When sampling works in data-driven control: Informativity for stabilization in continuous time

Jaap Eising Jorge Cortés

Abstract—**This paper introduces a notion of data informativity for stabilization tailored to continuous-time signals and systems. We establish results comparable to those known for discrete-time systems with sampled data. We justify that additional assumptions on the properties of the noise signals are needed to understand when sampled versions of continuous-time signals are informative for stabilization, thereby introducing the notions of square Lipschitzness and total bounded variation. This allows us to connect the continuous and discrete domains, yielding sufficient conditions to synthesize a stabilizing controller for the true continuous-time system on the basis of sampled data. Simulations illustrate our results.**

I. Introduction

Data-driven control has emerged as an appealing way of combining the use of data with solid theoretical principles from systems theory to synthesize controllers for unknown systems on the basis of measurements. The development of 'one-shot' controller design methods in particular has attracted significant interest, where data is directly employed for design without an intermediate system identification step. Owing to the discrete-time nature of sampled data, most of this progress has been for systems operating in discrete time. However, systems that evolve in continuous time are widespread across engineering disciplines due to the physical nature of real-world phenomena. Often times, such systems are interconnected with digital controllers that operate in discrete time. In the context of data-driven control, understanding the interface between the continuous and digital domains is particularly relevant as measurements come often in the form of samples. The goal of this paper is to understand to what extent continuous-time data and its samples are informative enough to ensure stabilizability of an unknown plant evolving in continuous time.

Literature review: Data-driven control has been particularly fruitful for linear systems, where the notion of persistency of excitation and specifically Willems' fundamental lemma [1] have allowed users to express any finite length trajectory in terms of sufficiently informative measurements. This has proven useful in a range of problems, including simulation [2], linear feedback design [3], predictive control [4], and optimal control laws [5], [6]. Aligned with this body of work, the informativity approach to data-driven control introduced in [7], [8] considers measurements that do not contain enough information to obtain a unique system. By making assumptions on the model class and noise model, this approach explicitly determines the set of all systems consistent with the measurements, thereby enabling the certification of desirable properties (e.g., stabilizability) for the measured system. Most of the aforementioned works deal with discrete-time systems, and correspondingly with measurements consisting of sequences of states and inputs. To our knowledge, the only works dealing with continuous-time systems do so on the basis of discretized measurements, see e.g., [9]–[12]. In line with this, [13] derives a variant of Willems' lemma for continuous-time systems on the basis of samples. Moreover, many real-world phenomena take place in continuous time and as such, the examples of [14]–[17] are found by discretizing a continuous-time system.

Statement of contributions: We deal with the model class of continuous-time linear systems and investigate the informativity of data for stabilization. First, we provide conditions for stabilizability with measurements in the form of continuous-time trajectories and noise models given in terms of integrals of the noise signal. Complementarily, we also derive conditions of when sampled data are informative for continuous-time stabilization. Through an example, we show how no connection between the two notions can be established without additional assumptions on the noise model, motivating our consideration of square Lipschitzness and bounded total square variation noise models. These notions allow us to establish several connections between the continuous and discrete domains, culminating in sufficient conditions to synthesize a stabilizing controller for the true continuous-time system on the basis of sampled data. Finally, we study the role of the sampling stepsize, provide a bound on it to guarantee the informativity of the sampled data and a criterion that enables us to remove a portion of the measurements without losing informativity. Simulations illustrate our results.

II. Problem formulation

Consider[1] the continuous-time system

$$\dot{x}(t) = A_s x(t) + B_s u(t) + w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $w(t) \in \mathbb{R}^n$ is a disturbance. Here $A_s : \mathbb{R}^n \to \mathbb{R}^n$ and $B_s : \mathbb{R}^n \to \mathbb{R}^m$ are unknown linear maps, and the sub-index s is used to denote the true *system* matrices. Given a finite-time horizon $T > 0$, we are interested in *absolutely continuous* state trajectories x of (1) on the interval $[0, T]$.

J. Eising and J. Cortés are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego. {jeising,cortes}@ucsd.edu

[1]We denote by $\mathbb{Z}_{>0}$ and $\mathbb{R}$ the set of positive integer and real numbers, respectively. For a vector $v \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, $\|v\|$ and $\|A\|$ denote the Euclidean norm and induced Euclidean norm, resp. The Moore-Penrose pseudo-inverse of A is denoted $A^\dagger$. We let I_n denote the $n \times n$ identity matrix. A property holds for almost all $t \in [0, T]$ if the set for which the property does not hold has Lebesgue measure 0. A function $z : [0, T] \to \mathbb{R}^n$ is L-Lipschitz if $\|z(t_1) - z(t_2)\| \leqslant L|t_1 - t_2|$ for $t_1, t_2 \in [0, T]$. If z is differentiable, this is equivalent to $\|z'(t)\| \leqslant L$ for all $t \in [0, T]$. z is absolutely continuous if there is an integrable function $\hat{z} : [0, T] \to \mathbb{R}^n$ such that $z(t) = z(0) + \int_0^t \hat{z}(\tau)d\tau$. Note that this means that z has a derivative $\hat{z}$ almost everywhere. We denote the set of square-integrable functions by $\mathcal{L}_2$.

Since A_s and B_s are unknown, we take a data-driven approach to determine properties of the system and control it. We consider continuous-time measurements over the interval $[0,T]$. Specifically, we consider measured state $x:[0,T]\to\mathbb{R}^n$ and input $u:[0,T]\to\mathbb{R}^m$ trajectories. We assume that the associated disturbance $w:[0,T]\to\mathbb{R}^n$ satisfies a *noise model*, denoted Δ, defined as follows: for $0\le Q\in\mathbb{R}^{n\times n}$, $w\in\Delta$ if and only if

$$\int_0^T w(t)w(t)^\top dt \leqslant Q. \tag{2}$$

Taking the trace of both sides, we see that (2) implies $\int_0^T w(t)^\top w(t)dt \leqslant \mathrm{tr}(Q)$, and therefore $\Delta\subseteq\mathcal{L}_2$.

This noise model captures the behavior of common assumptions on noise signals. For instance, if for almost all $t\in[0,T]$,

$$w(t)w(t)^\top \leqslant \tfrac{1}{T}Q, \tag{3}$$

then (2) holds. Moreover, if there is a bound on the norm of the values of the disturbance signal, this can be brought into the form above by observing that

$$w(t)^\top w(t)\leqslant k \iff w(t)w(t)^\top \leqslant kI_n,$$

We make the following assumption on the measurements.

Assumption 1 (Well-behavedness of the measurements)**.** The measurement signals $x:[0,T]\to\mathbb{R}^n$, $u:[0,T]\to\mathbb{R}^m$ satisfy

- The state signal x is absolutely continuous;
- The input signal u is square integrable;
- The corresponding noise signal $w:[0,T]\to\mathbb{R}^n$ belongs to Δ as defined by (2);
- The triplet (x,u,w) satisfies (1) for almost all $t\in[0,T]$.

This assumption is mild but necessary for our ensuing analysis. Since x is absolutely continuous on the compact interval $[0,T]$, it is bounded. As a consequence, $x\in\mathcal{L}_2$. This, together with the fact that (1) holds almost everywhere and $\Delta\subseteq\mathcal{L}_2$, implies that $\dot{x}\in\mathcal{L}_2$ too.

