Information Structures for Distributed k-dimensional Agreement ⋆

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Abstract

Given a network of agents, the agents are said to reach a k-dimensional agreement when the state variables agree within a k-dimensional linear subspace. This problem is a generalization of the well-studied average consensus problem, where the asymptotic states of the agents are not required to coincide, but rather to agree in a generalized sense. In this paper, we investigate what structural properties of the interaction graph are required to enable the agents to reach a k-dimensional agreement. We find that agreement protocols impose the use of communication graphs with a high network connectivity; more precisely, we show that the number of edges in the graph must grow linearly with the size of the agreement space k. We study under what conditions common graph topologies – such as line and circulant graphs – can sustain agreement protocols, and provide insights into the relationship between network connectivity and the space dimension k. Our characterization identifies the presence of cycles (precisely, of independent cycle families) in the network as a basic structural property that enables agents to reach an agreement. The applicability of the framework is illustrated via simulations on problems in robotic formation.

Key words: Multi-agent systems, Decentralized and distributed control, Networked robotic systems, Cooperative systems

1 Introduction

Distributed coordination algorithms play a fundamental role in several network synchronization problems, including rendezvous, distributed optimization, distributed computation and sensing, federated learning, and much more. A common objective in network coordination problems is that of making a group of agents agree on a common quantity. This problem is often referred to as consensus [28] and a vast body of literature has been developed on it – see, just as an example, the representative works [6, 28, 31]. In other cases, it is instead of interest to make the agents agree in a generalized sense: rather than on a common quantity, one may be interested in ensuring that the agents' states converge to a vector that belongs to a certain set (or vector space). When the agreement set is a linear subspace and the protocol used is linear, the problem is referred to as *k-dimensional agreement*. We recently proposed this problem in [5], where we provided algebraic characterizations of the matrices defining the agreement protocol and studied their design to optimize the rate of convergence. One outcome of our analysis in [5] is that k-agreement protocols, in general, require interaction graphs with higher connectivity as compared to those used for simpler coordination algorithms (such as average consensus [28]). In this paper, we seek to provide answers to the following question: what topological properties of the interaction graph ensure that a set of agents can reach an agreement? Our findings in this paper extend [5] in several directions: (i) we derive necessary conditions on the topology of the communication graph to enable an agreement; (ii) we show that agreement is possible when the interaction topology incorporates a sufficient number of independent cycles, and (iii) we provide insights into the design of graphs that support agreement protocols.

An important application of k-dimensional agreement problems is robotic formation control [9, 26], where achieving a certain configuration for the team amounts to ensuring that the joint state belongs to a certain set. In this work, we explore this application and we illustrate how k-dimensional agreement provides a natural framework to specify constraints to be satisfied by the team of robots at convergence.

Related work. Agreement problems are closely related to distributed consensus; consensus algorithms have been extensively studied in the literature. A (necessarily incomplete) list of studied topics includes: sufficient and necessary conditions to reach a consensus [10, 18, 19, 28, 30, 37], time delays [10], consensus with linear objective maps [13],

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the use of the alternating direction method of multipliers (ADMM) [7, 16, 34], dealing with quantized measurements [20], convergence rates [29, 39], robustness investigations [15, 21], among many others. Particularly related to the problem of k-dimensional agreement is that of constrained consensus [24,25] and distributed optimization with global constrains [38]. Differently from these approaches, agreement problems are characterized by constraints that apply not only during transients, but also asymptotically; also, agreement problems can be seen as constrained optimization problems with a non-separable cost function (see Section 3.2). In Pareto optimal distributed optimization [12], the group of agents cooperatively seeks to determine the minimizer of a cost function that depends on agent-dependent decision variables. Clustering-based consensus [1,4,23] is a closely-related problem where the states of agents in the same cluster are related and states of agents in different clusters are independent. Instead, in agreement problems, the state of each agent is dependent on every other agent in the network. Scaled consensus [32], is a special case of agreement to a subspace of dimension $k = 1$. Interestingly, strong connectivity of the interaction graph is necessary and sufficient for scaled consensus; in contrast, in this paper, we show that strong connectivity is no longer sufficient when the dimension of the agreement space is $k \geq 2$. Finally, while our previous work [5] investigates numerical algorithms to design agreement protocols, in this work we pose a fundamental question: under what structural conditions on the interaction topology it is possible to ensure the existence of agreement protocols?

