Performance-barrier-based Event-triggered Boundary Control of a Class of Reaction-Diffusion PDEs

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***Abstract*— This paper presents a novel event-triggered boundary control technique named performance-barrier-based event-triggered control for a class of reaction-diffusion PDEs under Neumann actuation of a Robin boundary condition. At its core, rather than insisting on a strictly monotonic decrease of the Lyapunov function of the closed-loop system, we allow it to increase as long as it remains within an established performance barrier. This approach integrates a performance residual—the difference between the performance barrier and the Lyapunov function—into the triggering mechanism. This integration provides the system's Lyapunov function with enhanced flexibility, thereby allowing for longer dwell-times compared to "regular" strategies demanding a monotonic decrease of the Lyapunov function. Notably, while adhering to the performance barrier, the closed-loop system globally exponentially converges to zero in the spatial L^2 norm without encountering Zeno phenomenon. We provide numerical simulations to illustrate the proposed technique and to compare it with the regular event-triggered control design, the latter being associated with strictly decreasing Lyapunov functions.**

I. INTRODUCTION

Event-triggered control offers an alternative to standard sampled-data control by updating the control input only after specific events, rather than on a fixed schedule. These events are determined by a specialized triggering mechanism that is influenced by the system's states. This method can be viewed as an advanced form of sampled-data control that adeptly merges feedback into both communication and control update procedures. By harnessing feedback, event-triggered control ensures that control input updates are made only when essential, reducing the frequency of updates while still maintaining satisfactory closed-loop system functionality [10].

At its core, event-triggered control consists of two main components: a feedback control law that guarantees the desired closed-loop functionality, and an event-triggering mechanism that specifies when to update the control input. For this system to operate effectively, it must avoid the phenomenon known as *Zeno behavior*, where an infinite number of control updates happen in a finite time frame. This challenge is typically addressed by carefully crafting the event triggering mechanism, ensuring the existence of a *minimum dwell-time*, *i.e.,* a positive lower bound for the period of time between successive events. Recently, there have been notable advancements in event-triggered control for systems steered by both linear and nonlinear ordinary differential equations [4], [8]–[10], [14], [15], [24], [25]. This progress has ignited interest in event-triggered control methods for systems characterized by partial differential equations (PDEs) [3], [5]–[7], [12], [17], [19]–[23], [26], [27]. Especially pertinent to this discussion are studies [21] and [22] that delve into event-triggered boundary control techniques for a class of reaction-diffusion PDEs, using dynamic event-triggers under anti-collocated and collocated boundary sensing and actuation.

In this study, we introduce an enhanced event-triggered boundary control method termed *performance-barrier-based event-triggered control (P-ETC)* for a class of reaction-diffusion PDEs. The underlying boundary control approach we employ is the infinite-dimensional backstepping boundary control. The proposed novel event-triggered boundary control approach offers significantly longer dwell-times between events in comparison to the recently devised dynamic event-triggered boundary control method for the identical class of reaction-diffusion PDEs [21], [22]. Longer dwell times lead to sparser control updates, which in turn results in saving communication bandwidth, using fewer computational resources for computing control inputs, and reducing actuator wear, among other desirable outcomes. Before delving into the specifics, it is essential to clarify the terminology, particularly the term *performance-barrier* which forms the crux of our novel approach.

The term *performance-barrier* is inspired by the safety-critical control literature [1], [2], [13], [25], although our context does not directly deal with safety. In our study, *performance* refers to the nominal decrease of the Lyapunov function, which serves as a measure of system convergence. On the other hand, *barrier* alludes to a boundary or threshold that the system should ideally not cross. Together, the *performance-barrier* terminology encapsulates the idea of comparing the Lyapunov function's behavior in relation to a predefined nominal decrease, treating it as a boundary that should not be violated.

In [21] and [22], the triggering mechanism imposes a monotonic decrease on the Lyapunov function of the closed-loop system. This is achieved by ensuring the time derivative of the Lyapunov function remains strictly negative. In terms of performance, this guarantees that the Lyapunov function decreases faster than a specific exponentially decaying signal, which incorporates the initial data. We refer to this signal as the *performance-barrier*. We posit that allowing the Lyapunov function to deviate from monotonically decreasing, while adhering to the performance barrier, could

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lead to an elongation in the dwell-times between events. This concept was influenced by the work of Ong *et al.* [16], in which the authors suggest combining derivative-based and function-based event-triggered designs. Their approach integrates both the time derivative and the value of the Lyapunov function into the triggering criterion. We introduce the concept of a *performance residual*, which is the difference between the performance barrier and the Lyapunov function. By incorporating this into the triggering mechanism, we allow for greater flexibility in the behavior of the Lyapunov function. Consequently, the Lyapunov function is not required to decrease monotonically at all times. By design, the performance-barrier-based approach provides a longer dwell-time at any given state compared to the previous *regular* method that forces the Lyapunov function to strictly decrease. Importantly, this is achieved while excluding Zeno behavior from the closed-loop system and still maintaining adherence to the performance barrier which leads to the global exponential convergence of the closed-loop system to zero in L^2-sense. The well-posedness of the closed-loop system is assured for the proposed performance-barrier-based boundary control method.

The rest of the paper is organized as follows. In Section II, we summarize the results of regular event-triggered control (R-ETC). Section III presents performance-barrier-based event-triggered control (P-ETC). A numerical example is provided in Section IV to illustrate the results, and the conclusion is provided in Section V.