Underlying the informativity approach is the observation that, on the basis of measurements, one can only conclude a property of interest of the true system (A_s,B_s) if *all* systems compatible with the measurements have such property. As such, we consider the set of all systems compatible with the measurement and noise model as defined by

$$\Sigma=\{(A,B)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\mid \dot{x}-Ax-Bu\in\Delta\}.$$

We are interested in finding a stabilizing controller for (A_s,B_s) on the basis of the measurements x and u. This leads to the following notion.

Definition II.1 (Informativity of continuous-time data for quadratic stabilization)**.** Data consisting of state $x:[0,T]\to\mathbb{R}^n$ and input $u:[0,T]\to\mathbb{R}^m$ trajectories are *informative for quadratic stabilization* if and only if there exists $K\in\mathbb{R}^{m\times n}$ and $P\in\mathbb{R}^{n\times n}$ such that $P>0$ and for all $(A,B)\in\Sigma$:

$$(A+BK)P+P(A+BK)^\top<0. \tag{4}$$

Our first objective is to provide necessary and sufficient conditions on the data (x,u) which ensure this notion of informativity is satisfied. Our second objective seeks to understand when sampled versions of the continuous-time data remain informative enough for stabilization. To formalize this objective, assume we have access to samples of the signals x and u at a number of discrete-time instants. We assume that the stepsize δ is a whole fraction of the time horizon[2], that is, $\frac{T}{\delta}\in\mathbb{Z}_{>0}$, which means that we consider samples at time instances $\{t_k=k\delta\}_{k=0}^{T/\delta-1}\subset[0,T]$. We collect the measurements and samples of the noise signal into matrices

$$\dot{X}_\delta=\begin{bmatrix}\dot{x}(0) & \cdots & \dot{x}(T-\delta)\end{bmatrix},\ X_\delta=\begin{bmatrix}x(0) & \cdots & x(T-\delta)\end{bmatrix}, \tag{5a}$$
$$U_\delta=\begin{bmatrix}u(0) & \cdots & u(T-\delta)\end{bmatrix},\ W_\delta=\begin{bmatrix}w(0) & \cdots & w(T-\delta)\end{bmatrix}. \tag{5b}$$

As before, $\dot{X}_\delta$, X_δ, and U_δ are known, but the samples of the noise, collected in the matrix W_δ, are unknown. However, we assume W_δ satisfies some noise model Δ_{disc}. In particular, as a special case of noise models considered in the discrete-time informativity literature [8], we assume that for some $0\le Q\in\mathbb{R}^{n\times n}$, $W_\delta\in\Delta_{\text{disc}}$ if and only if

$$\delta W_\delta W_\delta^\top\leqslant Q. \tag{6}$$

Note this holds for example if (3) is satisfied for all $t=k\delta$, where $k=0,\dots,{}^{T}\!/\!_{\delta}-1$. On the basis of the samples, we seek to find a stabilizing controller for all systems in the set

$$\Sigma^\delta=\{(A,B)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\mid \dot{X}_\delta-AX_\delta-BU_\delta\in\Delta_{\text{disc}}\}.$$

Our second objective can then be formalized as: provide conditions on the continuous-time measurements under which we can compare stabilizability properties of Σ and Σ^δ. We focus on understanding when the continuous-time measurements (x,u) are informative for quadratic stabilization on the basis of sampled data and on the stepsizes that make this happen.

III. Data informativity in continuous-time

Here we provide characterizations for when data, either in the form of continuous-time trajectories or sampled versions of it, is informative for continuous-time stabilization.

A. Informativity with continuous-time data

Here we address the first objective laid out in Section II and characterize when continuous-time data is informative for stabilization. We start by observing that the set Σ of systems compatible with the data can be defined via a Quadratic Matrix Inequality (QMI). Formally, consider measurements x and u satisfying Assumption 1, with noise model (2). For $N=N^\top\in\mathbb{R}^{(2n+m)\times(2n+m)}$, let

$$\mathcal{Z}(N):=\left\{(A,B)\mid\begin{bmatrix}I_n & A & B\end{bmatrix}N\begin{bmatrix}I_n & A & B\end{bmatrix}^\top\geqslant 0\right\}.$$

Then, one has $\Sigma=\mathcal{Z}(N_{\text{cont}}(Q))$, where

$$N_{\text{cont}}(Q):=\begin{bmatrix}Q&0&0\\0&0&0\\0&0&0\end{bmatrix}-\int_0^T\begin{pmatrix}\dot{x}(t)\\-x(t)\\-u(t)\end{pmatrix}\begin{pmatrix}\dot{x}(t)\\-x(t)\\-u(t)\end{pmatrix}^\top dt. \tag{7}$$

On the other hand, the stability condition (4) is equivalent to

$$\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}^\top\begin{bmatrix}0&-P&-PK^\top\\-P&0&0\\-KP&0&0\end{bmatrix}\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}>0, \tag{8}$$

for all $(A,B)\in\Sigma$. This means that we are interested in identifying conditions under which all (A,B) satisfying a QMI

[2]The choice of a uniform stepsize makes the notation simpler, but our results can be easily adapted to deal with more general sampling schemes.

(the one defined by (7)) satisfy another QMI (in this case, (8)). The following result achieves this.

Theorem III.1 (Necessary and sufficient conditions for informativity of continuous-time data)**.** *Suppose that the state and input trajectories $x:[0,T]\to\mathbb{R}^n$ and $u:[0,T]\to\mathbb{R}^m$ satisfy Assumption 1. Then the data (x,u) are informative for quadratic stabilization if and only if there exists $K\in\mathbb{R}^{m\times n}$, $P\in\mathbb{R}^{n\times n}$, and $\beta>0$ such that $P>0$ and*

$$-\begin{bmatrix} Q+\beta I_n & P & PK^\top \\ P & 0 & 0 \\ KP & 0 & 0 \end{bmatrix}+\int_0^T\begin{pmatrix}\dot x(t)\\ -x(t)\\ -u(t)\end{pmatrix}\begin{pmatrix}\dot x(t)\\ -x(t)\\ -u(t)\end{pmatrix}^\top dt\geqslant 0. \quad (9)$$

Proof. We partition $N_{\text{cont}}(Q)$ as

$$N_{\text{cont}}(Q)=:\begin{bmatrix} N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix},$$

where $N_{11}\in\mathbb{R}^{n\times n}$ and $N_{22}\in\mathbb{R}^{(n+m)\times(n+m)}$. Let

$$M:=\begin{bmatrix} 0 & -P & -PK^\top \\ -P & 0 & 0 \\ -KP & 0 & 0\end{bmatrix},$$

for which we consider a similar partition. To prove the result, we need to verify the hypotheses of [8, Corollary 4.13], which in this case take the form: $N_{22}\leqslant 0$, $N_{11}-N_{12}N_{22}^\dagger N_{21}\geqslant 0$, $M_{22}\leqslant 0$, and $\ker N_{22}\subseteq\ker N_{12}$. The first condition follows from the fact that

$$\int_0^T\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt\geqslant 0,$$

as it is the integral of positive semidefinite matrices. The second immediately follows from and the fact that $\mathcal{Z}(N_{\text{cont}}(Q))$ is nonempty (see [8, Eq. (3.5)]). The third is immediate since $M_{22}=0$. To show the fourth condition, we need to prove

$$\ker\int_0^T\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt\subseteq\ker\int_0^T\dot x\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt.$$

Let $v\in\ker\int_0^T\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt$. Then

$$\begin{aligned}0&=v^\top\left(\int_0^T\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt\right)v\\ &=\int_0^T v^\top\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top v dt.\end{aligned}$$

Therefore, $\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top v=0$ for almost all t, and hence $v\in\ker\int_0^T\dot x\begin{pmatrix}x(t)\\ u(t)\end{pmatrix}^\top dt$. We can invoke then [8, Corollary 4.13] to conclude that (8) holds for all $(A,B)\in\Sigma$ if and only if there exists $\alpha\geqslant 0$ and $\beta>0$ such that

$$M-\alpha N_{\text{cont}}(Q)\geqslant\begin{bmatrix}\beta I_n & 0\\ 0 & 0\end{bmatrix}.$$

Since $M\not\geqslant 0$, this requires $\alpha\neq 0$. Therefore, we can we can scale β and P by α, proving the statement. □

As presented, the equation (9) is not a linear matrix inequality (LMI) in the variables K, P, and β. However it can be rewritten as an LMI using the substitution $L:=KP$. This allows us to efficiently check for informativity by checking feasibility of an LMI in the variables L, P, and β. Afterwards, one can use the equation $K=LP^{-1}$ to find the corresponding stabilizing feedback.

