of points where the minimum-norm safe controller is discontinuous and show that it depends solely on the safe set and not on the particular CBF that describes it. Our analysis of the controller behavior as we approach a point of discontinuity allows us to identify sufficient conditions to ensure it grows unbounded or it remains bounded. Examples illustrate our results, providing insight into the conditions that lead to (un)bounded discontinuous minimum-norm controllers.**

I. Introduction

Safety-critical control for dynamical systems is an active area of research with applications to multiple domains such as transportation, autonomy, power systems, robotics, and manipulation. The notion of Control Barrier Function (CBF) has revealed to be a particularly useful tool as it provides a mathematically precise formulation of the range of design choices available to keep a desired set safe. This has spurred a flurry of activity aimed at synthesizing safe controllers as solutions to optimization-based formulations whose cost functions may encode energy considerations, minimal deviation from prescribed controllers, or other performance goals. A critical aspect in this endeavor is ensuring that safe controllers enjoy appropriate regularity (boundedness, continuity, Lipschitzness, smoothness) properties for ease of implementation and to ensure well-posedness of the resulting closed-loop system. Motivated by these observations, this work studies the continuity properties of the minimum-norm safe controller and analyzes conditions under which the existence of a bounded safe controller is guaranteed.

Literature Review: The notion of CBF builds on Nagumo's theorem [1], which establishes the invariance of a set with respect to trajectories of an autonomous system given suitable transversality conditions are satisfied on the boundary of that set. The extension to control systems introduced in [2] enforces a strict Nagumo-like condition to hold on the whole set to be made invariant. This condition was relaxed in [3] to arrive at the concept of CBF used here. The use of CBFs to enforce safety as forward set invariance has since expanded to many domains (we refer to [4], [5] for a comprehensive overview).

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Particularly useful is the fact that, if a CBF-certified safe controller is Lipschitz, then the closed-loop system is well posed and the superlevel set of the CBF is forward invariant. It is common to synthesize such controllers via optimization formulations which are examples of parametric optimization problems, with the optimization variable being the control signal and the parameter being the state. The resulting controller is well defined but is generally not guaranteed to be continuous, let alone Lipschitz. If the controller is discontinuous, then it might become unbounded even if the safe set is compact, violating hard limits imposed by hardware constraints or energy considerations. This has motivated the study in the literature of various sufficient conditions to ensure Lipschitzness or continuity of optimization-based controllers. One set of conditions [3] relies on assuming uniform relative degree 1 of the CBF with respect to the dynamical system. Another condition [10] asks that the properties defining the CBF are satisfied on an open set containing the safe set. Other works [11] derive continuity-ensuring conditions resorting to the classical parametric optimization literature [15], of which the optimization-based controller synthesis problem is a special case. In parametric optimization, the work [12] proves the continuity of the optimizer under continuity properties of the point-to-set map defined by the constraints. Other works derive continuity results under different types of constraint qualification conditions, including linear independence [13] and Mangasarian-Fromovitz [14]. The work [11] builds on this body of work to relax linear independence qualification for the special case of a convex linearly constrained quadratic parametric program. Our exposition here unifies these conditions under a common framework and provides a generalization, ensuring continuity of the min-norm safe controller under weaker conditions. We also analyze the boundedness of the controller when the conditions are not met and discontinuity arises. Finally, because of the connection with bounded control, relevant to the present work are methods for constructing CBFs under limited control authority [6]–[8] and the combination of CBFs with Hamilton-Jacobi reachability analysis to consider the impact of control bounds on the computation of safe sets [9].

Statement of Contributions: Given a CBF for a control-affine system, we study the boundedness properties of the associated minimum-norm safe controller. Apart from its intrinsic interest, the focus on this controller is justified by the fact that if it is not bounded, then no safe controller is. We start by explaining the limitations of the state of the art to guarantee the boundedness of safe controllers and illustrating them in two examples. Our first contribution is a rigorous characterization of the points of discontinuity of the minimum-norm safe controller. As a byproduct, this result allows us to generalize the known conditions for ensuring continuity. We show that the points of discontinuity are fully determined by the safe set and are independent of the specific choice of the CBF or the sensitivity to the violation of the CBF condition. These results sets the basis for our second contribution, which is the identification of tight conditions to ensure the (un)boundedness of the minimum-norm controller when approaching a point of discontinuity. We revisit the two examples in light of the technical discussion to explain the observed behavior of the minimum-norm controller. Our results are applicable to more general formulations of safety filters beyond the minimum-norm controller and have important implications for the synthesis of safe feedback controllers subject to hard constraints on the control effort.

Notation: The closure, interior, and boundary of a set $\mathcal{X}$ are denoted by $\bar{\mathcal{X}}$, $\mathrm{int}(\mathcal{X})$, and $\partial\mathcal{X}$, respectively. Given $s : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, $s \in C$ denotes that s is continuous and $s \in C^n$ denotes that s has a continuous n^{th} derivative. The gradient of $s \in C^1$ is denoted by ∇s and written as a row vector. A function s is locally Lipschitz at x with respect to $\mathcal{X}$ if there exists a neighborhood $\mathcal{N}$ and a constant $L \in \mathbb{R}$ such that $\|s(x_1) - s(x_2)\| \leq L\|x_2 - x_1\|$, for all $x_1, x_2 \in \mathcal{N} \cap \mathcal{X}$. A function s is locally Lispchitz on $\mathcal{X}'$ if it is locally Lispchitz at x with respect to $\mathcal{X}'$, for all $x \in \mathcal{X}'$. A function $\alpha : (-a, b) \to \mathbb{R}$ is an extended class-κ function if it is strictly increasing and $\alpha(0) = 0$.

II. Problem Statement

We consider a non-linear control affine system over an open set $\mathcal{X} \subseteq \mathbb{R}^n$

$$\dot{x} = f(x) + G(x)u, \tag{1}$$

where $x \in \mathcal{X}$ and $u \in \mathbb{R}^m$. Here, $f : \mathcal{X} \to \mathbb{R}^n$ and the column components $g_i : \mathcal{X} \to \mathbb{R}^n$, $i \in \{1, \ldots, m\}$ of G are locally Lipschitz on $\mathcal{X}$. Safety of the system can be certified through the following notion.

Definition II.1 (Control Barrier Function [4])**.** *Let $h : \mathcal{X} \to \mathbb{R}$ be C^1 and define its superlevel set $\mathcal{C} \triangleq \{x \in \mathbb{R}^n \mid h(x) \geq 0\} \subseteq \mathcal{X}$. The function h is a CBF if $\nabla h(x) \neq 0$ for all $x \in \partial\mathcal{C}$ and there exists a set $\mathcal{D} \subseteq \mathcal{X}$ such that $\mathcal{C} \subseteq \mathcal{D}$ and for all $x \in \mathcal{D}$, there exists $u \in \mathbb{R}^m$,*

$$\nabla h(x) f(x) + \alpha(h(x)) + \nabla h(x) G(x) u \geq 0. \tag{2}$$

where α is an extended class-κ function.