Contributions. The contribution of this work is fourfold. *(c1)* We provide a structural necessary condition for a certain graph to admit an agreement protocol. We apply this condition to study agreement protocols on basic graphs, such as line and circulant topologies. By drawing insights from our theorems, we show how these graphs can be modified to support agreement on high-dimensional subspaces. *(c2)* We provide a graph-theoretic sufficient condition that ensures that a graph admits an agreement protocol on arbitrary weights. Our analysis shows that agreement is made possible, graph-theoretically, by the presence of cycle families in the communication graph. *(c3)* We show how agreement algorithms can be adapted to account for cases where the local estimates are time-varying; in this case, we prove convergence of agreement algorithms and an input-to-state stability-type bound. *(c4)* We study the applicability of agreement protocols in robotic formation problems, and use these algorithms to constrain the asymptotic configuration of a team of robots.

Organization. Section 2 provides some terminology and basic results used throughout. Section 3 presents the problem of interest; Section 4 provides our graph-theoretic conditions for agreement that represent the main results of this paper. Section 5 extends the approach to tracking problems and Section 6 illustrates the techniques via numerical simulations. Conclusions are discussed in Section 7.

2 Preliminaries

In this section, we formalize basic notions used in the paper.

Notation. $\mathbb{N}_{>0} = \{1, 2, \dots\}$ denotes the set of positive natural numbers. For $x \in \mathbb{C}$, $\Re(x)$ and $\Im(x)$ denote, respectively, its real and imaginary parts. When $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^{n+m}$ denotes their concatenation. $\mathbb{1}_n \in \mathbb{R}^n$ is the vector of all ones; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; $\mathbb{0}_{n,m} \in \mathbb{R}^{n\times m}$ is the matrix of all zeros; for these matrices, subscripts are dropped when dimensions are clear from the context. Given $A \in \mathbb{R}^{n\times n}$, we often use the notation $A = [a_{ij}]$ to denote that a_{ij} is the element in row i and column j of A. $\sigma(A) = \{\lambda \in \mathbb{C} \ : \ \det(\lambda I - A) = 0\}$ is the spectrum, $\lambda_{\max}(A) = \max\{\Re(\lambda) \ : \ \lambda \in \sigma(A)\}$ is the spectral abscissa. When $A \in \mathbb{R}^{n\times m}$ is seen as a linear map, $\mathrm{Im}(A)$ denotes its image and $\ker(A)$ its null space. Given $p_1, \dots, p_n \in \mathbb{R}$, the polynomial $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_n$ is *stable* if all its roots have negative real part.

Algebraic graph-theoretic notions. A directed graph (or digraph) is $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $(i, j) \in \mathcal{E}$ denotes an edge from $j \in \mathcal{V}$ to $i \in \mathcal{V}$. We will leverage an equivalence class between matrices and digraphs as follows: given $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, there is a one-to-one correspondence between A and a weighted digraph that has n nodes $\mathcal{V} = \{1, \dots, n\}$ and a directed edge from j to i with edge weight a_{ij} if the matrix element a_{ij} is nonzero. Conversely, given $\mathcal{G}$, we say that a $A \in \mathbb{R}^{n\times n}$ is *consistent* with $\mathcal{G}$ if $(i, j) \notin \mathcal{E}$ implies $a_{ij} = 0$. For $a \in \mathbb{R}^{|\mathcal{E}|}$, $\mathbf{A}_{\mathcal{G}}(a)$ denotes the $\mathbb{R}^{n\times n}$ matrix consistent with $\mathcal{G}$ and parametrized by $a = (\{a_{ij}\}_{(i,j)\in\mathcal{E}})$. For example, for the graph in Fig. 1, $a = (a_{11}, a_{13}, a_{21}, a_{23}, a_{32}, a_{34}, a_{42})$ and

$$\mathbf{A}_{\mathcal{G}}(a) = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & 0 \end{bmatrix}.$$