Notation: $\mathbb{R}_+$ is the nonnegative real line whereas $\mathbb{N}$ is the set of natural numbers including zero. By $C^0(A;\Omega)$, the class of continuous functions on $A \subseteq \mathbb{R}^n$ is denoted, which takes values in $\Omega \subseteq \mathbb{R}$. By $C^k(A;\Omega)$, where $k \geq 1$, the class of continuous functions on A, which takes values in Ω and has continuous derivatives of order k, is denoted. $L^2(0,1)$ stands for the equivalence class of Lebesgue measurable functions $f:[0,1]\to\mathbb{R}$ such that $\|f\| = \big(\int_0^1 |f(x)|^2\big)^{1/2} < \infty$. Let $u:[0,1]\times\mathbb{R}_+\to\mathbb{R}$ be given. $u[t]$ represents the profile of u at certain $t\geq 0$, *i.e.*, $\big(u[t]\big)(x) = u(x,t)$, for all $x\in[0,1]$. For an interval $J\subseteq\mathbb{R}_+$, the space $C^0\big(J;L^2(0,1)\big)$ is the space of continuous mappings $J \ni t \to u[t] \in L^2(0,1)$. $I_m(\cdot)$ and $J_m(\cdot)$ with m being an integer respectively denote the modified Bessel and (nonmodified) Bessel functions of the first kind.

II. Regular Event-triggered Control (R-ETC)

Let us consider the following 1-D reaction-diffusion PDE under sampled-data boundary control:

$$u_t(x,t) = \varepsilon u_{xx}(x,t) + \lambda u(x,t), \tag{1}$$

$$u_x(0,t) = 0, \tag{2}$$

$$u_x(1,t) + qu(1,t) = U_j^r, \tag{3}$$

where $\varepsilon,\lambda > 0$, for $t\in[t_j^r, t_{j+1}^r)$ with $\{t_j^r\}_{j\in\mathbb{N}}$ being an increasing sequence generated by a suitable event-trigger and the initial condition $u[0]\in L^2(0,1)$. Here, U_j^r is the event-triggered boundary control input held constant for $t\in[t_j^r,t_{j+1}^r), j\in\mathbb{N}$.

The well-posedness of the boundary controlled plant (1)-(3) with piecewise constant inputs in between two sampling instants is presented in the following proposition.

Proposition 1. *For every $u[t_j^r]\in L^2(0,1)$, there exists a unique solution $u:[t_j^r,t_{j+1}^r]\times[0,1]\to\mathbb{R}$ between two time instants t_j^r and t_{j+1}^r such that $u\in C^0\big([t_j^r,t_{j+1}^r];L^2(0,1)\big)\cap C^1\big((t_j^r,t_{j+1}^r)\times[0,1]\big)$ with $u[t]\in C^2([0,1])$ which satisfies (2),(3) for $t\in(t_j^r,t_{j+1}^r]$ and (1) for $t\in(t_j^r,t_{j+1}^r], x\in(0,1)$.*

This result is a straightforward application of Theorem 4.11 in [11].

Assumption 1. *The parameters $q,\lambda,$ and ε satisfy the following relation:*

$$q > \frac{\lambda}{2\varepsilon} + \frac{1}{2}. \tag{4}$$

Remark 1. Under infinite-dimensional backstepping boundary control approach, Assumption 1 is pivotal in ensuring the stability of the target system. This is because we intentionally avoid using the signal $u(1,t)$ in the nominal control law. Such avoidance is crucial for event-triggered control design due to the challenges associated with obtaining a meaningful bound on the rate of change of $u(1,t)$. Furthermore, it is worth mentioning that an eigenfunction expansion of the solution of (1)-(3) with $U(t)=0$ (zero input) shows that the system is unstable when $\lambda > \varepsilon\pi^2/4$, no matter what $q>0$ (see Remark 1 in [21]). □

Consider the backstepping transformation defined as

$$w(x,t) = u(x,t) - \int_0^x K(x,y)u(y,t)dy, \tag{5}$$

where $K(x,y)$ given by

$$K(x,y) = -\frac{\lambda}{\varepsilon}x\frac{I_1\big(\sqrt{\lambda(x^2-y^2)/\varepsilon}\big)}{\sqrt{\lambda(x^2-y^2)/\varepsilon}}, \tag{6}$$

for $0\leq y\leq x\leq 1$. The control input is selected as

$$U_j^r = \int_0^1 k(y)u(y,t_j^r)dy, \tag{7}$$

for all $t\in[t_j^r,t_{j+1}^r), j\in\mathbb{N}$, where

$$k(y) = \wp K(1,y) + K_x(1,y), \tag{8}$$

with

$$\wp = q - \frac{\lambda}{2\varepsilon}. \tag{9}$$

Applying the transformation (5) and the control input (7) to the system (1)-(3) in $t\in[r_j^r,t_{j+1}^r), j\in\mathbb{N}$ results in the following target PDE:

$$w_t(x,t) = \varepsilon w_{xx}(x,t), \tag{10}$$

$$w_x(0,t) = 0, \tag{11}$$

$$w_x(1,t) = -\wp w(1,t) + d(t), \tag{12}$$

where

$$d(t) := \int_0^1 k(y)\big(u(y,t_j^r) - u(y,t)\big)dy, \tag{13}$$

for all $t \in [t_j^r, t_{j+1}^r), j \in \mathbb{N}$. This target PDE is used in the Lyapunov analysis to establish the relevant convergence properties. The inverse transformation of (5) is given by

$$u(x,t) = w(x,t) + \int_0^x L(x,y)w(y,t)dy, \tag{14}$$

where $L(x,y)$ is given by

$$L(x,y) = -\frac{\lambda}{\varepsilon} x \frac{J_1\big(\sqrt{\lambda(x^2-y^2)/\varepsilon}\big)}{\sqrt{\lambda(x^2-y^2)/\varepsilon}}, \tag{15}$$

for $0 \leq y \leq x \leq 1$.

In [21], the authors introduce an observer-based R-ETC method that guarantees the global exponential convergence of the combined system, encompassing both the plant and the observer, towards the equilibrium. In our study, we examine the equivalent full-state feedback scenario. Given that the findings from the observer-based approach are applicable to the full-state feedback scenario with the sole exception of eliminating observer-induced effects, we present the outcomes for the R-ETC in the full-state feedback context as summarized in [18] without delving into proofs.