If h admits an open set $\mathcal{D}$ satisfying the above definition, then we refer to it as a *strong CBF*, otherwise we call it a *weak CBF*. For each $x \in \mathcal{D}$, we denote by $K_{\text{cbf}}(x)$ the set of input values u satisfying (2) which, by Definition II.1, is nonempty.

The central result [4, Theorem 2] of CBF-based safety is that, if there exists a Lipschitz feedback controller $\bar{u} : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\bar{u}(x) \in K_{\text{cbf}}(x)$ in $\mathcal{D}$, then the set $\mathcal{C}$ is forward invariant with respect to the trajectories of the closed-loop system (1) under $u = \bar{u}(x)$. One particular choice of controller that satisfies the CBF condition (2) by construction is the so-called min-norm safe feedback controller $u^*(x) \triangleq \arg\min_{u \in K_{\text{cbf}}(x)} \|u\|^2$. In general, this controller is not necessarily Lipschitz. In fact, it might not even be bounded. This motivates our problem statement.

Problem 1. Let h be a CBF with a compact superlevel set $\mathcal{C}$. Determine the states in $\mathcal{C}$ where the min-norm safe feedback controller u^* is discontinuous and find conditions under which it is bounded/unbounded as the discontinuous states are approached. •

Our focus on establishing boundedness when continuity of the min-norm controller fails is motivated by three reasons. First, proving that the min-norm controller is unbounded shows that no safe bounded controller exists. This would also mean that there does not exit a continuous safe feedback controller. Second, if the min-norm is discontinuous but bounded, then there is room for finding a safe continuous controller. Finally, our investigation provides grounds for exploring whether the use of discontinuous controllers to ensure control-invariance for safety is applicable to a larger class of scenarios.

We end this section by noting that our results are directly applicable to safety filters based on quadratic programming (QP). In fact, any controller u that minimizes a cost function $\|u - u_{\text{nom}}(x)\|^2$ subject to (2), where u_{nom} is a predefined nominal controller, can be interpreted as a min-norm controller after the change of variables $u' = u - u_{\text{nom}}$.

III. Continuity of the Min-Norm Safe Controller: Limitations of the State of the Art

This section reviews known conditions in the literature that ensure the min-norm controller u^* is continuous and thus bounded in a compact set $\mathcal{C}$, and illustrates its limitations in a couple of simple examples. Considering the CBF condition (2), notice that if $\nabla h(x) f(x) + \alpha(h(x)) \geq 0$, then $u = 0$ validates (2). For such points, the min-norm controller $u^*(x) = 0$. On the other hand, when $\nabla h(x) f(x) + \alpha(h(x)) < 0$, a non-zero control is needed to ensure (2). We thus split $\mathcal{D}$ into the two sets

$$\mathcal{D}_+ \triangleq \{x \in \mathcal{D} \mid \nabla h(x) f(x) + \alpha(h(x)) \geq 0\}, \tag{3a}$$

$$\mathcal{D}_- \triangleq \{x \in \mathcal{D} \mid \nabla h(x) f(x) + \alpha(h(x)) < 0\}. \tag{3b}$$

Notice that u^* is defined as the optimizer of a quadratic program with one linear constraint. Such programs have a unique solution, cf. [16, 8.1.1], with the closed-form formula

$$u^*(x) = \begin{cases} 0, & x \in \mathcal{D}_+ \\ -\frac{\nabla h(x) f(x) + \alpha(h(x))}{\|\nabla h(x) G(x)\|^2} (\nabla h(x) G(x))^T, & x \in \mathcal{D}_-. \end{cases} \tag{4}$$

This expression is well defined on $\mathcal{D}$ since (2) implies that, if $\bar{x} \in \mathcal{D}_-$, then $\|\nabla h(\bar{x}) G(\bar{x})\| \neq 0$.

Lemma III.1 (Strong CBF Implies Continuous Min-Norm Controller [10, Thm. 5])**.** *Let h be a strong CBF with a compact superlevel set $\mathcal{C}$. Then u^* is continuous on $\mathcal{C}$.*

According to [3, Thm. 8], u^* is locally Lipschitz if the CBF h has relative degree 1, that is, for all $x \in \mathcal{D}$, $\|\nabla h(x)G(x)\| \neq 0$. The next result is a generalization of this fact.

Lemma III.2 (Generalization of Relative Degree 1 CBF Implies Continuous Min-Norm Controller)**.** *Let h be a CBF with compact superlevel set $\mathcal{C}$. If for all $x \in \partial\mathcal{C}$, $\|\nabla h(x)G(x)\| = 0$ implies $\nabla h(x)f(x) > 0$, then u^* is locally Lipschitz on $\mathcal{C}$.*

We postpone the proof of Lemma III.2 as it is a corollary of Lemma IV.1 below.

Remark III.3 (Assumption of uniform relative degree is limiting)**.** The assumption of uniform relative degree of the CBF, cf. [3, Thm. 8], has also been exploited for higher-order relative degree CBFs, cf. [17]. However, this assumption fails for the following two general cases:

(i) Let h be a continuously differentiable CBF with compact superlevel set $\mathcal{C}$. For such h, there always exists $y \in \text{int}(\mathcal{C})$ where $\|\nabla h(y)G(y)\| = 0$. To see that, note that by continuity of h and compactness of its superlevel set, h has a maximum value at some state $y \in \mathcal{C}$ [18, Thm. 4.16]. Recalling that $h(x) = 0$ at $\partial\mathcal{C}$ and $h(x) > 0$ in $\text{int}(\mathcal{C})$, we deduce that $y \in \text{int}(\mathcal{C})$. By differentiability and first-order optimality [16, 4.2.3], $\nabla h(y) = 0$ and, hence, $\|\nabla h(y)G(y)\| = 0$.

(ii) Consider the n-dimensional linear system (A, B), where B does not have full row rank. Let h be a continuously differentiable CBF with compact convex superlevel set $\mathcal{C}$. Then, there always exists $y \in \partial\mathcal{C}$ where $\|\nabla h(y)G(y)\| = \|\nabla h(y)B\| = 0$. To see this, note that since B is not full row rank, there is a unit vector $v \in \mathbb{R}^n$ such that $\|v^T B\| = 0$. By the surjectivity of the Gauss map [1] on the compact smooth surface $\partial\mathcal{C}$ [19, Thm. A], there is a point $y \in \partial\mathcal{C}$ at which the unit normal vector to $\partial\mathcal{C}$ is v. By [20, Thm. 3.15], $\nabla h(y)$ is normal to $\partial\mathcal{C}$ at y and thus parallel to v. Hence, $\|\nabla h(y)B\| = 0$. •

From the continuity of the min-controller on $\mathcal{C}$ ensured by either Lemmas III.1 or III.2, it follows from standard results in analysis, cf. [18, Thm. 5.15], that u^* is bounded if $\mathcal{C}$ is compact. As we will show later, the conditions of Lemmas III.1 and III.2 are not totally independent: rather, if the condition of Lemma III.1 is not met, i.e., h is weak, then the condition of Lemma III.2 is not met either.