A *path* in $\mathcal{G}$ is a sequence of edges $(e_1, e_2, \dots)$ such that the origin node of each edge is the destination node of the preceding edge. A graph is *strongly connected* if, for any $i, j \in \mathcal{V}$, there is a path from i to j. A graph is *complete* if there exists an edge connecting every pair of nodes, and it is *sparse* otherwise. A *closed path* is a path whose initial and final vertices coincide. A closed path is a *cycle* if, going along the path, one reaches no node, other than the initial-final node, more than once. A cycle of length equal to one is a *self cycle.* A set of cycles that have no nodes in common is a *cycle family.* With a slight abuse of notation, we will denote a cycle family by $f = \{e_1, e_2, \dots\}$, where $e_1, e_2, \dots \in \mathcal{E}$ are the edges involved in f. The *length of a cycle family* is the number of edges involved in all cycles (equivalently, the number of edges in $\{e_1, e_2, \dots\}$). The *weight of a cycle family* is given by the product of the weights of all edges in the cycle family (namely, $\prod_{(i,j)\in f} a_{ij}$). See Fig. 1 for an illustration.

Projections and linear subspaces. Given a linear subspace $\mathcal{M} \subset \mathbb{R}^n$, its *orthogonal complement* is $\mathcal{M}^\perp := \{x \in \mathbb{R}^n \ : \ x^\mathsf{T} y = 0, \forall\, y \in \mathcal{M}\}$. Given two subspaces $\mathcal{M}, \mathcal{N} \subseteq \mathbb{R}^n$, $\mathcal{M} \cap \mathcal{N} = \{0\}$, their *direct sum* is $\mathcal{W} := \{u + v \ : \ u \in \mathcal{M}, v \in \mathcal{N}\}$ and denoted by $\mathcal{W} = \mathcal{M} \oplus \mathcal{N}$; $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$ are *complementary* if $\mathcal{M} \oplus \mathcal{N} = \mathbb{R}^n$. Given complementary subspaces $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$, for any $z \in \mathbb{R}^n$, there exists a unique decomposition $z = x + y$, where $x \in \mathcal{M}$ and $y \in \mathcal{N}$. The transformation $\Pi_{\mathcal{M},\mathcal{N}}$, defined by $\Pi_{\mathcal{M},\mathcal{N}} z := x$, is called *projection onto $\mathcal{M}$ along $\mathcal{N}$*, and the transformation $\Pi_{\mathcal{N},\mathcal{M}}$ defined by $\Pi_{\mathcal{N},\mathcal{M}} z := y$ is called *projection onto $\mathcal{N}$ along $\mathcal{M}$*. Vector x is the projection of z onto $\mathcal{M}$ along $\mathcal{N}$, and y is the

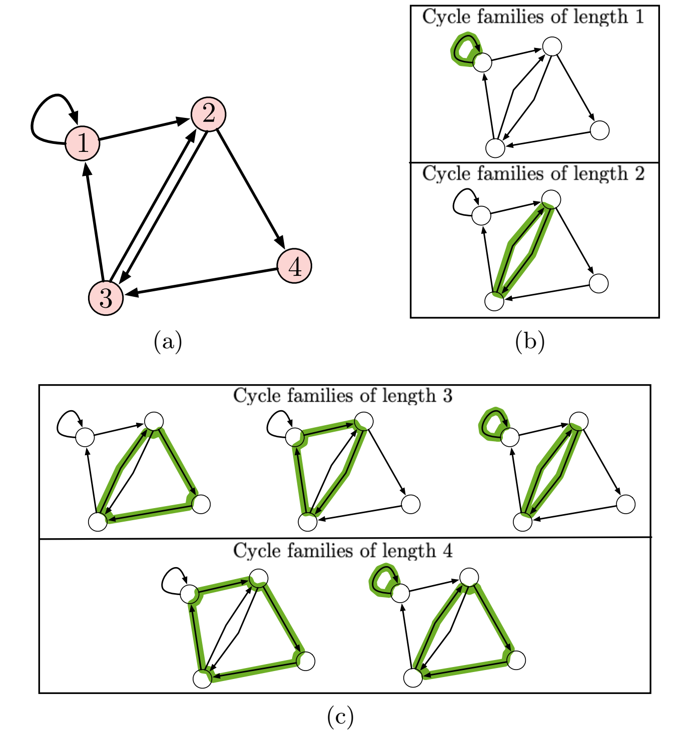


Fig. 1. (a) Illustration of a digraph and (b)-(c) associated ℓ-long cycle families, $\ell \in \{1, \dots, 4\}$ (a Hamiltonial ℓ-decomposition is a set of node-disjoint cycles such that the sum of the cycle lengths is equal to ℓ).