The R-ETC strategy [21] consists of two components:

1) An event-triggered boundary control input U_j^r (7)-(9) based on the infinite-dimensional backstepping technique,
2) An event-trigger determining event-times

$$t_{j+1}^r = \inf\big\{t \in \mathbb{R}_+ | t > t_j^r, \Gamma^r(t) > 0, j \in \mathbb{N}\big\}, \tag{16}$$

with $t_0^r = 0$ where

$$\Gamma^r\big(d(t), m^r(t)\big) := \Gamma^r(t) = d^2(t) - \gamma m^r(t), \tag{17}$$

and $\gamma > 0$ is an event-trigger design parameter. Here, $m^r(t)$ satisfies the ODE

$$\dot{m}^r(t) = -\eta m^r(t) - \rho d^2(t) + \beta_1 \|u[t]\|^2 + \beta_2 |u(1,t)|^2, \tag{18}$$

for all $t \in (t_j^r, t_{j+1}^r), j \in \mathbb{N}$ with $m^r(t_0) = m^r(0) > 0$, $m^r(t_j^{r-}) = m^r(t_j^r) = m^r(t_j^{r+})$, and $\eta, \rho, \beta_1, \beta_2 > 0$ being event-trigger parameters to be chosen.

Next, we present conditions on the selection of event-trigger parameters $\gamma, \eta, \beta_1, \beta_2, \rho > 0$ that ensure Zeno-free behavior and the global exponential convergence of the closed-loop system (1)-(3),(6)-(9),(16)-(18) to zero in the spatial L^2 norm. The arguments for parameter selection closely follow those in [21], and hence, we state the conditions on parameters in the following assumption without further details as summarized in [18].

Assumption 2 (Event-trigger parameter choice). *The parameters $\gamma, \eta > 0$ are arbitrary design parameters, and $\beta_1, \beta_2 > 0$ are chosen such that*

$$\beta_1 = \frac{\alpha_1}{\gamma(1-\sigma)}, \quad \beta_2 = \frac{\alpha_2}{\gamma(1-\sigma)}, \tag{19}$$

where $\sigma \in (0,1)$ and

$$\alpha_1 = 3\int_0^1 \Big(\varepsilon k''(y) + \varepsilon k(1)k(y) + \lambda k(y)\Big)^2 dy, \tag{20}$$

$$\alpha_2 = 3\big(\varepsilon q k(1) + \varepsilon k'(1)\big)^2, \tag{21}$$

with $k(y)$ given by (8). *Subject to Assumption 1, the parameter $\rho > 0$ is chosen as*

$$\rho = \frac{\varepsilon\kappa B}{2}, \tag{22}$$

for $B, \kappa > 0$ chosen such that

$$B\bigg(\varepsilon \min\Big\{\wp - \frac{1}{2}, \frac{1}{2}\Big\} - \frac{\varepsilon}{2\kappa}\bigg) - 2\beta_1 \tilde{L}^2 - 2\beta_2 - 4\beta_2 \int_0^1 L^2(1,y)dy > 0. \tag{23}$$

Note from Assumption 1 that $\wp > 1/2$, where $\wp$ is given by (9). *In the inequality above,*

$$\tilde{L} = 1 + \Big(\int_0^1\int_0^x L^2(x,y)dydx\Big)^{1/2}, \tag{24}$$

with $L(x,y)$ given by (15). □

We will summarize the main results of [21] in the following theorem.

Theorem 1 ([21]). *Consider the R-ETC approach* (7),(16)-(18) *under Assumption 1, which generates a set of event-times $I^r = \{t_j^r\}_{j\in\mathbb{N}}$ with $t_0^r = 0$. It holds that*

$$\Gamma^r(t) \leq 0 \text{ for all } t \in [0, \sup(I^r)). \tag{25}$$

Consequently, given appropriate choices for the event-trigger parameters γ,η,β_1,β_2, $\rho > 0$, the following results hold:

R1: The set of event-times $\{t_j^r\}_{j\in\mathbb{N}}$ with $t_0^r = 0$ generates an increasing sequence for any $\eta, \gamma, \rho > 0$ and $\beta_1, \beta_2 > 0$ satisfying (19) *in Assumption 2. Specifically, it holds that $t_{j+1}^r - t_j^r \geq \tau > 0$ where*

$$\tau = \frac{1}{a}\ln\Big(1 + \frac{\sigma a}{(1-\sigma)(a+\gamma\rho)}\Big). \tag{26}$$

Here $\sigma \in (0,1)$ appears in the relation (19) *and*

$$a = 1 + \rho_1 + \eta > 0, \tag{27}$$

where

$$\rho_1 = 3\varepsilon^2 k^2(1), \tag{28}$$

with $k(y)$ given by (8). *As $j \to \infty$, it follows that $t_j^r \to \infty$, thereby excluding Zeno behavior.*

R2: For every $u[0] \in L^2(0,1)$, there exists a unique solution $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ such that $u \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(J^r \times [0,1])$ with $u[t] \in C^2([0,1])$ which satisfies (2),(3),(7) *for all $t > 0$ and* (1) *for all $t > 0, x \in (0,1)$, where $J^r = \mathbb{R}_+ \backslash I^r$.*

R3: The dynamic variable $m^r(t)$ governed by (18) *with $m^r(0) > 0$ satisfies $m^r(t) > 0$ for all $t > 0$.*

R4: Consider the Lyapunov candidate given by

$$V^r(t) = \frac{B}{2}\|w[t]\|^2 + m^r(t), \tag{29}$$

where $B > 0$ satisfies (23) *and w is the target system state governed by* (10)-(12). *Under Assumption 2, it holds that*

$$V^r(t) \leq e^{-b^* t} V_0, \tag{30}$$

where $V_0 = V^r(0)$ and

$$b^* = \min\left\{\frac{2b}{B}, \eta\right\} > 0, \tag{31}$$

with $b > 0$ given by

$$b = \frac{\varepsilon B}{4} - \beta_1 \tilde{L}^2 - 2\beta_2 \int_0^1 L^2(1,y)dy. \tag{32}$$