CBFs that do not meet the conditions of these results are easy to encounter and arise in practice in contexts as simple as the problem of confining a double integrator to a circle centered at the origin. We next present two examples that do not satisfy the assumptions and generate discontinuous min-norm controllers: one being bounded and the other one unbounded.

[1]The Gauss map assigns points on the manifold $\partial\mathcal{C}$ to the unit sphere embedded in $\mathbb{R}^n$ such that the image of any point in $\partial\mathcal{C}$ is the unit vector normal to $\partial\mathcal{C}$ at that point.

Example III.4 (Weak CBF with Bounded Min-Norm Controller)**.** Consider the double-integrator dynamics on $\mathbb{R}^2$ defined by $f(x) = (x_2, 0)$ and $G(x) = (0, 1)$. The function $h(x) = 1 - x_1^2 - x_2^2$ is a CBF with any extended class-κ function α. Notice further that h is a weak CBF. To see this, let $\bar{x} = (1 + \epsilon, 0)$ with any arbitrarily small $\epsilon > 0$. Since $\bar{x} \notin \mathcal{C}$, we have $\nabla h(\bar{x})f(\bar{x}) + \alpha(h(\bar{x})) + \nabla h(\bar{x})G(\bar{x})u = \alpha(h(\bar{x})) < 0$, and therefore condition (2) cannot be satisfied at $\bar{x}$. Therefore, h does not admit an open set $\mathcal{D}$ satisfying Definition II.1. In addition, the condition of Lemma III.2 is not satisfied at the boundary point $(1, 0)$. Consider now the norm of the min-norm safe controller (4) defined on $\mathcal{D} = \mathcal{C}$,

$$|u_1^*(x)| = \begin{cases} 0, & x \in \mathcal{D}_+, \\ \frac{2x_1x_2 - \alpha(h(x))}{2x_2}, & x \in \mathcal{D}_-. \end{cases}$$

Note that u_1^* is continuous on $\mathcal{C} \setminus \{(\pm 1, 0)\}$. However, choosing $\alpha(r) = r$, we have that $\limsup_{x \to (1,0), x \in \mathcal{D}_-} |u_1^*(x)|$ and $\lim_{x \to (1,0), x \in \mathcal{D}_+} |u_1^*(x)| = 0$. Thus, although discontinuous at $(1, 0)$, u_1^* is bounded at this point, cf. top plot in Figure 1. •

Example III.4 shows that the min-norm safe controller might be bounded even if the CBF does not satisfy the continuity conditions in the literature. The next example shows this fact is not generic.

Example III.5 (Weak CBF with Unbounded Min-Norm Controller)**.** Consider the dynamics $f(x) = (x_2, 0)$ and $G(x) = (0, x_2^2)$. With the same reasoning as in Example III.4, $h(x) = 1 - x_1^2 - x_2^2$ is a weak CBF that does not satisfy the requirement of Lemma III.2. The norm of the min-norm safe controller is:

$$|u_2^*(x)| = \begin{cases} 0, & x \in \mathcal{D}_+, \\ \frac{2x_1x_2 - \alpha(h(x))}{2x_2^3}, & x \in \mathcal{D}_-. \end{cases}$$

Observe that u_2^* is continuous on $\mathcal{C} \setminus \{(\pm 1, 0)\}$. However, with the choice $\alpha(r) = r$, $\limsup_{x \to (1,0), x \in \mathcal{D}_-} |u_2^*(x)| = \infty$ and $\lim_{x \to (1,0), x \in \mathcal{D}_+} |u_2^*(x)| = 0$. Thus, u_2^* is neither continuous nor bounded on $\mathcal{C}$, cf. bottom plot in Figure 1.•

IV. Points of Discontinuity of The Min-Norm Safe Controller

Here we characterize the points of (dis)continuity of the min-norm controller u^* in $\mathcal{C}$. This is motivated by the fact that if u^* goes unbounded when approaching a point in $\mathcal{C}$, then it is discontinuous at it. Therefore, the results of this section are a stepping stone towards the identification of conditions for (un)boundedness of u^*.

Lemma IV.1 (Points of discontinuity of u^* in $\mathcal{C}$)**.** *Let h be a CBF for a system (1) with a Lipschitz gradient and an associated Lipschitz class-κ function α, and let u^* be the min-norm controller given by (4). Define $\mathcal{Z}_{h,\alpha} \triangleq \{x \in \mathcal{C} \mid \nabla h(x)f(x) + \alpha(h(x)) = 0 = \|\nabla h(x)G(x)\|\}$. Then, u^* is locally Lipschitz on $\mathcal{C} \setminus \mathcal{Z}_{h,\alpha}$.*

Proof. The proof is an extension of the proof of [3, Thm. 8]. Note that Since h is a CBF, (2) is satisfied for $\mathcal{D} = \mathcal{C}$

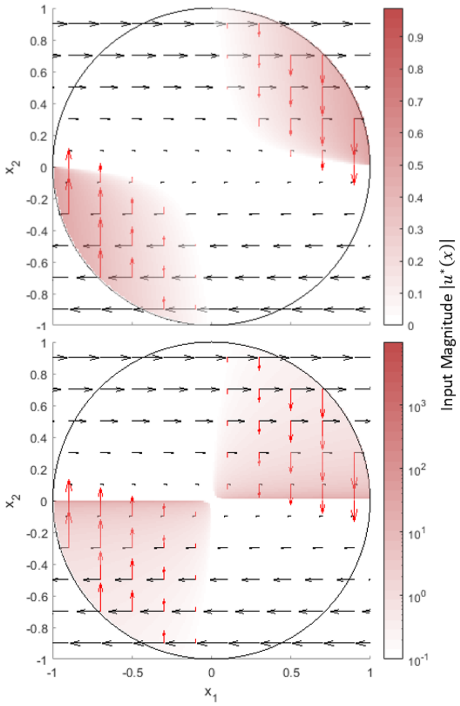

Fig. 1: Illustration of boundedness of the min-norm safe controller. Top (resp. bottom) plot corresponds to Example III.4 (resp., Example III.5). In each case, the unit circle is the superlevel set of the weak CBF h, black arrows show the vector field $f(x)$, red arrows show $G(x)u^*(x)$, and the color map shows the magnitude of the input u^*.