projection of z onto $\mathcal{N}$ along $\mathcal{M}$. A matrix $\Pi \in \mathbb{R}^{n\times n}$ is a projection onto some subspace if and only if $\Pi^2 = \Pi$. The projection $\Pi_{\mathcal{M},\mathcal{M}^\perp}$ onto $\mathcal{M}$ along $\mathcal{M}^\perp$ is called *orthogonal projection onto* $\mathcal{M}$. Because the subspace $\mathcal{M}$ uniquely determines $\mathcal{M}^\perp$, we will denote in compact form $\Pi_{\mathcal{M},\mathcal{M}^\perp}$ by $\Pi_{\mathcal{M}}$. Projections that are not orthogonal are called *oblique projections*.

Lemma 2.1 ***[17, Thm. 2.11 and Thm. 2.31]*** *Let $\Pi \in \mathbb{R}^{n\times n}$ be a projection with* $\operatorname{rank}(\Pi) = k$. *There exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$\Pi = T \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Moreover, if Π is an orthogonal projection, then T can be chosen to be an orthogonal matrix, i.e., $TT^\top = I$. □

Lemma 2.2 ***[17, Thm. 2.26]*** *Let $\mathcal{M}, \mathcal{N}$ be complementary subspaces and the columns of $M \in \mathbb{R}^{n\times k}$ and $N \in \mathbb{R}^{n\times k}$ form a basis for $\mathcal{M}$ and $\mathcal{N}^\perp$, respectively. Then,*

$$\Pi_{\mathcal{M},\mathcal{N}} = M(N^\top M)^{-1}N^\top. \qquad \square$$

We recall the following known properties [17, Thm. 1.60]:

$$\operatorname{Im}(M^\top) = \operatorname{Im}(M^\dagger) = \operatorname{Im}(M^\dagger M) = \operatorname{Im}(M^\top M),$$
$$\ker(M) = \operatorname{Im}(M^\top)^\perp = \ker(M^\dagger M) = \operatorname{Im}(I - M^\dagger M).$$

From these properties and Lemma 2.2, given $M \in \mathbb{R}^{m\times n}$, we have

$$\Pi_{\operatorname{Im}(M)} = MM^\dagger, \qquad \Pi_{\ker(M)} = I - M^\dagger M,$$

where $M^\dagger \in \mathbb{R}^{n\times m}$ is the Moore-Penrose inverse of M.

3 Problem setting

In this section, we formalize the problem of interest and motivate its applicability in multi-agent robotics.

3.1 Problem formulation

Consider a set of agents $\mathcal{V} = \{1, \dots, n\}$, each characterized by a scalar state $x_i \in \mathbb{R}, i \in \mathcal{V}$. The agents interact with each other to update their states, as described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Precisely, at every time, each agent $i \in \mathcal{V}$ exchanges its state with its neighbors and updates it as follows:

$$\dot{x}_i = a_{ii}x_i + \sum_{j\in\mathcal{N}_i} a_{ij}x_j, \tag{1}$$

where $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ is the set of in-neighbors of i, and $a_{ij} \in \mathbb{R}$ are parameters describing the magnitude of the couplings. By setting $A = [a_{ij}]$, where $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, and $x = (x_1, \dots, x_n)$, the network dynamics are:

$$\dot{x} = Ax. \tag{2}$$

We say that the network reaches a k-dimensional agreement if the state of the agents converge to k independent weighted sums of the initial conditions.