Here, $L(x,y)$ and $\tilde{L}$ are given by (15) *and* (24)*, respectively. Note from* (23) *that $b > 0$.*

R5: Subject to Assumption 2, the closed-loop system (1)-(3),(6)-(9),(16)-(18) *globally exponentially converges to zero in L^2-sense satisfying the following estimate*

$$\|u[t]\| \leq Me^{-\frac{b}{2}^* t}\sqrt{\|u[0]\|^2 + m^r(0)}, \tag{33}$$

where b^* is given by (31) *and*

$$M = \sqrt{\frac{2\tilde{L}^2}{B}\max\left\{\frac{B\tilde{K}^2}{2}, 1\right\}}. \tag{34}$$

Here, $\tilde{K} = 1 + \left(\int_0^1\int_0^x K^2(x,y)dydx\right)^{1/2}$ where $K(x,y)$ is given by (6).

Remark 2. Selecting the event-trigger parameters $\beta_1, \beta_2 > 0$ as per (19) ensures that the set of event-times $\{t_j^r\}_{j\in\mathbb{N}}$ with $t_0^r = 0$, forms an increasing sequence. This remains true irrespective of the values chosen for the parameters $\eta, \gamma, \rho > 0$. However, for the closed-loop system to converge globally and exponentially to zero in L^2-sense satisfying (33),(34),(31),(32), the event-trigger parameter $\rho > 0$ has to be selected as in (22). The parameters $\eta, \gamma > 0$ can be chosen freely. The parameter γ can be chosen to scale the values of β_1, β_2 given by (19). The parameter η characterizes the decay rate of $m^r(t)$ governed by (18). Thus, η may be used to adjust the speed at which events are generated. A smaller η can often result in less frequent events and, consequently, less frequent control updates, as noted [21]. The P-ETC approach, to be introduced in Section III, produces even sparser control updates than the R-ETC for any given set of event-trigger parameters. □

Remark 3. By differentiating (29) in $t \in (t_j^r, t_{j+1}^r), j \in \mathbb{N}$ along the solutions of (10)-(13), and subject to Assumption 1 along with the event-trigger parameters chosen in Assumption 2, we can obtain that

$$\dot{V}^r(t) \leq -b^* V^r(t), \tag{35}$$

for $t \in (t_j^r, t_{j+1}^r), j \in \mathbb{N}$. The relation (35) indicates that the R-ETC forces the Lyapunov function (29) to strictly decrease along the system trajectories. This stringent condition may limit our ability to harness the full potential of event-triggered control for achieving sparse control updates. A more flexible approach might involve a triggering mechanism that permits the Lyapunov function to deviate from a monotonic decrease, yet remain compliant with the R-ETC performance barrier $e^{-b^* t}V_0$ in (30). Such flexibility could potentially result in longer intervals between events, *i.e.,* an increase in dwell-times. In Section III, we introduce a design that embodies this flexible approach. □

III. Performance-barrier-based Event-triggered Control (P-ETC)

In this section, we discuss the design of P-ETC. We introduce a *performance residual*, defined as the difference between the performance barrier $e^{-b^* t}V_0$ and the Lyapunov function, into the triggering mechanism. This inclusion is made with the intention of imparting greater flexibility to the behavior of the Lyapunov function, thereby permitting it to deviate from a monotonic decrease while adhering to the performance barrier.

Let $I^p = \{t_0^p, t_1^p, t_2^p, \ldots\}$ denote the sequence of event-times associated with P-ETC. Let the parameters $\eta, \gamma, \beta_1, \beta_2, \rho > 0$ be selected as outlined in Assumption 2, and let $c > 0$ be a design parameter. The proposed P-ETC strategy consists of two components:

1) An event-triggered boundary control input U_j^p

$$U_j^p = \int_0^1 k(y, t_j^p)u(y, t_j^p)dy, \tag{36}$$

for all $t \in [t_j^p, t_{j+1}^p), j \in \mathbb{N}$. Accordingly, the boundary condition (3) becomes

$$u_x(1,t) + qu(1,t) = U_j^p. \tag{37}$$

2) An event-trigger determining the event-times

$$t_{j+1}^p = \inf\left\{t \in \mathbb{R}_+ | t > t_j^p, \Gamma^p(t) > 0\right\}, \tag{38}$$

with $t_0^p = 0$ and $\Gamma^p(t)$ defined as

$$\Gamma^p(t) := d^2(t) - \gamma m^p(t) - \frac{c}{\rho}\Big(e^{-b^* t}V_0 - V^p(t)\Big). \tag{39}$$

Here, $m^p(t)$ satisfies

$$\begin{aligned}\dot{m}^p(t) = &- \eta m^p(t) - \rho d^2(t) + \beta_1\|u[t]\|^2 + \beta_2 u^2(1,t)\\ &+ c\Big(e^{-b^* t}V_0 - V^p(t)\Big),\end{aligned} \tag{40}$$

where $m^p(t_0^p) = m^r(t_0^r) > 0$, $m^p(t_j^{p-}) = m^p(t_j^p) = m^p(t_j^{p+})$, $b^* > 0$ is given by (31), $d(t)$ is defined as (13) for $t \in [t_j^p, t_{j+1}^p), j \in \mathbb{N}$, and $V^p(t)$ is defined as

$$V^p(t) = \frac{B}{2}\|w[t]\|^2 + m^p(t), \tag{41}$$

with

$$V_0 = V^p(0) = V^r(0), \tag{42}$$

w satisfying the target PDE (10) for $t \in [t_j^p, t_{j+1}^p), j \in \mathbb{N}$, and $B > 0$ chosen to satisfy (23).