Remark III.2 (Comparison of computational complexity with discrete-time case)*.* The condition (9) of Theorem III.1 takes the form of the scalar inequality $\beta>0$, the $n\times n$ LMI $P>0$ and an LMI of dimensions $(2n+m)\times(2n+m)$. Instead, the condition of informativity for quadratic stabilization in the discrete-time case, cf. [8, Theorem 5.1], requires $\beta>0$, $P>0$, and an LMI of dimensions $(3n+m)\times(3n+m)$. •

Remark III.3 (General noise models)*.* One can extend Theorem III.1 for noise models more general than (2) without significant additional effort. Let $\Pi:[0,T]\to\mathbb{R}^{(n+1)\times(n+1)}$ be a matrix-valued function and partition it as

$$\Pi(t)=\begin{bmatrix}\Pi_{11}(t) & \Pi_{12}(t)\\ \Pi_{21}(t) & \Pi_{22}(t)\end{bmatrix},\ \text{with } \Pi_{11}(t)\in\mathbb{R}^{n\times n}.$$

Consider the generalized noise model: $w\in\Delta$ if and only if

$$\int_0^T\begin{bmatrix}I_n\\ w(t)^\top\end{bmatrix}^\top\Pi(t)\begin{bmatrix}I_n\\ w(t)^\top\end{bmatrix}dt\geqslant 0.$$

Under more general assumptions than those made above, an extension of Theorem III.1 can be derived analogously. •

B. Informativity with sampled data

Here, we analyze when sampled versions of continuous-time data are sufficiently informative for stabilization. Let the state and input trajectories $x:[0,T]\to\mathbb{R}^n$ and $u:[0,T]\to\mathbb{R}^m$ satisfy Assumption 1. Recall the definitions of the matrices $\dot X_\delta$, X_δ, U_δ, and W_δ in (5), and consider a noise model Δ_{disc} as in (6). Note that the set of systems compatible with the sampled data $(\dot X_\delta,X_\delta,U_\delta)$ and noise model (6) can be described by $\Sigma^\delta=\mathcal{Z}(N_\delta(Q))$, where

$$N_\delta(Q):=\begin{bmatrix}Q & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\delta\begin{pmatrix}\dot X_\delta\\ -X_\delta\\ -U_\delta\end{pmatrix}\begin{pmatrix}\dot X_\delta\\ -X_\delta\\ -U_\delta\end{pmatrix}^\top.$$

As before, we are interested in finding a stabilizing controller for (A_s,B_s) on the basis of the discrete measurements, leading to the following notion.

Definition III.4 (Informativity of discrete-time data for quadratic stabilization, cf. [8, Def. 2.1])**.** The sampled data $(\dot X_\delta,X_\delta,U_\delta)$ are *informative for continuous-time quadratic stabilization* if and only if there exists $K\in\mathbb{R}^{m\times n}$ and $P\in\mathbb{R}^{n\times n}$ such that for all $(A,B)\in\Sigma^\delta$:

$$P>0,\quad (A+BK)P+P(A+BK)^\top<0.$$

We now provide a characterization for informativity of discrete-time data for stabilization of continuous-time systems:

Theorem III.5 (Necessary and sufficient conditions for informativity of discrete-time data)**.** *Suppose the data $(\dot X_\delta,X_\delta,U_\delta)$ sampled from the system* (1) *correspond to noise model* (6)*. Then, the data $(\dot X_\delta,X_\delta,U_\delta)$ are informative for continuous-time quadratic stabilization if and only if there exists $K\in\mathbb{R}^{m\times n}$, $P\in\mathbb{R}^{n\times n}$, and $\beta>0$ such that $P>0$ and*

$$\begin{bmatrix}-Q-\beta I_n & -P & -PK^\top\\ -P & 0 & 0\\ -KP & 0 & 0\end{bmatrix}+\delta\begin{bmatrix}\dot X_\delta\\ -X_\delta\\ -U_\delta\end{bmatrix}\begin{bmatrix}\dot X_\delta\\ -X_\delta\\ -U_\delta\end{bmatrix}^\top\geqslant 0. \quad (10)$$

The proof of this result is similar to that of Theorem III.1 and we omit it for brevity. Note that Theorem III.5 complements the result in Theorem III.1, which characterizes informativity of continuous-time data for stabilization of continuous-time systems. Together with [8, Thm. 5.1], which characterizes informativity of discrete measurements for stabilization of discrete systems, these paint a complete picture.

Given these characterizations, a natural question is to figure out the relationship between continuous-time data (x, u) being informative, as in Theorem III.1, and sampled versions of it being informative, as in Theorem III.5. As it turns out, without additional assumptions, there is no implication between the two notions: data (x, u) can meet the conditions (9) but not those in (10), and vice versa. The reason for this can be tracked back to comparing the terms

$$\int_0^T \begin{pmatrix} \dot{x}(t) \\ -x(t) \\ -u(t) \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ -x(t) \\ -u(t) \end{pmatrix}^\top dt \text{ and } \delta \begin{bmatrix} \dot{X}_\delta \\ -X_\delta \\ -U_\delta \end{bmatrix} \begin{bmatrix} \dot{X}_\delta \\ -X_\delta \\ -U_\delta \end{bmatrix}^\top . \quad (11)$$

The issue at hand stems from the fact that, if the signal w (or equivalently the measurement signals $\dot{x}$, x, or u) is changed on a measure zero set, the integral on the left remains the same, whereas the individual samples on the right might change. The following example illustrates some of the difficulties in comparing the quantities in (11).

Example III.6 (Comparing noise models)**.** Consider the continuous-time linear system with noise given by $\dot{x}(t) = w(t)$. We consider measurements of this system over the time interval $[0, 2]$. Define $\mathcal{S}_0 = (1, 2)$, $\mathcal{S}_1 = (1, 2]$, $\mathcal{S}_2 = [1, 2]$, with corresponding noise signals,

$$w_\alpha(t) = \begin{cases} 1 & \text{for } t \in \mathcal{S}_\alpha \\ 0 & \text{otherwise} \end{cases},$$

for each $\alpha \in \{0, 1, 2\}$. Note that, for each α, we have $\int_0^3 w_\alpha(t) w_\alpha(t)^\top dt = 1$. Given initial condition $x(0) = 1$, each of these noise signals leads to the same state trajectory $x(t)$. Now suppose we sample the system at $t = 0$, $t = 1$ and $t = 2$. Defining matrices W_α as in (5) corresponding to the noise signals w_α, respectively, we obtain

$$W_0 W_0^\top = 0, \quad W_1 W_1^\top = 1, \quad W_2 W_2^\top = 2.$$

More generally, this shows that without making further assumptions on the signal w, we can not necessarily conclude that certain bounds hold for the sampled data. •

IV. Linking informativity of continuous and discrete measurements

In this section we study the relationship between informativity for stabilization of continuous and discrete measurements.