Definition 3.1 ***(k-dimensional agreement)*** *Let $n, k \in \mathbb{N}_{>0}$, and $W \in \mathbb{R}^{n\times n}$, with* $\operatorname{rank}(W) = k$. *We say that* (2) *globally asymptotically reaches a k-dimensional agreement on W if, for any $x(0) \in \mathbb{R}^n$,*

$$\lim_{t\to\infty} x(t) = Wx(0). \tag{3}$$

When this holds, A is called a k-dimensional agreement algorithm (or protocol). □

Definition 3.1 formalizes a notion of agreement between the agents whereby, at convergence, the network's state is constrained to a k-dimensional space (precisely, the space $\operatorname{Im}(W)$). As observed in [5, Rem. 3.2], agreement is a generalization of the classical average consensus problem [28, 32]. Indeed, consensus amounts to $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t)$, $\forall i, j \in \mathcal{V}$, which is achieved when $k = 1$, $W = vw^\top$, and $v = \mathbb{1}$; hence, average consensus is a special case of agreement.

According to [5, Prop. 4.2], linear protocols of the form (2) can agree only on matrices W that are oblique projections. Based on this conclusion, we make the following assumption.

Assumption 1 ***(Matrix of weights is a projection)*** *The matrix W in* (3) *satisfies $W^2 = W$ and* $\operatorname{rank}(W) = k$. □

Accordingly, agreement protocols shall be utilized in distributed computation tasks where it is of interest to compute some oblique projection of the network's initial conditions. See Fig. 2 for an illustration.

Remark 3.2 ***(Importance of projections in the applications)*** Orthogonal and oblique projections emerge naturally in many practical engineering problems [2]: orthogonal projections are used to compute solution to regression problems (e.g., see Section 7); oblique projections solve weighted or constrained least-squares regression problems [2, 11], and are widely used in signal processing [3], subspace identification [14], and more [2]. □

As observed in [5, Ex. 3.4], whether a group of agents can reach an agreement depends on the choice of W and on the topology $\mathcal{G}$ of the underlying interaction graph. In this work, we are interested in addressing the following question: given an arbitrary, pre-specified, matrix of weights W (as in Assumption 1), what are the graph topologies $\mathcal{G}$ that admit an algorithm (2) that reaches an agreement on W? This question inspires the following definition.

Definition 3.3 ***(Agreement on arbitrary weights)*** *Let $k \in \mathbb{N}_{>0}$ and $\mathcal{G}$ be fixed. The set of agents is said to be globally k-agreement reachable on arbitrary weights if, for any $W \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(W) = k$, there exists A such that (2) globally asymptotically reaches a k-dimensional agreement on W.* □

A necessary condition for a set of agents to be agreement reachable on arbitrary weights is that the underlying interaction graph $\mathcal{G}$ is strongly connected (see [5, Lem. 4.5]). Hence, we impose the following.

Assumption 2 ***(Strong connectivity)*** *The communication digraph $\mathcal{G}$ is strongly connected.* □

Intuitively, because W is an arbitrary matrix, each entry of the vector $\lim_{t\to\infty} x(t)$ in general depends on the *entire* vector of initial conditions $x(0)$ (cf. (3)). As a result, each entry of the vector $x(0)$ must be able to propagate to the entire graph, hence requiring strong connectivity. Notice, however, that strong connectivity is not sufficient for achieving agreement, as the following example shows.

Example 3.4 ***(Strong connectivity does not imply agreement reachability)*** Assume that a network of $n = 3$ agents is interested in agreeing on a space with $k = 2$ by using a non-complete communication graph $\mathcal{G}$. By using [5, Lem. 4.1], the agents are agreement reachable on arbitrary weights only if:

$$A = \underbrace{\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}}_{=T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \underbrace{\begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \end{bmatrix}^{\mathsf{T}}}_{=T^{-1}} = \beta t_3 \tau_3^{\mathsf{T}}, \qquad (4)$$

for some β such that $\Re(\beta) < 0$ and some $T \in \mathbb{R}^{3\times 3}$. By (4), A must be a rank-one matrix and, since $\mathcal{G}$ is not complete, at least one of the entries of A must be identically zero. These two properties imply that at least one of the rows or columns of A must be identically zero, and thus that $\mathcal{G}$ cannot be strongly connected. Since $\mathcal{G}$ is not strongly connected, by [8, Cor 4.5] at least one of the rows or columns of $W = \lim_{t\to\infty} e^{At}$ must be identically zero. In summary, the agents are globally 2-agreement reachable on arbitrary weights only if $\mathcal{G}$ is the complete graph. □

In this work, we seek to characterize the structural properties of interaction graphs support agreement protocols on arbitrary weights. Formally, we study the following problem.