Lemma 1. *Under the P-ETC event-trigger* (38)-(42)*, it holds that $d^2(t) \leq \gamma m^p(t) + \frac{c}{\rho}\Big(e^{-b^* t}V_0 - V^p(t)\Big)$, and consequently $m^p(t) > 0$, for $t \in [0, \sup(I^p))$.*

Proof. P-ETC events are triggered to guarantee $\Gamma^p(t) \leq 0$, *i.e.,* $d^2(t) \leq \gamma m^p(t) + \frac{c}{\rho}\Big(e^{-b^* t}V_0 - V^p(t)\Big)$ for $t \in [0, \sup(I^p))$. This inequality in combination with (40) yields:

$$\begin{aligned}\dot{m}^p(t) \geq &- (\eta + \gamma\rho)m^p(t) + \beta_1\|u[t]\|^2 + \beta_2 u^2(1,t)\\ \geq &- (\eta + \gamma\rho)m^p(t),\end{aligned} \tag{43}$$

for $t \in (t_j^p, t_{j+1}^p), j \in \mathbb{N}$. Thus, considering the time-continuity of $m^p(t)$, we can obtain the following estimate:

$$m^p(t) \geq m^p(t_j^p)e^{-(\eta+\gamma\rho)(t-t_j^p)}, \tag{44}$$

for $t \in [t_j^p, t_{j+1}^p], j \in \mathbb{N}$. Recall that we have chosen $m^p(t_0) = m^p(0) > 0$. Therefore, it follows from (44) that $m^p(t) > 0$ for all $t \in [0, t_1^p]$. Again using (44) on $[t_1^p, t_2^p]$, we can show that $m^p(t) > 0$ for all $t \in [t_1^p, t_2^p]$. Applying the same reasoning successively to the future intervals, it can be shown that $m^p(t) > 0$ for $t \in [0, \sup(I^p))$. □

Lemma 2. *Assume that an event has occurred at* $t = t^* \geq 0$ *under P-ETC event-trigger* (38)-(42). *If the next event time* $t = t^p$ *generated by P-ETC event-trigger* (38)-(42) *is finite, then the next event time* $t = t^r$ *generated by R-ETC event-trigger* (16)-(18) *is less than or equal to* t^p, *i.e.,* $t^r \leq t^p$, *provided that* $m^r(t^*) = m^p(t^*) > 0$ *and* $e^{-b^*t}V_0 \geq V^p(t)$ *for all* $t \in [t^*, t^r]$. *The equality holds if* $e^{-b^*t}V_0 = V^p(t)$ *for all* $t \in [t^*, t^r = t^p]$.

Proof. Consider the time period $t \in [t^*, \min\{t^r, t^p\}]$. Then, subtracting (18) from (40) and assuming $e^{-b^*t}V_0 \geq V^p(t)$ for $t \in [t^*, \min\{t^r, t^p\}]$, we can obtain that

$$\begin{aligned}\dot{m}^p - \dot{m}^r &= -\eta\big(m^p(t) - m^r(t)\big) + c\big(e^{-b^*t}V_0 - V^p(t)\big)\\ &\geq -\eta\big(m^p(t) - m^r(t)\big),\end{aligned} \tag{45}$$

where the equality holds if $e^{-b^*t}V_0 = V^p(t)$ for all $t \in [t^*, \min\{t^r, t^p\}]$. Application of the Comparsion Principle on (45) between $t \in [t^*, \min\{t^r, t^p\}]$ leads to

$$m^p(t) - m^r(t) \geq e^{-\eta(t-t^*)}\big(m^p(t^*) - m^r(t^*)\big) = 0, \tag{46}$$

as $m^p(t^*) = m^r(t^*)$, and therefore,

$$m^p(t) \geq m^r(t), t \in [t^*, \min\{t^r, t^p\}], \tag{47}$$

where the equality holds if $e^{-b^*t}V_0 = V^p(t)$ for all $t \in [t^*, \min\{t^r, t^p\}]$.

Assume that $t^r > t^p$. Then, we have from (47) that

$$m^p(t) \geq m^r(t), [t^*, t^p], \tag{48}$$

and from (38),(39) that

$$d^2(t^p) = \gamma m^p(t^p) + \frac{c}{\rho}\big(e^{-b^*t^p}V_0 - V^p(t^p)\big), \tag{49}$$

and from (16),(17) that

$$d^2(t^p) < \gamma m^r(t^p). \tag{50}$$

But (48)-(50) is a contradiction. Thus, $t^r \leq t^p$, with the equality being true if $e^{-b^*t}V_0 = V^p(t)$ for $t \in [t^*, t^r = t^p]$. This completes the proof. □

We use Lemma 1 and 2 to prove the following major result.

Theorem 2. *Consider the P-ETC approach* (36)-(42) *under Assumption 1, which generates a set of event-times* $I^p = \{t_j^p\}_{j\in\mathbb{N}}$ *with* $t_0^p = 0$. *It holds that*

$$\Gamma^p(t) \leq 0 \; for \;\; all \; t \in [0, \sup(I^p)). \tag{51}$$

Consequently, if the event-trigger parameters $\gamma,\eta,\beta_1,\beta_2,\rho > 0$ *are chosen as outlined in Assumption 2 and* $c > 0$, *the following results hold:*

R1: The set of event-times I^p *generates an increasing sequence. Specifically, it holds that* $t_{j+1}^p - t_j^p \geq \tau > 0$ *where* τ *is given by* (26). *Thus* $t_j^p \to \infty$ *as* $j \to \infty$, *excluding Zeno behavior.*

R2: For every $u[0] \in L^2(0,1)$, *there exists a unique solution* $u : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ *such that* $u \in C^0(\mathbb{R}_+; L^2(0,1) \cap C^1(J^p \times [0,1])$ *with* $u[t] \in C^2([0,1])$ *which satisfies* (2),(36),(37) *for all* $t > 0$ *and* (1) *for all* $t > 0, x \in (0,1)$, *where* $J^p = \mathbb{R}_+ \backslash I^p$.