A. Connections between noise models

As illustrated by Example III.6, we need to make additional assumptions on the noise signal in order to link informativity of continuous and discrete measurements. Here, we consider two alternative models: square Lipschitzness and bounded total square variation.

Definition IV.1 (Square Lipschitzness)**.** For $L \geqslant 0$, $w : [0, T] \to \mathbb{R}^n$ is *L-square Lipschitz* if for all $t_1, t_2 \in [0, T]$:

$$\|w(t_1)w(t_1)^\top - w(t_2)w(t_2)^\top\| \leqslant L|t_1 - t_2|. \quad (12)$$

This property can be guaranteed on the basis of common assumptions on the signal w.

Lemma IV.2 (Square Lipschitzness from common assumptions)**.** *Let $w : [0, T] \to \mathbb{R}^n$ be differentiable, bounded and Lipschitz, that is, $\|w(t)\| \leqslant L_1$ and $\|\dot{w}(t)\| \leqslant L_2$ for all $t \in [0, T]$. Then w is $2L_1L_2$-square Lipschitz.*

Proof. For $t_1, t_2 \in [0, T]$, using that w is differentiable,

$$\begin{aligned} w(t_1)w(t_1)^\top - w(t_2)w(t_2)^\top &= \int_{t_2}^{t_1} \frac{d}{dt} \left(w(t)w(t)^\top \right) dt, \\ &= \int_{t_2}^{t_1} \dot{w}(t)w(t)^\top + w(t)\dot{w}(t)^\top dt. \end{aligned}$$

Thus $\|w(t_1)w(t_1)^\top - w(t_2)w(t_2)^\top\| \leqslant \int_{t_2}^{t_1} 2\|\dot{w}(t)w(t)^\top\| dt$. The result follows by noting that $\|\dot{w}(t)w(t)^\top\| \leqslant L_1L_2$. □

Note that the conditions of Lemma IV.2 are not necessary. In particular, w need not be differentiable everywhere. The following result establishes a relationship between continuous- and discrete-time noise models.

Lemma IV.3 (Continuous- and discrete-time noise models under square Lipschitzness)**.** *Suppose that $w : [0, T] \to \mathbb{R}^n$ is L-square Lipschitz and δ is such that $\frac{T}{\delta} \in \mathbb{Z}_{>0}$. Then*

$$\| \int_0^T w(t)w(t)^\top dt - \delta W_\delta W_\delta^\top \| \leqslant \tfrac{1}{2}\delta T L.$$

Proof. Note that we can write

$$\begin{aligned} &\int_0^T w(t)w(t)^\top dt - \delta W_\delta W_\delta^\top \\ &= \sum_{k=0}^{T/\delta - 1} \int_{k\delta}^{(k+1)\delta} (w(t)w(t)^\top - w(k\delta)w(k\delta)^\top) dt. \end{aligned} \quad (13)$$

Since w is L-square Lipschitz, $\|w(t)w(t)^\top - w(k\delta)w(k\delta)^\top\| \leqslant |t - k\delta| L$ for $t \in [k\delta, (k+1)\delta]$. Hence,

$$\begin{aligned} &\| \int_0^T w(t)w(t)^\top dt - \delta W_\delta W_\delta^\top \| \\ &\leqslant \sum_{k=0}^{T/\delta - 1} \int_{k\delta}^{(k+1)\delta} \|w(t)w(t)^\top - w(k\delta)w(k\delta)^\top\| dt \\ &\leqslant L \sum_{k=0}^{T/\delta - 1} \int_{k\delta}^{(k+1)\delta} |t - k\delta| dt = \tfrac{1}{2}\delta T L. \end{aligned}$$ □

Square Lipschitzness requires the noise signal w to be continuous. As an alternative, the following concept allows us to consider discontinuous signals.

Definition IV.4 (Total square variation)**.** Let $\mathcal{P}$ denote the set of all partitions of $[0, T]$, that is,

$$\mathcal{P} = \{\pi = \{t_0, \ldots, t_{n_\pi}\} \mid 0 = t_0 \leqslant \ldots \leqslant t_{n_\pi} = T\}.$$

The *total square variation* of the signal $w : [0, T] \to \mathbb{R}^n$ is

$$V_0^T(w) = \sup_{\pi \in \mathcal{P}} \sum_{i=0}^{n_\pi - 1} \|w(t_{i+1})w(t_{i+1})^\top - w(t_i)w(t_i)^\top\|.$$

The step function is an example of a discontinuous signal that has a finite total square variation. The following result establishes another relationship between continuous- and discrete-time noise models.

Lemma IV.5 (Continuous- and discrete-time noise models under bounded total variation)**.** *Suppose that $w : [0,T] \to \mathbb{R}^n$ has $V_0^T(w)$ finite and let δ be such that $\frac{T}{\delta} \in \mathbb{Z}_{>0}$. Then,*

$$\| \int_0^T w(t)w(t)^\top dt - \delta W_\delta W_\delta^\top \| \leqslant \delta V_0^T(w).$$

Proof. Let $V_k := V_{k\delta}^{(k+1)\delta}(w)$. By definition, $V_0^T(w) = \sum_{k=0}^{T/\delta-1} V_k$. Now, for any $t \in [k\delta, (k+1)\delta]$, consider the partition $\{k\delta, t, (k+1)\delta\}$. Then,

$$\begin{aligned} V_k &\geqslant \|w(t)w(t)^\top - w((k+1)\delta)w((k+1)\delta)^\top\| \\ &\quad + \|w(t)w(t)^\top - w(k\delta)w(k\delta)^\top\| \\ &\geqslant \|w(t)w(t)^\top - w(k\delta)w(k\delta)^\top\|. \end{aligned}$$

This, combined with (13), yields the result. □

Note that if w is L-square Lipschitz, then $V_0^T(w) \leqslant LT$, and in this case the result in Lemma IV.5 (bound with δLT) is weaker than that of Lemma IV.3 (bound with $\frac{1}{2}\delta LT$). Lemmas IV.3 or IV.5 allow us to bound the deviation of the continuous-time signal to its samples and draw conclusions regarding the noise model (2) and its counterpart (6).

Corollary IV.6 (Relations between noise models)**.** *Suppose δ is such that $\frac{T}{\delta} \in \mathbb{Z}_{>0}$ and let $L \geqslant 0$ be such that $w : [0,T] \to \mathbb{R}^n$ is either (i) L-square Lipschitz or (ii) $V_0^T(w) \leqslant \frac{1}{2}LT$. Then, the following two statements hold:*

$$\delta W_\delta W_\delta^\top \leqslant Q \Rightarrow \int_0^T w(t)w(t)^\top dt \leqslant Q + \tfrac{1}{2}\delta TLI_n,$$

$$\int_0^T w(t)w(t)^\top dt \leqslant Q \Rightarrow \delta W_\delta W_\delta^\top \leqslant Q + \tfrac{1}{2}\delta TLI_n.$$

B. Inclusions between sets of consistent systems

Here we address the second objective laid out in Section II and compare the stabilizability properties of the sets $\mathcal{Z}(N_{\text{cont}}(Q))$ and $\mathcal{Z}(N_\delta(Q))$. To tackle this, note that the additional assumptions on the noise signal described in Section IV-A shrink the set of systems consistent with the data and we formalize this next. Given state and input trajectories $x : [0,T] \to \mathbb{R}^n$ and $u : [0,T] \to \mathbb{R}^m$ that satisfy Assumption 1, we define the sets

$$\begin{aligned} \mathcal{M}_{x,u}^L &:= \{(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid \dot{x} - Ax - Bu \\ &\qquad \text{is } L\text{-square Lipschitz}\}, \\ \mathcal{N}_{x,u}^L &:= \{(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid \\ &\qquad V_0^T(\dot{x} - Ax - Bu) \leqslant \tfrac{1}{2}LT\}. \end{aligned}$$

Then, the set of all systems compatible with the measurement, the noise model (2), and for which the noise is L-square Lipschitz is

$$\mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{M}_{x,u}^L. \tag{14a}$$

In a similar fashion, the set of all systems compatible with the measurement, the noise model (2), and for which the total square variation of the noise is less than or equal to $\frac{1}{2}LT$ is

$$\mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{N}_{x,u}^L. \tag{14b}$$

The true system from which the measurements are taken is contained in the intersections in (14) if the true realization of the noise has the corresponding property. The following result is a consequence of Corollary IV.6.