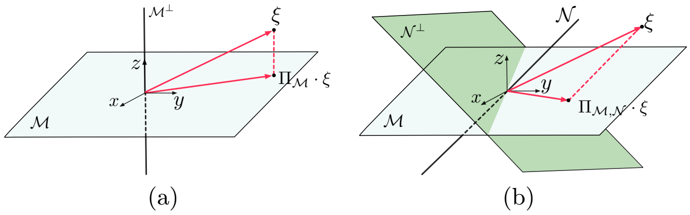


Fig. 2. (a) Geometric interpretation of orthogonal projections: a vector $\xi \in \mathbb{R}^3$ is projected onto $\mathcal{M} \subset \mathbb{R}^3$. (b) Geometric interpretation of oblique projections: $\mathcal{M}$ and $\mathcal{N} \in \mathbb{R}^3$ are complementary subspaces and ξ is projected on $\mathcal{M}$ along $\mathcal{N}$. Notice that the projection ray belongs to $\operatorname{span}(\mathcal{N})$.

Problem 1 ***(Characterization of the class of communication graphs that enable agreement)*** *Determine a class of communication graphs such that, for any graph in the class, the group of agents is globally k-agreement reachable on arbitrary weights.* □

The answer to Problem 1 is fully known in the case of consensus problems: a set of agents can reach a consensus on some weights if and only if the interaction graph admits a spanning tree [30] and a consensus on arbitrary weights if and only if the graph is strongly connected [27, 32]. By contrast, the question stated here of what topologies are agreement reachable on arbitrary weights is novel and has not been addressed in the literature.

3.2 Motivating application: mobile robotic formation

To illustrate the importance of designing agreement algorithms, we next demonstrate how this problem provides a natural solution to enforce a desired configuration in multi-agent mobile robotics. Consider a group of $n = 4$ robots modeled using single-integrator dynamics. Let $x(0) \in \mathbb{R}^4$ denote the **x**-coordinates of the robots' positions at time 0 (we refer to Section 6 for a generalization to the two-dimensional case), and assume that the group is interested in achieving a final formation x^* such that $x_1^* = x_2^*, x_3^* = x_4^*$ and that the control energy used to reach such a formation is minimized. Formally, the desired configuration is given by the solution to:

$$\begin{aligned} x^* = \arg\min_{x\in\mathbb{R}^4} \quad & \|x(0) - x\|_R^2 \\ \text{subject to:} \quad & x_1 = x_2, \ x_3 = x_4. \end{aligned} \qquad (5)$$

where $\|\cdot\|_R^2$ denotes the square weighted norm defined by $\mathbb{R}^{4\times 4} \ni R \succ 0$. Further, because each robot has no knowledge of global coordinates, this must be achieved through a distributed coordination algorithm. By letting

$$D = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

the formation requirements can be encapsulated by the constraint $Dx = 0$, and the solution to (5) with $R = I$ is given by $x^\star = \Pi_{\ker(D)} x_0$. It is now immediate to see that x^* can be computed using an agreement algorithm, with $W = \Pi_{\ker(D)}$.

4 Structural necessary and sufficient conditions for agreement

In this section, we provide necessary and a sufficient condition for agreement reachability on arbitrary weights. To present our results, it will be useful to interpret the agreement protocol A in (2) as a matrix to be designed, parametrized on the vector of edge weights $a \in \mathbb{R}^{|\mathcal{E}|}$, denoted by $\mathbf{A}_{\mathcal{G}}(a)$. See Section 2 for a presentation of the notation.