and therefore $\|\nabla h(x)G(x)\| \neq 0$, for all $x \in \mathcal{D}_-$. Thus, on $\mathcal{D}_-$, u^* is a quotient with a non-zero Lipschitz denominator and a Lipschitz numerator. Hence, both expressions in the piecewise definition of u^* in (4) are locally Lipschitz on their respective domains $\mathcal{D}_+$ and $\mathcal{D}_-$. It remains to prove that u^* is locally Lipschitz with respect to $\mathcal{C}$ at all the points in the boundary between $\mathcal{D}_+$ and $\mathcal{D}_-$ that are not in $\mathcal{Z}_{h,\alpha}$. For a point x in the boundary between $\mathcal{D}_+$ and $\mathcal{D}_-$, $\nabla h(x)f(x) + \alpha(h(x)) = 0$. If at such a point $\|\nabla h(x)G(x)\| \neq 0$ (i.e., $x \notin \mathcal{Z}_{h,\alpha}$), then there is a neighborhood $\mathcal{N}$ of x such that $\|\nabla h(y)G(y)\| \neq 0$ for all $y \in \mathcal{N}$. Thus $u^*(x) = \omega(\frac{\nabla h(x)f(x)+\alpha(h(x))}{\|\nabla h(x)G(x)\|})(\nabla h(x)G(x))^T$ for $x \in \mathcal{N}$, where

$$\omega(r) = \begin{cases} 0, & r \geq 0, \\ -r, & r < 0, \end{cases}$$

which is locally Lipschitz on $\mathbb{R}$. That u^* is locally Lipschitz at x follows from the facts that the composition and product of locally Lipschitz functions is locally Lipschitz, and the quotient of locally Lipschitz functions is locally Lipschitz provided that the denominator is not zero. □

Lemma IV.1 can be seen as an extension of previous results, cf. [3, Thm. 8], establishing local Lipschitzness of u^* by assuming uniform relative degree 1 of h. If this is the case, then $\mathcal{Z}_{h,\alpha}$ is empty and thus u^* is locally Lipschitz on $\mathcal{C}$. Given the dependency of $\mathcal{Z}_{h,\alpha}$ on h and α, one might consider the possibility that a suitable choice of these functions might eliminate the potential points of discontinuity. The following results rule this out.

Lemma IV.2 (Discontinuity Points Are Independent of α)**.** *Let h be a CBF. Then there exists an extended class-κ function α that validates the CBF condition (2) and such that $\mathcal{Z}_{h,\alpha} \subseteq \partial\mathcal{C}$. Moreover, let α_1 and α_2 be two extended class-κ functions that validate the CBF definition for h. Then $\mathcal{Z}_{h,\alpha_1} \cap \partial\mathcal{C} = \mathcal{Z}_{h,\alpha_2} \cap \partial\mathcal{C}$.*

Proof. We prove that if α validates Definition II.1 for h, then any class-κ function $\bar{\alpha}$ that satisfies $\bar{\alpha}(r) > \alpha(r)$ for all $r > 0$ validates Definition II.1 for h and gives $\mathcal{Z}_{h,\bar{\alpha}} \cap \text{int}(\mathcal{C}) = \emptyset$. That $\bar{\alpha}$ validates the CBF condition (2) is immediate. Now let $\bar{x} \in \text{int}(\mathcal{C})$ be such that $\nabla h(\bar{x})f(\bar{x}) + \bar{\alpha}(h(\bar{x})) = 0$. We show that $\|\nabla h(\bar{x})G(\bar{x})\| \neq 0$ and thus $\bar{x} \notin \mathcal{Z}_{\bar{\alpha},h}$. Since $\bar{\alpha}(r) > \alpha(r)$ for $r > 0$, $\nabla h(\bar{x})f(\bar{x}) + \alpha(h(\bar{x})) < 0$ because $h(\bar{x}) > 0$ as $\bar{x} \in \text{int}(\mathcal{C})$. But α validates condition (2) and thus $\|\nabla h(\bar{x})G(\bar{x})\| \neq 0$. The proof of the last claim in the statement is immediate from the fact that $\alpha_1(h(x)) = \alpha_2(h(x)) = 0$ on $\partial\mathcal{C}$. □

If we thus define

$$\mathcal{Z}_h \triangleq \{x \in \partial\mathcal{C} \mid \nabla h(x)f(x) = \|\nabla h(x)G(x)\| = 0\}, \quad (5)$$

then Lemmas IV.1 and IV.2 justify stating that u^* is continuous on $\mathcal{C} \setminus \mathcal{Z}_h$. This shows that u^* is continuous on $\text{int}(\mathcal{C})$ and that the possible points of discontinuity are independent of the choice of α.

Lemma IV.3 (Discontinuity Points Are Independent of h)**.** *Let h_1, $h_2 \in C^1$ be CBFs with the same superlevel set $\mathcal{C}$. Then, $\mathcal{Z}_{h_1} = \mathcal{Z}_{h_2}$.*

Proof. By Definition II.1, $\nabla h_i(x) \neq 0$, $i \in \{1, 2\}$ on $\partial\mathcal{C}$. By [21, Thm. 5.1], both $h_1 = 0$ and $h_2 = 0$ define the same differentiable manifold $\partial\mathcal{C}$ of dimension $n - 1$ embedded in $\mathbb{R}^n$. By [20, Thm. 3.15], the tangent space T_x of this manifold at a point x is given by $T_x = \text{kernel}(\nabla h_1(x)) = \text{kernel}(\nabla h_2(x))$. Thus $\nabla h_1(x)$ and $\nabla h_2(x)$ are parallel, and the result follows using the definition of $\mathcal{Z}_h$. □

Lemma IV.3 shows that $\mathcal{Z}_h$ is associated to the set $\mathcal{C}$ and is independent of the CBF that has this set as its superlevel set. We thus write $\mathcal{Z}$ to denote $\mathcal{Z}_h$ without loss of generality.

Lemma III.2 can now be readily proved: in fact, the hypotheses there imply that $\mathcal{Z}$ is empty, and therefore, by Lemma IV.1, u^* is continuous on $\mathcal{C}$. Now that it is proved that the non-emptiness of the set $\mathcal{Z}$ implies potential discontinuity; one might then hope that boundedness of u^* can be established for a weak CBF h by ensuring that $\mathcal{Z}$ is empty. The next result shows that the latter is never the case.

Lemma IV.4 (Weak CBF Implies Possible Discontinuity)**.** *If h is a weak CBF, then $\mathcal{Z}$ is nonempty.*

Proof. Define the sequence of sets $\mathcal{D}_n \triangleq \{x \in \mathbb{R}^n \mid \text{d}(x, \mathcal{C}) < 1/n\}$, where $\text{d}(x, \mathcal{C})$ is the distance function from x to set $\mathcal{C}$, which is continuous, cf. [22, Thm. 3.1]. Note that $\mathcal{C} \subset \mathcal{D}_n$ and $\mathcal{D}_n$ is open for all $n \in \mathbb{N}$. Since h is a weak CBF, for each $n \in \mathbb{N}$, there exists $x_n \in \mathcal{D}_n \setminus \mathcal{C}$ such that for all $u \in \mathbb{R}^m$ and all class-κ functions α, $\nabla h(x_n)f(x_n) +$