Corollary IV.7 (Inclusion relationships between sets of consistent systems)**.** *Let $x : [0,T] \to \mathbb{R}^n$ and $u : [0,T] \to \mathbb{R}^m$ be state and input trajectories satisfying Assumption 1 and let δ be such that $\frac{T}{\delta} \in \mathbb{Z}_{>0}$. Then*

- *[L-square Lipschitz noise:]*

$$\mathcal{Z}(N_\delta(Q)) \cap \mathcal{M}_{x,u}^L \subseteq \mathcal{Z}(N_{\text{cont}}(Q + \tfrac{1}{2}\delta TLI_n)), \tag{15a}$$

$$\mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{M}_{x,u}^L \subseteq \mathcal{Z}(N_\delta(Q + \tfrac{1}{2}\delta TLI_n)). \tag{15b}$$

  Moreover, if the noise signal corresponding to the measurements, $w : [0,T] \to \mathbb{R}^n$, is L-square Lipschitz, then the set on the left-hand side in (15b) *is non-empty and contains the true system.*

- *[Noise of bounded total square variation:]*

$$\mathcal{Z}(N_\delta(Q)) \cap \mathcal{N}_{x,u}^L \subseteq \mathcal{Z}(N_{\text{cont}}(Q + \tfrac{1}{2}\delta TLI_n)), \tag{16a}$$

$$\mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{N}_{x,u}^L \subseteq \mathcal{Z}(N_\delta(Q + \tfrac{1}{2}\delta TLI_n)). \tag{16b}$$

  Moreover, if the noise signal corresponding to the measurements, $w : [0,T] \to \mathbb{R}^n$, is such that $V_0^T(w) \leqslant \frac{1}{2}LT$, then the set on the left-hand side in (16b) *is non-empty and contains the true system.*

Recall that stabilizing controllers for the sets in the right-hand sides of (15) or (16) can be found using either Theorems III.1 or III.5. In particular, this means that Corollary IV.7 allows us to find a stabilizing controller for all systems in (14).

Theorem IV.8 (Sufficient conditions for sampled data)**.** *Consider state and input trajectories $x : [0,T] \to \mathbb{R}^n$ and $u : [0,T] \to \mathbb{R}^m$ such that Assumption 1 holds. Suppose there exists $K \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{n\times n}$, and $\beta > \frac{1}{2}\delta TL$ such that $P > 0$ and* (10) *is satisfied. Then,* (4) *holds for all $(A,B) \in \mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{M}_{x,u}^L$ and all $(A,B) \in \mathcal{Z}(N_{\text{cont}}(Q)) \cap \mathcal{N}_{x,u}^L$.*

This result follows from combining Corollary IV.7 and Theorem III.5. Theorem IV.8 provides conditions under which the *true* system can be stabilized and, importantly, this can be checked with only samples of the measurements (i.e., the conditions *do not* require knowledge of the continuous-time signals themselves). In contrast, Theorem III.5 also only relies on samples, but it only guarantees stabilization of all systems in Σ^δ. As discussed, for a given δ, the set $\mathcal{Z}(N_{\text{cont}})$ is *not* necessarily contained in Σ^δ. Given that we cannot distinguish the true system from any other system in $\mathcal{Z}(N_{\text{cont}})$, this means that Theorem III.5 might not guarantee the stabilization of the true system. Comparing Theorems IV.8 and III.5, we note that both require the satisfaction of the same LMI, but that Theorem IV.8 specifies $\beta > \frac{1}{2}\delta TL$ instead of $\beta > 0$. This can be interpreted as requiring a *margin of stability*, given that (10) implies that

$$(A+BK)P + P(A+BK)^\top < -\beta I_n < -\tfrac{1}{2}\delta TLI_n,$$

for all $(A,B) \in \mathcal{Z}(N_\delta(Q))$. Theorem IV.8 can then be restated as follows: if the closed-loop systems resulting from all systems compatible with the sampled measurements are stable 'enough', then all systems compatible with the continuous measurements are stabilized as well.

A natural question arising from the result in Theorem IV.8 is how small should the stepsize be to ensure the samples from the continuous-time signals remain informative. Intuitively, if we sample very coarsely, e.g., $\delta = T$, then this will be unlikely. The following result settles this question.

Corollary IV.9 (Bound on stepsize for informativity of sampled data)**.** *Consider state and input trajectories $x:[0,T]\to\mathbb{R}^n$ and $u:[0,T]\to\mathbb{R}^m$ such that Assumption 1 holds. Assume the corresponding noise signal w is either L-square Lipschitz or such that $V_0^T(w)\leqslant\frac{1}{2}LT$. Suppose that (x,u) are informative for quadratic stabilization and let $\hat\beta$ be the largest $\beta>0$ such that there exists $K\in\mathbb{R}^{m\times n}$, $P\in\mathbb{R}^{n\times n}$, with $P>0$ and* (9)*. If $\delta<\frac{1}{TL}\hat\beta$, then* (10) *holds with $\beta=\hat\beta-\frac{1}{2}\delta TL>\frac{1}{2}\delta TL$.*

The proof of this result leverages the margin of stability associated to informative continuous-time data (x,u) and follows from Corollary IV.7. As a consequence, we deduce that, under the assumptions of Corollary IV.9, there *always* exists a stepsize small enough to conclude continuous-time quadratic stabilization.

Next, we identify conditions under which we can verify, on the basis of data, that the noise signal corresponding to the measurements and the true system is either L-square Lipschitz or has bounded total square variation. Even more, the following result establishes sufficient conditions to determine these properties for all possible noise corresponding to *every* system compatible with the measurements.