R3: The dynamic variable $m^p(t)$ *governed by* (40)-(42) *with* $m^p(0) = m^r(0) > 0$ *satisfies* $m^p(t) > 0$ *for all* $t > 0$.

R4: The Lyapunov candidate $V^p(t)$ *given by* (41),(42) *satisfies*

$$V^p(t) \leq e^{-b^*t}V_0, \tag{52}$$

for all $t > 0$, *where* $b^* > 0$ *is given by* (31).

R5: The closed-loop system (1),(2),(36)-(42) *globally exponentially converges to zero in* L^2*-sense satisfying the estimate* (33),(34),(31),(32).

Proof. Under P-ETC, recall from Lemma 1 that it holds that $\Gamma^p(t) \leq 0$ and $m^p(t) > 0$ for $t \in [0, \sup(I^p))$. Consider the time period $t \in [0, \sup(I^p))$. If the event-trigger parameters $\eta, \gamma, \beta_1, \beta_2, \rho$ are selected as in Assumption 2, then we can show that $\dot{V}^p$ satisfies

$$\dot{V}^p \leq -b^*V^p + c\Big(e^{-b^*t}V_0 - V^p(t)\Big), \tag{53}$$

for $t \in (t_j^p, t_{j+1}^p)$ where V^p and b^* are given by (41) and (31), respectively. Let

$$W^p(t) := e^{-b^*t}V_0 - V^p(t). \tag{54}$$

Taking the time derivative of $W^p(t)$ and using (53), we can show that

$$\begin{aligned}\dot{W}^p &= -b^*e^{-b^*t}V_0 - \dot{V}^p\\ &\geq -(b^* + c)W^p,\end{aligned} \tag{55}$$

for $t \in (t_j^p, t_{j+1}^p)$. Then, noting that $W^p(t)$ is continuous and $W^p(0) = 0$, we can obtain that

$$\begin{aligned}W^p(t) &\geq e^{-(b^*+c)(t-t_j^p)}W^p(t_j^p)\\ &\geq e^{-(b^*+c))(t-t_j^p)} \times \prod_{i=1}^{i=j} e^{-(b^*+c)(t_i^p - t_{i-1}^p)}W^p(0)\\ &\geq e^{-(b^*+c))t}W^p(0) = 0.\end{aligned} \tag{56}$$

for all $t \in [0, \sup(I^p))$, *i.e.,* $e^{-b^*tV_0} \geq V^p(t)$ for all $t \in [0, \sup(I^p))$. Thus, recalling Lemma 2 and R1 of Theorem 1, we can state that $t_{j+1}^p - t_j^p \geq \tau$ and $t_j^p \to \infty$ as $j \to \infty$ excluding the Zeno behavior. The well-posedness of the closed-loop system (1),(2),(36)-(42) in the sense of R2 in Theorem 2 is a direct consequence of Proposition 1. The solution is constructed iteratively between consecutive triggering times. Since the system is Zeno-free, we can obtain

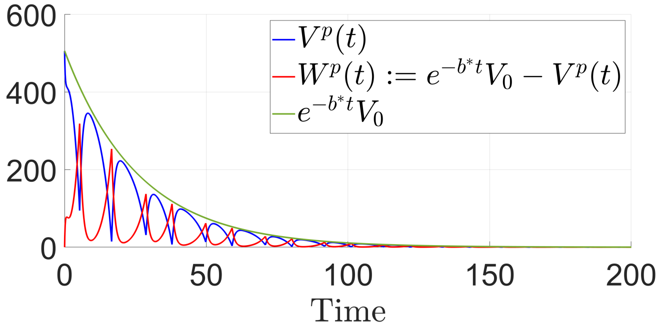

Fig. 1: Evolution of the Lyapunov function $V^p(t)$ and the performance residual $W^p(t)$.

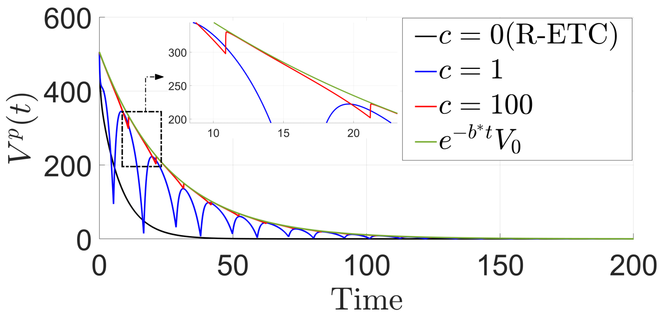

Fig. 2: Behavior of the Lyapunov function $V^p(t)$ under different choices of c.

that $m^p(t)$ governed by (40) with $m^p(0) > 0$ satisfies $m^p(t) > 0$ and $W^p(t) \geq 0$, *i.e.*, $e^{-b^*t}V_0 \geq V^p(t)$ for all $t > 0$. Thus, following classical arguments involving the bounded invertibility of the transformations (5),(6) and (14),(15), we can obtain the global exponential convergence of the closed-loop system (1),(2),(36)-(42) satisfying the decay estimate (33),(34),(31),(32). $\square$

Remark 4. Unlike R-ETC as discussed in Remark 3, which mandates the Lyapunov function to strictly decrease, P-ETC offers more flexibility to the Lyapunov function. As observed from (53), we have $\dot{V}^p(t) \leq -b^*V^p(t) + c\big(e^{-b^*t}V_0 - V^p(t)\big)$. This implies that the time derivative of the Lyapunov function does not have to be negative at all times. Specifically, when $e^{-b^*t}V_0 - V^p(t)$ is large, meaning the Lyapunov function is significantly below the performance barrier, $\dot{V}^p(t)$ can be positive, allowing $V^p(t)$ to increase. Conversely, when $e^{-b^*t}V_0 - V^p(t)$ is getting small, indicating the Lyapunov function is approaching the performance barrier, $\dot{V}^p(t)$ is compelled to decrease, becoming negative. If $e^{-b^*t}V_0 = V^p(t)$, then $\dot{V}^p(t)$ is definitely negative, preventing the Lyapunov function from breaching the performance barrier and ensuring it remains below this threshold. Fig. 1 illustrates the evolution of $V^p(t)$ and the residual $W^p(t) = e^{-b^*t}V_0 - V^p(t)$ in the simulation example considered in Section IV.