$\alpha(h(x_n)) + \nabla h(x_n)G(x_n)u < 0$. This implies that necessarily $\|\nabla h(x_n)G(x_n)\| = 0$ and $\nabla h(x_n)f(x_n) + \alpha(h(x_n)) < 0$. Consider the sequence $\{x_n\}$. Since $\mathcal{C}$ is compact, the closure of $\mathcal{D}_1$, namely $\bar{\mathcal{D}}_1$, is compact. Since $\{x_n\} \subseteq \bar{\mathcal{D}}_1$, there exists, cf. [18, Thm. 3.6], a convergent subsequence of $\{x_n\}$, denoted $\{y_n\}$, whose limit is $\bar{y}$. By the definition of $\{y_n\}$, we have $\mathrm{d}(y_n, \mathcal{C}) \to 0$, and by continuity, $\mathrm{d}(\bar{y}, \mathcal{C}) = 0$, and so $\bar{y} \in \mathcal{C}$. Since $h(y_n) < 0$ for all n, it follows that $h(\bar{y}) \leq 0$, and therefore it must be that $h(\bar{y}) = 0$, i.e., $\bar{y} \in \partial\mathcal{C}$. Continuity and the fact that $\|\nabla h(y_n)G(y_n)\| = 0$ for all n implies $\|\nabla h(\bar{y})G(\bar{y})\| = 0$. Similarly, continuity and the fact that $\nabla h(y_n)f(y_n) + \alpha(h(y_n)) < 0$ implies that $\nabla h(\bar{y})f(\bar{y}) + \alpha(h(\bar{y})) = \nabla h(\bar{y})f(\bar{y}) \leq 0$. Since h is a CBF and $\bar{y} \in \mathcal{C}$, we have $\nabla h(\bar{y})f(\bar{y}) + \alpha(h(\bar{y})) = \nabla h(\bar{y})f(\bar{y}) \geq 0$. Therefore $\nabla h(\bar{y})f(\bar{y}) = 0$ and thus, $\bar{y} \in \mathcal{Z}$, implying $\mathcal{Z} \neq \emptyset$. ☐

Lemma IV.4 provides an important connection between the conditions for continuity presented in Section III. In fact, if the CBF is not strong, but weak (i.e., the condition of Lemma III.1 is not met), then Lemma IV.4 implies that the condition of Lemma III.2 is not satisfied either.

V. (Un)Boundedness Conditions For The Min-Norm Safe Controller

This section identifies conditions to determine when the min-norm controller is bounded. For a compact safe set $\mathcal{C}$, the controller can go unbounded only if approaching a state at which it is discontinuous (see e.g., Example III.5 for an illustration). From the exposition in Section IV, we know that the points of discontinuity of the min-norm controller are contained in $\mathcal{Z}$, cf. (5). The following result provides computable sufficient conditions for (un)boundedness when approaching a point in $\mathcal{Z}$.

Theorem V.1 ((Un)Boundedness Conditions of Min-Norm Controller). *Let $h \in C^2$ be a CBF with compact superlevel set $\mathcal{C}$ and an associated α that is differentiable at 0. Assume f and G are differentiable at $\bar{x} \in \mathcal{Z}$ and let $H_h(\bar{x})$, $J_f(\bar{x})$, and $J_{g_i}(\bar{x})$ denote the Hessian of h and the Jacobians of f and g_i, respectively. Consider the linear equation*

$$Av = \begin{bmatrix} c_1 \\ c_2 \\ \mathbf{0} \end{bmatrix}, \tag{6}$$

with $v \in \mathbb{R}^n$, $c_1, c_2 \in \mathbb{R}$. Here, $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$, $A \triangleq \begin{bmatrix} \nabla h(\bar{x})^T & \beta_f(\bar{x}) & \beta_G(\bar{x}) \end{bmatrix}^T$ and

$$\beta_f(x) \triangleq H_h(x)f(x) + (J_f^T(x) + \alpha'(h(x))I_n)\nabla h(x)^T \in \mathbb{R}^n,$$

$$\beta_{g_i}(x) \triangleq H_h(x)g_i(x) + J_{g_i}^T(x)\nabla h(x)^T \in \mathbb{R}^n,$$

$$\beta_G(x) \triangleq \begin{bmatrix} \beta_{g_1}(x) & \dots & \beta_{g_m}(x) \end{bmatrix} \in \mathbb{R}^{n\times m}.$$

Then, the following statements hold:

(i) if (6) has a solution v with $c_1 \geq 0$ and $c_2 < 0$, then u^* is not bounded as $x \to \bar{x}$ in $\mathcal{C}$ from the direction of v, i.e., $u^*(\bar{x} + vt)$ goes unbounded as $t \to 0^+$.

(ii) if (6) does not have any non-trivial solution with $c_1 \geq 0$ and $c_2 \leq 0$, then u^* is bounded as it approaches $\bar{x}$ from all possible directions in $\mathcal{C}$.

Proof. The proof proceeds by examining the limit $\limsup_{t\to 0} \|u^*(\bar{x} + vt)\|$ for $v \in \mathbb{R}^n$. In doing so, we face the challenge that u^* is given by a piecewise expression that is generally discontinuous at $\bar{x}$. In addition, when computing the limit, one finds an indeterminate form of the type $0/0$. This leads us to the use of a particular form of L'Hôpital's rule [18] that can handle the discontinuous piecewise expression and the presence of the $\limsup$.

For brevity, we use $\bar{x}_t \triangleq \bar{x} + vt$, $h_G(t) \triangleq \nabla h(\bar{x}_t)G(\bar{x}_t)$, $N(t) \triangleq \nabla h(\bar{x}_t)f(\bar{x}_t) + \alpha(h(\bar{x}_t))$, and $D(t) \triangleq \|h_G(t)\|$. According to (3b), $\|u^*(\bar{x}_t)\| = \frac{-N(t)}{D(t)}$ for $\bar{x}_t \in \mathcal{D}_-$.

(i) Let v be a solution of (6) with $c_1 \geq 0$ and $c_2 < 0$. Because of the first row of (6), we have that $\frac{d}{dt}h(\bar{x}_t) = \nabla h(\bar{x})v = c_1 \geq 0$. If $\nabla h(\bar{x})v > 0$, then by continuity, $\nabla h(\bar{x}_t) > 0$ for small enough t. Thus by [18, Thm. 5.11], $h(\bar{x}_t) > 0$, i.e., $\bar{x}_t \in \mathcal{C}$, for small enough t. If $\nabla h(\bar{x})v = 0$, then v is tangential to $\mathcal{C}$. Hence $\bar{x}_t$ approaches $\bar{x}$ from within $\mathcal{C}$ or tangentially to it, meaning that v is a valid direction of approach to consider. The second row of (6) ensures that $\frac{d}{dt}N(t)|_{t=0^+} = v^T\beta_f(\bar{x}) = c_2 < 0$, which again by [18, Thm. 5.11] proves that $N(t) < 0$, i.e., $\bar{x}_t \in \mathcal{D}_-$ by (3b), for sufficiently small t. Hence, $\lim_{t\to 0^+} \|u^*(\bar{x}_t)\| = \lim_{t\to 0^+} \frac{-N(t)}{D(t)}$. Direct evaluation of this expression at $t = 0$ (where $\bar{x}_t = \bar{x}$) yields an indeterminate form of the type $0/0$. We therefore resort to L'Hôpital's rule [18, Thm. 5.13], which requires the existence of the limit of the derivative of the numerator $-N(t)$ and denominator $D(t)$. For the numerator, we have already established $\lim_{t\to 0^+} \frac{d}{dt}N(t) = c_2$. As for the denominator, it is the norm of the differentiable function $h_G(t)$, and its derivative exists at t where $h_G(t) \neq 0$. But since $\bar{x}_t \in \mathcal{D}_-$ for small enough t, the CBF condition (2) ensures that $h_G(t) \neq 0$ for sufficiently small t. Thus, the derivative of the denominator exists for sufficiently small $t > 0$. A proof of the existence of the limit of this derivative $\lim_{t\to 0^+} \frac{d}{dt}(D(t)) = \lim_{t\to 0^+} v^T\beta_G(\bar{x}_t)\frac{h_G(t)}{\|h_G(t)\|}$ follows. By Hölder's inequality,