Lemma IV.10 (Verifying the assumptions using data)**.** *Let the state $x:[0,T]\to\mathbb{R}^n$ and input $u:[0,T]\to\mathbb{R}^m$ trajectories satisfy Assumption 1. Suppose there exists $\lambda\geqslant 1$ such that*

$$AA^\top+BB^\top<(\lambda-1)I_n, \tag{17}$$

for all $(A,B)\in\mathcal{Z}(N_{\text{cont}}(Q))$,

(i) If the signal $\begin{pmatrix}\dot x^\top & -x^\top & -u^\top\end{pmatrix}^\top$ is L-square Lipschitz, then

$$\mathcal{Z}(N_{\text{cont}}(Q))\subseteq\mathcal{M}_{x,u}^{\lambda L}. \tag{18}$$

(ii) If $V_0^T\left(\begin{pmatrix}\dot x^\top & -x^\top & -u^\top\end{pmatrix}^\top\right)\leqslant\frac{1}{2}LT$, then

$$\mathcal{Z}(N_{\text{cont}}(Q))\subseteq\mathcal{N}_{x,u}^{\lambda L}. \tag{19}$$

Moreover, such λ exists if and only if

$$\int_0^T\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\begin{pmatrix}x(t)\\u(t)\end{pmatrix}^\top dt>0. \tag{20}$$

Proof. To prove statements (i) and (ii), let $w_{(A,B)}(t):=\dot x(t)-Ax(t)-Bu(t)$. Then, to prove that (18) hold, we need to show that $w_{(A,B)}(t)$ is λL-square Lipschitz continuous for all $(A,B)\in\mathcal{Z}(N_{\text{cont}}(Q))$. Similarly, (19) is equivalent to $V_0^T(w_{(A,B)})\leqslant\frac{1}{2}\lambda LT$ for all $(A,B)\in\mathcal{Z}(N_{\text{cont}}(Q))$. Note that (17) is equivalent to

$$\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}^\top\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}<\lambda I_n, \tag{21}$$

for all $(A,B)\in\mathcal{Z}(N_{\text{cont}}(Q))$. This implies that

$$\begin{aligned}&w_{(A,B)}(t_1)w_{(A,B)}(t_1)^\top-w_{(A,B)}(t_2)w_{(A,B)}(t_2)^\top\\
=&\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}^\top\left(\begin{pmatrix}\dot x(t_1)\\-x(t_1)\\-u(t_1)\end{pmatrix}\begin{pmatrix}\dot x(t_1)\\-x(t_1)\\-u(t_1)\end{pmatrix}^\top-\begin{pmatrix}\dot x(t_2)\\-x(t_2)\\-u(t_2)\end{pmatrix}\begin{pmatrix}\dot x(t_2)\\-x(t_2)\\-u(t_2)\end{pmatrix}^\top\right)\begin{bmatrix}I_n\\A^\top\\B^\top\end{bmatrix}\\
\leqslant&\,\lambda\left(\begin{pmatrix}\dot x(t_1)\\-x(t_1)\\-u(t_1)\end{pmatrix}\begin{pmatrix}\dot x(t_1)\\-x(t_1)\\-u(t_1)\end{pmatrix}^\top-\begin{pmatrix}\dot x(t_2)\\-x(t_2)\\-u(t_2)\end{pmatrix}\begin{pmatrix}\dot x(t_2)\\-x(t_2)\\-u(t_2)\end{pmatrix}^\top\right).\end{aligned}$$

Taking the norm on both sides of this inequality, and applying this to the definition of L-square Lipschitz continuity (resp. total square variation) immediately yields (i) (resp. (ii)). To prove the last statement, we apply [8, Cor. 4.13] to see that (21) holds for all $(A,B)\in\mathcal{Z}(N_{\text{cont}}(Q))$ if and only if there exists $\alpha\geqslant 0$ and $\beta>0$ such that

$$\begin{bmatrix}(\lambda-1-\beta)I_n-\alpha Q & 0 & 0\\0 & -I_n & 0\\0 & 0 & -I_n\end{bmatrix}+\alpha\int_0^T\begin{pmatrix}\dot x(t)\\-x(t)\\-u(t)\end{pmatrix}\begin{pmatrix}\dot x(t)\\-x(t)\\-u(t)\end{pmatrix}^\top dt\geqslant 0.$$

Zooming in on the right-lower block, we see that this requires (20). Conversely, if (20) holds, there exists α such that

$$\alpha\int_0^T\begin{pmatrix}x(t)\\u(t)\end{pmatrix}\begin{pmatrix}x(t)\\u(t)\end{pmatrix}^\top dt\geqslant I_{2n}.$$

Then, for large enough $\lambda\geq 1$, the LMI is satisfied, concluding the proof. ☐

The combination of Lemma IV.10 and Theorem IV.8 yields the following result.

Corollary IV.11 (Sufficient conditions for informativity)**.** *Suppose that the state $x:[0,T]\to\mathbb{R}^n$ and input $u:[0,T]\to\mathbb{R}^m$ trajectories satisfy Assumption 1 and that* (20) *holds. Take $\lambda\geqslant 1$ such that* (17) *holds. Assume there exists $K\in\mathbb{R}^{m\times n}$, $P\in\mathbb{R}^{n\times n}$, and $\beta>\frac{1}{2}\delta TL$ such that $P>0$ and* (10) *holds. If either (i) the signal $\begin{pmatrix}\dot x^\top & -x^\top & -u^\top\end{pmatrix}^\top$ is $\frac{L}{\lambda}$-square Lipschitz, or (ii) $V_0^T\left(\begin{pmatrix}\dot x^\top & -x^\top & -u^\top\end{pmatrix}^\top\right)\leqslant\frac{1}{2}\frac{L}{\lambda}T$, then (x,u) are informative for quadratic stabilization.*

C. Refining and coarsening sampled data

Here we examine the impact that the stepsize has on the informativity of sampled data and its relationship with the informativity of the continuous-time data. Indeed, as suggested by Corollary IV.6, decreasing the stepsize brings both notions of informativity closer together. In this section, we instead consider increasing the stepsize and examine to what extent the number of samples can be reduced while retaining informativity.

Let δ and γ be stepsizes satisfying $\frac{T}{\delta},\frac{T}{\gamma}\in\mathbb{Z}_{>0}$. Using the triangle inequality and Lemma IV.3, we can conclude that, if w is L-square Lipschitz,

$$\|\delta W_\delta W_\delta^\top-\gamma W_\gamma W_\gamma^\top\|\leqslant\tfrac{1}{2}(\delta+\gamma)TL.$$

This result can be applied similarly to Corollary IV.6 to obtain results comparing the respective noise models, in turn linking their respective informativity properties. However, if we instead refine (respectively, coarsen) the sampling by multiplying the stepsize with a constant, we can obtain less conservative bounds.

Lemma IV.12 (Bounds on noise model under different stepsizes)**.** *Let $w:[0,T]\to\mathbb{R}^n$ be L-square Lipschitz, δ and γ such that $\gamma=(\ell+1)\delta$ with $\frac{T}{\delta},\frac{T}{\gamma}\in\mathbb{Z}_{>0}$ and $\ell\in\mathbb{Z}_{\geqslant 0}$. Then*

$$\|\delta W_\delta W_\delta^\top-\gamma W_\gamma W_\gamma^\top\|\leqslant\tfrac{1}{2}(\gamma-\delta)TL=\tfrac{1}{2}\ell\delta TL.$$

Proof. Note that

$$W_\delta W_\delta^\top=\sum_{k=0}^{T/\gamma-1}\sum_{j=0}^{\ell}w(k\gamma+j\delta)w(k\gamma+j\delta)^\top.$$

On the other hand, we can expand

$$(\ell+1)W_\gamma W_\gamma^\top = \sum_{k=0}^{T/\gamma-1}\sum_{j=0}^{\ell} w(k\gamma)w(k\gamma)^\top.$$

The fact that w is L-square Lipschitz yields

$$\|w(k\gamma+j\delta)w(k\gamma+j\delta)^\top - w(k\gamma)w(k\gamma)^\top\| \leqslant j\delta L.$$

Combining the above, we get

$$\|\delta W_\delta W_\delta^\top - \gamma W_\gamma W_\gamma^\top\| \leqslant \delta\tfrac{T}{\gamma}\Big(\sum_{j=0}^{\ell} j\delta L\Big) = \tfrac{1}{2}(\gamma-\delta)TL,$$

proving the result. □

This result allows us to link properties of the noise models under different sampling rates.