Theorem 4.1 ***(Graph-theoretic necessary conditions)*** *Consider the protocol $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x$ and let Assumptions 1-2 hold. The set of agents is globally k-agreement reachable on arbitrary weights only if*

$$|\mathcal{E}| \geq kn. \tag{6}$$

□

PROOF. By [5, Lem. 2.1], W admits the decomposition:

$$W = T \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T^{-1}. \tag{7}$$

Let $T = \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix}, t_1, \ldots, t_n \in \mathbb{R}^n$; it follows from [5, Thm. 5.3] that $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x$ reaches an agreement if and only if the following set of equations admits a solution a:

$$0 = \mathbf{A}_{\mathcal{G}}(a)t_i, \qquad i \in \{1, \ldots, k\}, \tag{8a}$$

$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(c)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \ldots, n-k\}. \tag{8b}$$

The system of equations (8) to be solved consists of nk linearly independent linear equations and $n-k$ nonlinear equations with $|\mathcal{E}|$ unknowns and $n-k$ arbitrarily chosen real numbers $p_1, \ldots, p_{n-k}$. Due to the invertibility of matrix T, the equations (8a) are linearly independent and thus generic solvability of (8) requires the following necessary condition: $|a| = |\mathcal{E}| \geq nk$, from which the claim follows. Ξ

The inequality (6) provides two important types of bounds on the structural properties of graphs that can sustain an agreement protocol. First, for given n and k, (6) gives a lower bound on the minimal graph connectivity required for agreement. When k increases, (6) states that the number of edges in $\mathcal{G}$ must grow at least linearly with k. Second, for a given network topology $\mathcal{G}$ (and hence given n and $\mathcal{E}$), (6) gives an upper bound on the dimension of the allowable agreement space: $k \leq |\mathcal{E}|/n$. These bounds can be used to derive useful insights into the relationship between agreement spaces and graph topologies, as illustrated in the following examples.

Example 4.2 ***(Necessary conditions for agreement using circulant digraphs)*** Consider the *one-directional* circulant topology of Fig. 3(a). In this case, $|\mathcal{E}| = 2n$ and thus (6) gives $k \leq 2$. Next, consider the *bi-directional* circulant topology of Fig. 3(b). Here, $|\mathcal{E}| = 3n$, and (6) gives $k \leq 3$. In words, the one-directional circulant digraph can support agreement on subspaces of dimension at most 2, and the bi-directional circulant digraph on subspaces of dimension at most 3.

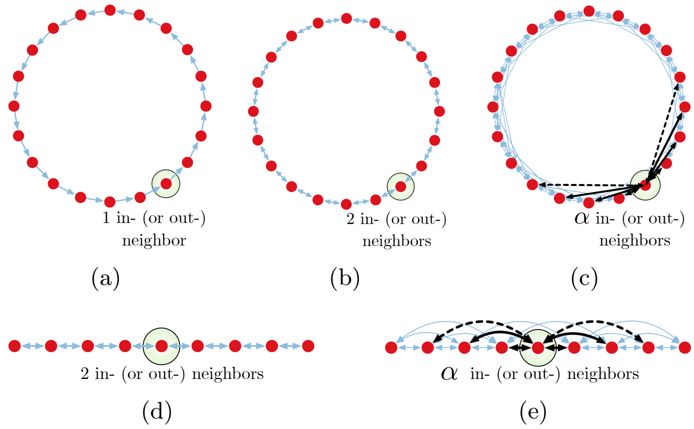


Fig. 3. (a) One-directional circulant topology; (b)–(c) bi-directional circulant topology; (d)-(e) bi-directional line topology. The graph in (a) is the least-connected graph topology that can reach a 1-dimensional agreement on arbitrary weights (see Remark 4.4). (b) and (d) also admit agreement protocols on arbitrary weights within subspaces of dimension at most $k = 1$ (see Examples 4.2, 4.3). For (c) and (d), the connectivity of each node α must scale proportionally with k. (see examples 4.2 and 4.3). In all plots, all nodes have self-cycles, which are omitted here for illustration purposes. Dashed lines illustrate the trend of edge increase as a function of α.