$$\left|\frac{v^T\beta_G(\bar{x}_t)h_G(t)}{\|h_G(t)\|}\right| \leq \|v^T\beta_G(\bar{x}_t)\|\frac{\|h_G(t)\|}{\|h_G(t)\|} = \|v^T\beta_G(\bar{x}_t)\|.$$

Hence, using the last m rows of (6), the assumption of continuous differentiability, and the sandwich theorem for limits [23, Thm. 3.3.3], $\lim_{t\to 0^+} \frac{d}{dt}D(t) = 0$. By L'Hôpital, $\lim_{t\to 0^+} \|u^*(\bar{x}_t)\| = \lim_{t\to 0^+} \frac{-N(t)}{D(t)} = \lim_{t\to 0^+} \frac{-N'(t)}{D'(t)} = \infty$.

(ii) We prove the contrapositive: assume there exists a vector v such that $\bar{x}_t = \bar{x} + vt$ approaches $\bar{x}$ from within $\mathcal{C}$ or tangent to it as $t \to 0^+$ and $\limsup_{t\to 0^+} \|u^*(\bar{x}_t)\| = \infty$, and let us show that then v solves (6) with $c_1 \geq 0$ and $c_2 \leq 0$. Note that $\nabla h(\bar{x})v \geq 0$, since otherwise, under the theorem assumptions, for sufficiently small t, $\bar{x}_t \notin \mathcal{C}$, i.e., $\bar{x}_t$ would approach $\bar{x}$ from outside $\mathcal{C}$, which is a contradiction. This ensures the satisfaction of the first row in (6). Similarly, if $v^T\beta_f(\bar{x}) > 0$, then under the theorem assumptions, for sufficiently small t, $\bar{x}_t \in \mathcal{D}_+$ and thus $\limsup_{t\to 0^+} u^*(\bar{x}_t) = 0$, which contradicts $\limsup_{t\to 0^+} \|u^*(\bar{x}_t)\| = \infty$. This ensures the satisfaction of the second row in (6). According to Lemma I.3, there exists a sequence $\{\bar{t}_i\} \to 0^+$ with

$\{\bar{x}_{\bar{t}_i}\} \subset \mathcal{D}_-$ such that $D'(\bar{t}_i) = v^T \beta_G(\bar{x}_{\bar{t}_i}) \frac{h_G(\bar{t}_i)}{\|h_G(\bar{t}_i)\|} \to 0$. It remains to show that this implies $v^T \beta_G(\bar{x}) = \mathbf{0}^T$. We reason by contradiction and assume $v^T \beta_G(\bar{x}) \neq \mathbf{0}^T$. Without loss of generality, we can assume that the limit of $\frac{h_G(\bar{t}_i)}{\|h_G(\bar{t}_i)\|}$, denoted $\zeta \in \mathbb{R}^n$, exists (this can be done because $\{\frac{h_G(\bar{t}_i)}{\|h_G(\bar{t}_i)\|}\}$ is a sequence from the set of unit vectors in $\mathbb{R}^n$, which is compact, so there exists a convergent subsequence [18, Thm. 3.6]). This and the continuity of β_G imply that $D'(\bar{t}_i) \to v^T \beta_G(\bar{x})\zeta = 0$. Without loss of generality, assume $\frac{\|h_G(\bar{t}_i)\|}{\|h_G(\bar{t}_{i+1})\|} \to \infty$ (that this does not undermine generality is shown by Lemma I.1(i)). Now, Lemma I.1(ii) applied element-wise gives

$$\frac{h_G(\bar{t}_i) - h_G(\bar{t}_{i+1})}{\|h_G(\bar{t}_i)\| - \|h_G(\bar{t}_{i+1})\|} \to \zeta. \quad (7)$$

The sequence in (7) can be written as

$$\frac{h_G(\bar{t}_{i+1}) - h_G(\bar{t}_i)}{\bar{t}_{i+1} - \bar{t}_i} \frac{\bar{t}_{i+1} - \bar{t}_i}{\|h_G(\bar{t}_{i+1})\| - \|h_G(\bar{t}_i)\|}. \quad (8)$$

Using the continuous differentiability of ∇h and G at $\bar{x}$, the first term of (8)

$$\frac{h_G(\bar{t}_{i+1}) - h_G(\bar{t}_i)}{\bar{t}_{i+1} - \bar{t}_i} \to \frac{d}{dt}(h_G(t))\Big|_{t=0} = (v^T \beta_G(\bar{x}))^T \neq \mathbf{0},$$

by hypothesis of contradiction. Consequently, the second term in (8) converges to a non-zero scalar, which we denote by a. Therefore, $\zeta = a(v^T \beta_G(\bar{x}))^T$. This implies that $D(\bar{t}_i) \to v^T \beta_G(\bar{x})\zeta = a\|v^T \beta_G(\bar{x})\|^2 \neq 0$, which is a contradiction. □

Theorem V.1 provides sufficient conditions for boundedness of the min-norm controller at a point of possible discontinuity. Note that the second row of the the matrix A in (6) is the gradient of $\nabla h(x) f(x) + \alpha(h(x))$. Similarly the $(2+i)^{\text{th}}$ row is the gradient of $\nabla h(x) g_i(x)$. Each of the two equations $\nabla h(x) f(x) + \alpha(h(x)) = 0$ and $\nabla h(x) g_i(x) = 0$ defines a differentiable $(n-1)$-dimensional surface embedded in $\mathbb{R}^n$. Thus, the existence of a solution v for (6) with $c_1 > 0$ and $c_2 < 0$ amounts to the existence of a vector that

(i) points to the region in $\mathcal{C}$ that requires non-zero control for safety, and
(ii) is perpendicular to the surfaces defined by $\nabla h(x) G(x) = 0$.

This provides with a geometric intuition for the conditions identified in Theorem V.1. Figure 2(a)-(b) illustrates them for a generic two-dimensional single-input system.