Remark 5. The parameter c in the P-ETC event-trigger (38)-(42) plays a pivotal role in shaping the behavior of the Lyapunov function $V^p(t)$ given by (41). For any c such

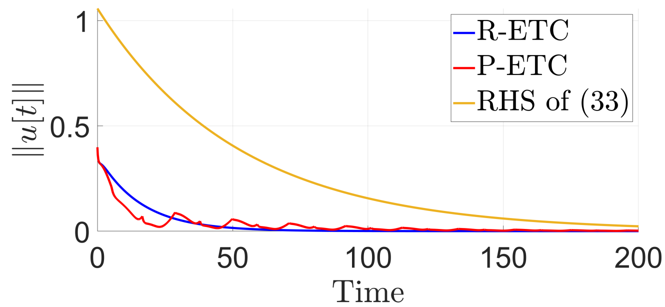

Fig. 3: Evolution of $\|u[t]\|$ under R-ETC and P-ETC.

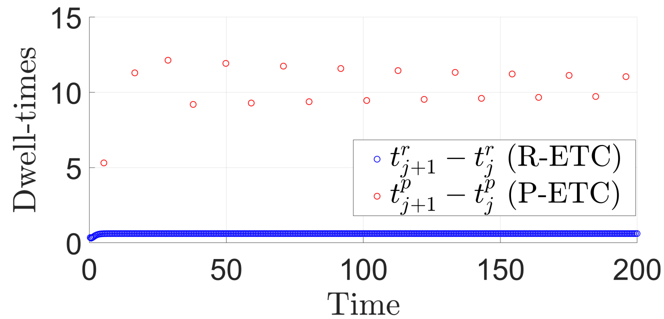

Fig. 4: Dwell-times under R-ETC and P-ETC.

that $0 < c < \infty$, it follows that $\dot{V}^p(t) \leq -b^*V^p(t) + c\big(e^{-b^*t}V_0 - V^p(t)\big)$, for all $t \in (t^p_j, t^p_{j+1}), j \in \mathbb{N}$, and consequently, $V^p(t) \leq e^{-b^*t}V_0$, for all $t > 0$ (see Theorem 2 and its proof). As described earlier, this allows the Lyapunov function to increase during certain periods as long as it remains below the nominal performance-barrier. As $c \to \infty$, the Lyapunov function approaches the performance-barrier, *i.e.*, $V^p(t) \to e^{-b^*t}V_0$. Conversely, setting $c = 0$ results in $\dot{V}^p(t) \leq -b^*V^p(t)$, reducing the P-ETC to the R-ETC. This demands a strict decrease in the Lyapunov function while ensuring $V^p(t) \leq e^{-b^*t}V_0$ for all $t > 0$. Fig. 2 illustrates the behavior of the Lyapunov function for different choices of c in the simulation example considered in Section IV.

IV. Numerical Simulations

We consider a reaction-diffusion system with $\varepsilon = 0.1, \lambda = 0.25, q = 2$, and the initial conditions $u[0] = 10x^2(x-1)^2$. The parameters for the event-triggers are chosen as follows: $m(0) = 10^{-4}, \gamma = 1, \eta = 0.0383, c = 1$, and $\sigma = 0.9$. It can

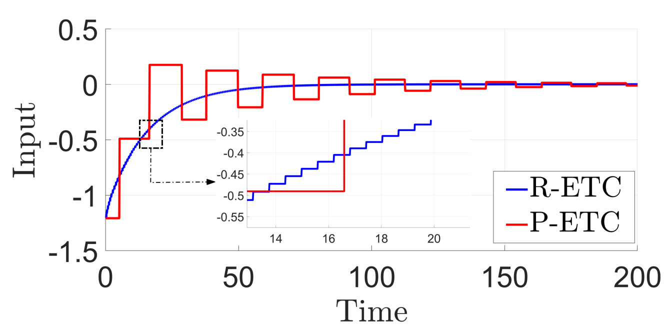

Fig. 5: Control inputs under R-ETC and P-ETC.

be shown using (20),(21) that $\alpha_1 = 0.3466, \alpha_2 = 0.5405$. Therefore, from (19), we can obtain $\beta_1 = 3.4665; \beta_2 = 5.4055$. Let us choose B and κ as $B = 3308.7$ and $\kappa = 5$ so that (23) is satisfied. Then, from (22), we can obtain $\rho = 827.1872$. We use $\Delta t = 0.001$s to time discretize the plant dynamics using the implicit Euler scheme. Note that the event-triggers are discretized at $\Delta t = 0.001s$ as well. Space discretization is done using a step size of $\Delta x = 0.005$.

Fig. 2 illustrates the evolution of the Lyapunov functions under R-ETC and P-ETC. We can observe that $V^r(t)$ under R-ETC monotonically decreases over time whereas $V^p(t)$ is allowed to increase under P-ETC while respecting the performance barrier $e^{-b^* t}V_0$. As evident in Fig. 3, $\|u[t]\|$ under P-ETC takes longer to converge than R-ETC. This is because the Lyapunov function in P-ETC is permitted to deviate from a monotonically decreasing pattern. However, as observed from Fig. 4, the dwell-times of P-ETC are significantly larger than those of R-ETC, resulting in less frequent control updates that compensate for the extended convergence time. The control inputs for both R-ETC and P-ETC are depicted in Fig. 5.