Corollary IV.13 (Relations between noise models with different stepsizes)**.** *Let $w:[0,T]\to\mathbb{R}^n$ be L-square Lipschitz, δ and γ such that $\gamma=(\ell+1)\delta$ with $\frac{T}{\delta},\frac{T}{\gamma}\in\mathbb{Z}_{>0}$, and $\ell\in\mathbb{Z}_{\geqslant 0}$. Then*

$$\delta W_\delta W_\delta^\top \leqslant Q \Rightarrow \gamma W_\gamma W_\gamma^\top \leqslant Q + \tfrac{1}{2}(\gamma-\delta)TLI_n,$$
$$\gamma W_\gamma W_\gamma^\top \leqslant Q \Rightarrow \delta W_\delta W_\delta^\top \leqslant Q + \tfrac{1}{2}(\gamma-\delta)TLI_n.$$

We are now ready to provide a criterion to increase the sampling stepsize without losing informativity.

Theorem IV.14 (Coarsening measurements)**.** *Consider state $x:[0,T]\to\mathbb{R}^n$ and input $u:[0,T]\to\mathbb{R}^m$ trajectories such that Assumption 1 holds. Assume the corresponding noise signal w is L-square Lipschitz. Suppose that the data $(\dot X_\delta, X_\delta, U_\delta)$ are informative for continuous-time quadratic stabilization and let $\hat\beta$ the largest $\beta>0$ such that there exists $K\in\mathbb{R}^{m\times n}$, $P\in\mathbb{R}^{n\times n}$, with $P>0$ and* (10)*. Then, the data $(\dot X_\gamma, X_\gamma, U_\gamma)$ are informative for continuous-time quadratic stabilization for $\gamma=(\ell+1)\delta$, with $\ell<\frac{2}{\delta TL}\hat\beta$.*

Note that, under the assumptions of Theorem IV.14, the samples $(\dot X_\gamma, X_\gamma, U_\gamma)$ are contained in those of $(\dot X_\delta, X_\delta, U_\delta)$. This means that, given informative data, the result allows to find a subset of it which remains informative. In particular, to determine continuous-time quadratic stabilization, we can draw conclusions from data that contains ℓ times less samples. One can derive similar results for the case of noise with bounded total variation, but we omit them for brevity.

V. Scalar system with square Lipschitz noise

Here, we show that nontrivial behavior arises even for a scalar system with well-behaved noise and input signals. Consider the scalar linear system

$$\dot x(t) = -x(t) + \tfrac{1}{10}u(t) + w(t),$$

with initial condition $x(0)=1$. The time horizon is $T=1$. We consider noise signals of the form (2) with $Q=1$. We excite the system with a uniform input $u(t)=1$ and the (piecewise linear) noise signal

$$w(t) = \max\{0, 2-4t\} = \begin{cases} 0 & t\leqslant \frac{1}{2} \\ 2-4t & t>\frac{1}{2}\end{cases}.$$

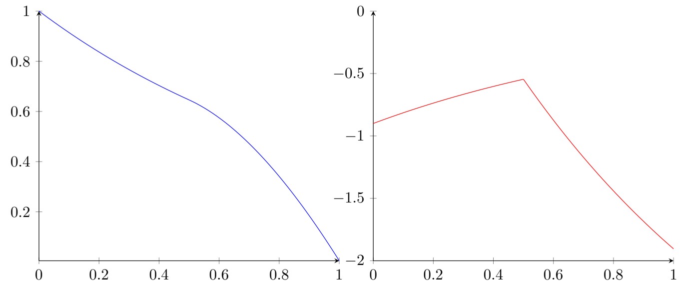

Figure 1: Measured state $x(t)$ (left) and derivative $\dot x(t)$ (right) signals. Together with the input $u(t)=1$, these constitute the continuous-time data considered in Section V.

It is straightforward to show that $\int_0^1 w(t)^2dt = \frac{2}{3}\leqslant 1$, and that w is 16-square Lipschitz. Solving for the dynamics yields

$$x(t) = \begin{cases} \frac{1}{10}e^{-t}(9+e^t) & t\leqslant\frac{1}{2} \\ \frac{1}{10}e^{-t}(9-40\sqrt{e}+e^t(61-40t) & t>\frac{1}{2}\end{cases}.$$

Figure 1 shows the signal x and its derivative.

Any system, given in terms of state and input matrices (a,b), is compatible with the measurements if and only if $(a,b)\in\mathcal{Z}(N_{\text{cont}}(1))\cap\mathcal{M}^{16}_{x,u}$, where $N_{\text{cont}}(1)$ is given in (7). Calculating the relevant integrals yields

$$N_{\text{cont}}(1) \approx \begin{bmatrix} -0.154 & -0.500 & -0.995 \\ -0.500 & -0.422 & -0.595 \\ -0.995 & -0.595 & -1\end{bmatrix}.$$

Now, note that for $P=\frac{1}{2}>0$, $K=2$, and $\beta=\frac{1}{10}$, the LMI (9) holds. Using Theorem III.1, this allows us to conclude that the data (x,u) is informative for quadratic stabilization. Indeed, the true, measured system is indeed stabilized by the feedback gain $K=2$.

Next, we turn our attention to sampling the data. We take δ equal to 2^{-i}, for $i=1,\dots 6$, and show the corresponding matrices $N_\delta(1)$ in (22). We first consider whether the samples are informative for continuous-time quadratic stabilization. Note that, for each $i\leqslant 3$, the left-upper block of $N_{2^{-i}}(1)$ is greater than 0. This implies that $(0,0)\in\mathcal{Z}(N_{2^{-i}}(1))$, and therefore the data cannot be informative for continuous-time quadratic stabilization. Figure 2 illustrates this, showing the sets of systems consistent with the continuous measurements and with sampled data for $\delta=\frac{1}{2},\frac{1}{8},\frac{1}{16}$, and $\frac{1}{64}$. Using Matlab with YALMIP [18] and MOSEK, we can check the conditions in Theorem III.5 for different values of δ. This yields that the data are informative for continuous-time quadratic stabilization for $\delta=\frac{1}{16}$ and smaller values. As argued above, this does not yet allow us to conclude that the continuous-time measurements are informative for quadratic stabilization of the true system on the basis of sampled data. To illustrate this, recall that, on the basis of the measurements (x,u), we cannot distinguish the true system from any of those in $\mathcal{Z}(N_{\text{cont}}(1))$. In Figure 2, we see that the system $(4.35,-3)$, for example, is compatible with the continuous measurements, but not with any of the sampled data, that is, $(4.35,-3)\notin\mathcal{Z}(N_\delta(1))$. This shows that even if all systems in $\mathcal{Z}(N_\delta(1))$ can be stabilized, this does not imply that the measurements (x,u) are informative for quadratic stabilizability.