We note that condition (ii) (with $c_2 \leq 0$) in Theorem V.1 is *almost* a negation of condition (i) (with $c_2 < 0$). This shows that (i) is almost a sufficient and necessary condition for unboundedness of u^*. The gap between both conditions stems from the fact that L'Hôpital's rule is indeterminate when both derivatives of the numerator and the denominator approach 0. A geometric interpretation of this situation is depicted in Figure 2c.

Corollary V.2 (Condition for Boundedness of Min-Norm Controller on $\mathcal{C}$)**.** *If condition (ii) in Theorem V.1 holds for all $\bar{x} \in \mathcal{Z}$, then u^* is bounded on $\mathcal{C}$.*

We revisit now Examples III.4 and III.5 in light of the above results. Notice that in both cases $\mathcal{Z} = \{(1,0), (-1,0)\}$. Taking α with $\alpha'(0) = 1$, (6) at $\bar{x} = (1,0)$ becomes

$$-2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & d \end{bmatrix} v = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix},$$

where $d = 1$ for Example III.4 and $d = 0$ for Example III.5. It is clear that the only possible solution for this system of equations with $c_1 \geq 0$ and $c_2 \leq 0$ with $d = 1$ is the trivial solution $v = \mathbf{0}$. Thus, by Theorem V.1(ii), u_1^* from Example III.4 is bounded as its argument approaches $\bar{x}$, as we would expect by our analysis of Example III.4. However, a solution $v = (0,1)$ solves the system with $d = 0$, $c_1 = 0 \geq 0$ and $c_2 = -2 < 0$. By Theorem V.1(i), u_2^* from Example III.5 goes unbounded as it approaches $\bar{x}$ from the direction of v, which is tangential to $\mathcal{C}$. This is also expected by our analysis of Example III.5.

Remark V.3 (When Unbounded Min-Norm Is Inevitable)**.** The system of linear equations in (6) has a coefficient matrix A with $m+2$ rows and n columns. A non-trivial solution v to (6) exists if the first two rows of A are linearly independent and the remaining rows are linearly independent of the first two. This shows that, if the system data is such that the matrix A satisfies these independence properties, then an unbounded min-norm controller is inevitable. •

VI. CONCLUSIONS

We have studied the continuity and boundedness properties of the min-norm safe feedback controller for general control-affine systems within the framework of control barrier functions (CBF). After re-interpreting the known results in the literature in light of the notion of strong and weak CBFs, we have characterized the set of possible points of discontinuity of the minimum-norm safe controller and shown that it only depends on the safe set (and not on the specific CBF or the sensitivity to the violation of the CBF condition). Based on this characterization, we have generalized the known conditions to guarantee the continuity of the min-norm safe controller and identified sufficient conditions for its (un)boundedness. Our results have important implications for the synthesis of safe feedback controllers subject to hard constraints on control effort. Future work will explore questions about the existence of continuous safe controllers when the min-norm controller is discontinuous but bounded, the modification of CBFs that admit safe controllers when no control bounds are present to incorporate such limits, and the design of discontinuous (but bounded) safe controllers.

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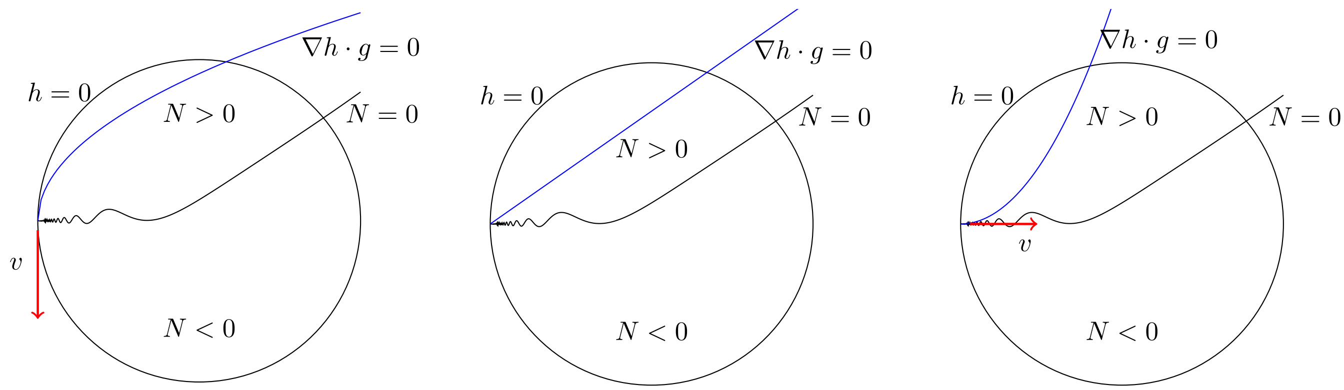

(a) Condition (i) in Theorem V.1 holds.

(b) Condition (ii) in Theorem V.1 holds.

(c) Neither condition in Theorem V.1 holds.

Fig. 2: Illustration of the conditions identified in Theorem V.1. In (a), v solves (6) with $c_1 = 0$ and $c_2 < 0$. In (b), there is no vector v that points to the region in $\mathcal{C}$ where $N < 0$ and is also tangential to the curve $\{\nabla h \cdot g = 0\}$. In (c), v solves (6) with $c_1 > 0$ but with $c_2 = 0$, and thus neither condition in Theorem V.1 is satisfied.

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Appendix I

The next results are exploited in the proof of Theorem V.1.

Lemma I.1 (Basic facts on real sequences)**.** *The following facts hold:*

(i) Any sequence $\{a_n\} \subset \mathbb{R}_{>0}$ convergent to 0 contains a subsequence $\{\bar{a}_n\}$ with $\frac{\bar{a}_n}{\bar{a}_{n+1}} \to \infty$.

(ii) If the sequences $\{a_n\}, \{b_n\} \subset \mathbb{R}$ both converge to 0, $\frac{a_n}{b_n} \to L$, and $\frac{b_n}{b_{n+1}} \to c \neq 1$ (not excluding $c = \infty$), then $\frac{a_n - a_{n+1}}{b_n - b_{n+1}} \to L$.

Proof. To prove *(i)*, the subsequence $\{\bar{a}_n\}$ can be constructed as follows. Take $\bar{a}_1 = a_1$. By definition of convergence, for any $\bar{a}_n > 0$, there is $a_m \le \bar{a}_n / n$. Taking $\bar{a}_{n+1} = a_m$ gives $\frac{\bar{a}_n}{\bar{a}_{n+1}} \ge n$. Statement *(ii)* follows directly from noting that $\frac{a_n - a_{n+1}}{b_n - b_{n+1}} - \frac{a_n}{b_n} = \left(\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}}\right) / \left(\frac{b_n}{b_{n+1}} - 1\right) \to 0$. $\square$

The following generalized version of L'Hôpital's rule is convenient for our purposes.