V. Conclusion

This paper has introduced a novel event-triggered boundary control strategy for a class of reaction-diffusion PDEs, termed performance-barrier-based event-triggered boundary control. Unlike *regular* methods that impose a constant decrease in the Lyapunov function, our approach permits deviations, as long as they remain within a defined performance barrier. Central to this innovative design is the concept of the *performance residual*, which represents the gap between the barrier and the Lyapunov function. Incorporating this idea allows for enhanced flexibility in the behavior of the Lyapunov function while ensuring that the prescribed performance is met, which leads to notably longer dwell-times between events compared to *regular* event-triggered control. Our strategy guarantees global exponential convergence of the closed-loop system to zero in the spatial L^2 norm and ensures Zeno-free behavior. We have conducted a numerical simulation that illustrates the effectiveness of the proposed event-triggered boundary control approach.

References

[1] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *2019 18th European control conference (ECC)*. IEEE, 2019, pp. 3420–3431.

[2] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2017.

[3] M. Diagne and I. Karafyllis, "Event-triggered boundary control of a continuum model of highly re-entrant manufacturing systems," *Automatica*, vol. 134, p. 109902, 2021.

[4] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on automatic control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[5] N. Espitia, "Observer-based event-triggered boundary control of a linear 2× 2 hyperbolic systems," *Systems & Control Letters*, vol. 138, p. 104668, 2020.

[6] N. Espitia, A. Girard, N. Marchand, and C. Prieur, "Event-based control of linear hyperbolic systems of conservation laws," *Automatica*, vol. 70, pp. 275–287, 2016.

[7] N. Espitia, I. Karafyllis, and M. Krstic, "Event-triggered boundary control of constant-parameter reaction–diffusion PDEs: a small-gain approach," *Automatica*, vol. 128, p. 109562, 2021.

[8] A. Girard, "Dynamic triggering mechanisms for event-triggered control," *IEEE Transactions on Automatic Control*, vol. 60, no. 7, pp. 1992–1997, 2015.

[9] W. H. Heemels, M. Donkers, and A. R. Teel, "Periodic event-triggered control for linear systems," *IEEE Transactions on automatic control*, vol. 58, no. 4, pp. 847–861, 2013.

[10] W. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *2012 51st IEEE Conference on Decision and Control (CDC)*. IEEE, 2012, pp. 3270–3285.

[11] I. Karafyllis and M. Krstic, *Input-to-state stability for PDEs*. Springer, 2019.

[12] R. Katz, E. Fridman, and A. Selivanov, "Boundary delayed observer-controller design for reaction–diffusion systems," *IEEE Transactions on Automatic Control*, vol. 66, no. 1, pp. 275–282, 2021.

[13] M. Krstic, "Inverse optimal safety filters," *IEEE Transactions on Automatic Control*, vol. 69, no. 1, pp. 16–31, 2024.

[14] T. Liu and Z.-P. Jiang, "A small-gain approach to robust event-triggered control of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2072–2085, 2015.

[15] M. Mazo and P. Tabuada, "On event-triggered and self-triggered control over sensor/actuator networks," in *2008 47th IEEE conference on decision and control*. IEEE, 2008, pp. 435–440.

[16] P. Ong and J. Cortés, "Performance-barrier-based event-triggered control with applications to network systems," *IEEE Transactions on Automatic Control*, 2023.

[17] B. Rathnayake and M. Diagne, "Event-based boundary control of one-phase Stefan problem: A static triggering approach," in *2022 American Control Conference (ACC)*. IEEE, 2022, pp. 2403–2408.

[18] ——, "Periodic event-triggered boundary control of a class of reaction-diffusion PDEs," in *2023 American Control Conference (ACC)*. IEEE, 2023, pp. 1800–1806.

[19] ——, "Self-triggered boundary control of a class of reaction-diffusion PDEs," in *2023 62nd IEEE Conference on Decision and Control (CDC)*. IEEE, 2023, pp. 6887–6892.

[20] ——, "Observer-based event-triggered boundary control of the one-phase Stefan problem," *International Journal of Control*, pp. 1–12, 2024.

[21] B. Rathnayake, M. Diagne, N. Espitia, and I. Karafyllis, "Observer-based event-triggered boundary control of a class of reaction–diffusion PDEs," *IEEE Transactions on Automatic Control*, vol. 67, no. 6, pp. 2905–2917, 2022.

[22] B. Rathnayake, M. Diagne, and I. Karafyllis, "Sampled-data and event-triggered boundary control of a class of reaction–diffusion PDEs with collocated sensing and actuation," *Automatica*, vol. 137, p. 110026, 2022.

[23] T. Strecker, M. Cantoni, and O. M. Aamo, "Event-triggered boundary control of semilinear hyperbolic systems," *IEEE Transactions on Automatic Control*, vol. 69, no. 1, pp. 418–425, 2024.

[24] P. Tallapragada and J. Cortés, "Event-triggered stabilization of linear systems under bounded bit rates," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1575–1589, 2015.

[25] A. J. Taylor, P. Ong, J. Cortés, and A. D. Ames, "Safety-critical event triggered control via input-to-state safe barrier functions," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 749–754, 2020.

[26] J. Wang and M. Krstic, "Event-triggered adaptive control of a parabolic PDE–ODE cascade with piecewise-constant inputs and identification," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5493–5508, 2023.

[27] ——, "Event-triggered adaptive control of coupled hyperbolic PDEs with piecewise-constant inputs and identification," *IEEE Transactions on Automatic Control*, vol. 68, no. 3, pp. 1568–1583, 2023.