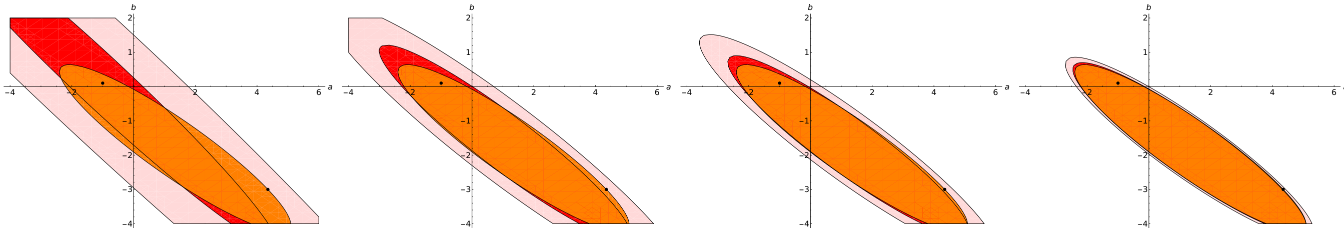

Figure 2: The sets of systems (a, b) that are compatible with the measurements and with sampled data. In orange the set $\mathcal{Z}(N_{\text{cont}}(1))$. In red the set $\mathcal{Z}\left(N_\delta(1)\right)$ for $\delta = \frac{1}{2}, \frac{1}{8}, \frac{1}{16}$, and $\frac{1}{64}$ from left to right. In light red the set $\mathcal{Z}\left(N_\delta(1 + \frac{1}{2}\delta T L)\right)$ for the same values of δ, $T = 1$, and $L = 16$. The black dots denotes the true system $(-1, \frac{1}{10})$ and the (indistinguishable on the basis of the measurements) system $(4.35, -3)$.

$$N_{\frac{1}{2}}(1) \approx \begin{bmatrix} 0.446 & -0.626 & -0.723 \\ -0.626 & -0.709 & -0.823 \\ -0.723 & -0.823 & -1 \end{bmatrix}, \quad N_{\frac{1}{4}}(1) \approx \begin{bmatrix} 0.171 & -0.588 & -0.864 \\ -0.588 & -0.557 & -0.714 \\ -0.864 & -0.714 & -1 \end{bmatrix}, \quad N_{\frac{1}{8}}(1) \approx \begin{bmatrix} 0.0152 & -0.550 & -0.931 \\ -0.550 & -0.487 & -0.656 \\ -0.931 & -0.656 & -1 \end{bmatrix}, \tag{22a}$$

$$N_{\frac{1}{16}}(1) \approx \begin{bmatrix} -0.068 & -0.526 & -0.963 \\ -0.526 & -0.454 & -0.626 \\ -0.963 & -0.626 & -1 \end{bmatrix}, \quad N_{\frac{1}{32}}(1) \approx \begin{bmatrix} -0.111 & -0.514 & -0.979 \\ -0.514 & -0.438 & -0.610 \\ -0.979 & -0.610 & -1 \end{bmatrix}, \quad N_{\frac{1}{64}}(1) \approx \begin{bmatrix} -0.132 & -0.507 & -0.987 \\ -0.507 & -0.430 & -0.603 \\ -0.987 & -0.603 & -1 \end{bmatrix}. \tag{22b}$$

To determine for the stepsizes for which sampled versions of the continuous-time measurements are informative for quadratic stabilization of the true system, we employ the additional knowledge on the noise signal and resort to Theorem IV.8. In this case, the fact that w is L-square Lipschitz with $L = 16$ (alternatively, a more conservative bound for L could be obtained from Lemma IV.10). Note in particular that the set inclusions displayed in Figure 2, where $\mathcal{Z}(N_{\text{cont}}(1))$ is contained in each of the sets $\mathcal{Z}\left(N_\delta(1 + \frac{1}{2}\delta T L)\right)$, are consistent with (15) in Corollary IV.7.

Using Matlab, we verify that the required LMI of Theorem IV.8 is feasible for $\delta = \frac{1}{64}$. This guarantees the existence of a stabilizing feedback K for all systems $(a, b) \in \mathcal{Z}(N_{\text{cont}}(1)) \cap \mathcal{M}^L_{x,u}$ on the basis of sampled data with $\delta = \frac{1}{64}$. This is consistent with the bound for the stepsize obtained in Corollary IV.9, which guarantees samples from the continuous-time signals are informative for $\delta < \frac{1}{16}\hat{\beta} \approx 0.0096 \approx \frac{1}{104}$.

VI. Conclusions

We have studied the informativity problem for continuous-time signals and systems. We first characterized when continuous-time data is informative for continuous-time stabilization and then focused on understanding the informativity of sampled data. After motivating the need for additional assumptions on the noise signal, we have introduced the notions of square Lipschitzness and bounded total square variation. Under these noise models, we have provided sufficient conditions to deduce stabilizability properties of the set of systems compatible with the continuous-time measurements on the basis of sampled data and characterized the role of the sampling stepsize. Future research will include the investigation of necessary conditions, the study of informativity under other noise models, and the generalization of our results to problems beyond stabilization like $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance.

References

[1] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[2] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *International Journal of Control*, vol. 81, no. 12, pp. 1946–1959, 2008.

[3] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," *IEEE Transactions on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2020.

[4] J. Coulson, J. Lygeros, and F. Dörfler, "Data-enabled predictive control: in the shallows of the DeePC," in *European Control Conference*, Naples, Italy, June 2019, pp. 307–312.

[5] G. R. Gonçalves da Silva, A. S. Bazanella, C. Lorenzini, and L. Campestrini, "Data-driven LQR control design," *IEEE Control Systems Letters*, vol. 3, no. 1, pp. 180–185, 2018.

[6] G. Baggio, V. Katewa, and F. Pasqualetti, "Data-driven minimum-energy controls for linear systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 589–594, 2019.

[7] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel, "Data informativity: a new perspective on data-driven analysis and control," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4753–4768, 2020.

[8] H. J. van Waarde, M. K. Camlibel, J. Eising, and H. L. Trentelman, "Quadratic matrix inequalities with applications to data-based control," *arXiv preprint arXiv:2203.12959*, 2022.

[9] M. Bianchi, S. Grammatico, and J. Cortés, "Data-driven stabilization of switched and constrained linear systems," *IEEE Transactions on Automatic Control*, 2022, submitted. Available at https://arxiv.org/abs/2208.11392.

[10] J. Miller and M. Sznaier, "Data-driven gain scheduling control of linear parameter-varying systems using quadratic matrix inequalities," *arXiv preprint arxiv:2209.06251*, 2022.

[11] A. Bisoffi, C. De Persis, and P. Tesi, "Data-based stabilization of unknown bilinear systems with guaranteed basin of attraction," *Systems & Control Letters*, vol. 145, p. 104788, 2020.

[12] J. Berberich, S. Wildhagen, M. Hertneck, and F. Allgöwer, "Data-driven analysis and control of continuous-time systems under aperiodic sampling," in *IFAC Symposium on System Identification*, vol. 54, no. 7, 2021, pp. 210–215.

[13] V. G. Lopez and M. A. Müller, "On a continuous-time version of Willems' lemma," *arXiv preprint arxiv:2203.03702*, 2022.

[14] J. Berberich, C. W. Scherer, and F. Allgöwer, "Combining prior knowledge and data for robust controller design," *IEEE Transactions on Automatic Control*, pp. 1–16, 2022.

[15] H. J. van Waarde, J. Eising, M. K. Camlibel, and H. L. Trentelman, "A behavioral approach to data-driven control with noisy input-output data," *arXiv preprint arXiv:2206.08408*, 2022.

[16] H. J. van Waarde, C. De Persis, M. K. Camlibel, and P. Tesi, "Willems' fundamental lemma for state-space systems and its extension to multiple datasets," *IEEE Control Systems Letters*, vol. 4, no. 3, pp. 602–607, 2020.

[17] C. De Persis and P. Tesi, "Low-complexity learning of linear quadratic regulators from noisy data," *Automatica*, vol. 128, p. 109548, 2021.

[18] J. Lofberg, "YALMIP: a toolbox for modeling and optimization in MATLAB," in *IEEE Int. Conf. on Robotics and Automation*, 2004, pp. 284–289.