Lemma I.2 (Generalized L'Hôpital [24, Thm. II])**.** *Let the functions $f, g : (a,b) \to \mathbb{R}$ be continuously differentiable on (a,b), with neither g nor g vanishing on (a,b). Then* $\liminf_{t \to a^+} \frac{f'(t)}{g'(t)} \le \liminf_{t \to a^+} \frac{f(t)}{g(t)} \le \limsup_{t \to a^+} \frac{f(t)}{g(t)} \le \limsup_{t \to a^+} \frac{f'(t)}{g'(t)}$.

The following result shows a key property in the technical argumentation of the proof of Theorem V.1.

Lemma I.3. *Under the assumptions of Theorem V.1, let* $v \in \mathbb{R}^n$ *be such that* $\limsup_{t\to 0^+} \|u^*(\bar{x}_t)\| = \infty$ *(recall* $\bar{x}_t = \bar{x} + vt$*) and* $v^T\beta_f(\bar{x}) \leq 0$*. Then, there exists a sequence* $\{t_i\} \to 0^+$ *with* $\{\bar{x}_{t_i}\} \subset \mathcal{D}_-$ *such that* $v^T\beta_G(\bar{x}_{t_i})\frac{(\nabla h(\bar{x}_{t_i})G(\bar{x}_{t_i}))^T}{\|\nabla h(\bar{x}_{t_i})G(\bar{x}_{t_i})\|} \to 0$.

Proof. We utilize the abbreviations introduced at the beginning of the proof of Theorem V.1 for convenience. We consider the cases $v^T\beta_f(\bar{x}) < 0$ and $v^T\beta_f(\bar{x}) = 0$ separately.

Case 1: If $v^T\beta_f(\bar{x}) < 0$, then by [18, Thm. 5.11], $\bar{x}_t \in \mathcal{D}_-$ for sufficiently small t. Hence, $\limsup_{t\to 0^+} \|u^*(\bar{x}_t)\| = \lim_{t\to 0^+} \frac{-N(t)}{D(t)} = \infty$. Direct evaluation gives a $\frac{0}{0}$ type of limit. By Lemma I.2, $\limsup_{t\to 0^+} \frac{-d/dt(N(t))}{d/dt(D(t))} = \infty$. Since $\frac{d}{dt}(N(t)) = v^T\beta_f(\bar{x}_t)$ is continuous at $t = 0$, it is bounded on a small enough interval $t \in [0, \epsilon]$. Thus, the only way the $\limsup$ approaches ∞ is that there exists a sequence $\{t_i\}$ such that $\frac{d}{dt}(D(t))|_{t=t_i} = v^T\beta_G(\bar{x}_{t_i})\frac{h_G(t_i)}{\|h_G(t_i)\|} \to 0$.

Case 2: If $v^T\beta_f(\bar{x}) = 0$, then for small enough positive t, either $\bar{x}_t \in \mathcal{D}_+$, $\bar{x}_t \in \mathcal{D}_-$, or $\bar{x}_t$ alternates between $\mathcal{D}_-$ and $\mathcal{D}_+$ indefinitely. The first case is impossible if u^* goes unbounded as $t \to 0^+$. The second case can be handled analogously to Case 1. Hence, we focus on the last case, where $\bar{x}_t$ alternates between $\mathcal{D}_-$ and $\mathcal{D}_+$ indefinitely as $t \to 0^+$. This means that the continuous function N approaches 0 by alternating between positive and negative values indefinitely as $t \to 0^+$. By assumption, there exists $\{t_i\}$ such that $\|u^*(\bar{x}_{t_i})\| \to \infty$. Without loss of generality, we assume that $\bar{x}_{t_i} \in \mathcal{D}_-$ for all i and that $u^*(\bar{x}_{t_i})$ grows monotonically (a subsequence satisfying these assumptions can always be found). By continuity of $N(t)$ and the intermediate value theorem [18, Thm. 4.23], for every t_i there is an interval $(t_{1,i}, t_{2,i})$ such that $\bar{x}_t \in \mathcal{D}_-$ for all $t \in (t_{1,i}, t_{2,i})$ and $N(t_{1,i}) = N(t_{2,i}) = 0$. We distinguish two cases.

Case 2.1: Assume there exists $\bar{n}$ such that $D(t_{1,i})D(t_{2,i}) \neq 0$ for all $i \geq \bar{n}$. Thus, $\frac{N(t_{1,i})}{D(t_{1,i})} = \frac{N(t_{2,i})}{D(t_{2,i})} = 0$. This together with the continuity of $N(t)/D(t)$ in $(t_{1,i}, t_{2,i})$ implies that there exists $\bar{t}_i \in (t_{1,i}, t_{2,i})$ where $-N(t)/D(t)$ attains its maximum in $(t_{1,i}, t_{2,i})$. Therefore, the sequence $-\frac{N(\bar{t}_i)}{D(\bar{t}_i)}$ approaches ∞ since $\frac{-N(\bar{t}_i)}{D(\bar{t}_i)} \geq \frac{-N(t_i)}{D(t_i)} \to \infty$. The continuous differentiability of $-N(t)/D(t)$ in $(t_{1,i}, t_{2,i})$ for all i, ensured by the lemma's assumptions, implies that $0 = \frac{d}{dt}\left(\frac{-N(t)}{D(t)}\right)\big|_{t=\bar{t}_i} = \frac{N'(\bar{t}_i)D(\bar{t}_i) - N(\bar{t}_i)D'(\bar{t}_i)}{D(\bar{t}_i)^2}$. Keeping in mind that $\bar{x}_{\bar{t}_i} \in \mathcal{D}_-$ and thus $D(\bar{t}_i) \neq 0$, it follows that $N'(\bar{t}_i) = D'(\bar{t}_i)\frac{N(\bar{t}_i)}{D(\bar{t}_i)}$. Now since $\frac{N(\bar{t}_i)}{D(\bar{t}_i)} \to -\infty$ but $N'(t) = v^T\beta_f(\bar{x}_t) \to 0$, it should be that $D'(\bar{t}_i) = v^T\beta_G(\bar{x}_{\bar{t}_i})\frac{h_G(\bar{t}_i)}{\|h_G(\bar{t}_i)\|} \to 0$, which proves the statement.

Case 2.2: Assume that for all n, there exists $i > n$ such that $D(t_{1,i})D(t_{2,i}) = 0$. Without loss of generality, assume $D(t_{1,i}) = 0$ for all i (the same reasoning can be applied when $D(t_{2,i}) = 0$ for all i or when this alternates). By assumption, both functions N and D are continuous on $[t_{1,i}, t_i]$ and differentiable on $(t_{1,i}, t_i)$. Using [18, Theorem 5.9], for each i, there exists $\bar{t}_i$ such that $N'(\bar{t}_i) = \frac{N(t_i)-N(t_{1,i})}{D(t_i)-D(t_{1,i})}D'(\bar{t}_i) = \frac{N(t_i)}{D(t_i)}D'(\bar{t}_i)$. A reasoning similar to that of Case 2.1 now yields $D'(\bar{t}_i) \to 0$. □