Generalizing this idea, consider a bi-directional circulant digraph where each agent communicates with $\alpha \in \mathbb{N}_{>0}$ nearest neighbors (see Fig. 3(c)). Using (6) with $|\mathcal{E}| = n(\alpha + 1)$ gives $\alpha \geq k - 1$; in words, to support an agreement protocol on a k-dimensional space, each agent must communicate with at least $k - 1$ independent neighbors. □

Example 4.3 ***(Necessary conditions for agreement using line digraphs)*** Consider the *bi-directional* line topology of Fig. 3(d). In this case, $|\mathcal{E}| = n + 2(n-1)$ and (6) yields $k \leq \left\lfloor \frac{3n-2}{n} \right\rfloor \leq 3$. Similarly to the bi-directional circulant digraph, the bi-directional line topology can support agreement protocols on subspaces of dimension at most 3.

Generalizing, consider bi-directional line digraphs where each agent communicates with $\alpha \in \mathbb{N}_{>0}$ nearest neighbors (see Fig. 3(e)). Using (6) with $|\mathcal{E}| = n + \alpha n - \frac{\alpha}{2}(\frac{\alpha}{2} + 1)$, gives $\alpha \geq 2k - 1$. By comparison, we conclude that agreement protocols on line topologies (requiring $\alpha \geq 2k - 1$) need a higher connectivity in comparison to agreement protocols on circulant digraphs (requiring $\alpha \geq k - 1$). The reason being that the cardinality of the edge sets of line topologies is smaller than that of circulant topologies. □

Finally, we discuss the relationship between (6) and established conditions for consensus in the following remark.

Remark 4.4 ***(Strong connectivity implies*** (6) ***when*** $k = 1$***)*** By interpreting the consensus problem as a special case of agreement, it is easy to relate condition (6) with established necessary conditions for consensus. Recall that consensus on arbitrary weights can be reached if and only if the underlying interaction graph is strongly connected [27, 32]. It is immediate to see that the graph with the least number of edges that contains self-cycles and is strongly connected is the circulant digraph (see Fig. 3(a)). This graph has $|\mathcal{E}| = 2n$. In this case, and assuming $k = 1$ (consensus), (6) reads $2n \geq 1 \cdot n$, which holds true for any n. We have thus found that condition (6) is automatically satisfied for any digraph that can sustain an agreement algorithm. □

The following result provides structural sufficient conditions for agreement.

Theorem 4.5 ***(Graph-theoretic sufficient conditions)*** *Consider the protocol $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x$ and let Assumptions 1-2 hold and $|\mathcal{E}| \geq nk + n - k$. If there exists a partitioning of the edge parameters $a = (\{a_{ij}\}_{(i,j)\in\mathcal{E}})$ into two disjoint sets:*

$$a_v = \{a_1, \dots, a_{n-k}\} \quad \text{and} \quad a_c = \{a_{n+1}, \dots, a_{|\mathcal{E}|}\},$$

such that:

(i) For all $\ell \in \{1, \dots, n-k\}$, there exists an ℓ-long cycle family $\mathcal{C}_\ell^*$, such that $a_\ell \in \mathcal{C}_\ell^*$;
(ii) Any edge in $\mathcal{C}_\ell^*$ other than a_ℓ belongs to a_c,
(iii) Any ℓ-long cycle family other than $\mathcal{C}_\ell^*$ that contains edges in a_v also contains at least one edge in a_c that does not appear in C_ℓ^*,

then the set of agents is globally k-agreement reachable on arbitrary weights. □

PROOF. Recall from [5, Thm. 5.3] that $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x$ reaches an agreement if and only if there exists a stable polynomial:

$$P(\lambda) = \lambda^{n-k-1} + p_1\lambda^{n-k-2} + \dots + p_{n-k-1}. \tag{9}$$

with coefficients $p = (p_1, \dots, p_{n-k})$ such that there exists a solution a^* to the following set of algebraic equations:

$$0 = \mathbf{A}_{\mathcal{G}}(a)t_i, \qquad i \in \{1, \dots, k\}, \tag{10a}$$
$$p_\ell = \sum_{\xi\in\mathcal{C}_\ell(\mathcal{G})} (-1)^{d(c)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \dots, n-k\}, \tag{10b}$$

where $t_1, \dots, t_k$ are as in (7). We will prove this claim by showing that there exists a stable $P(\lambda)$ such that (10) admit a solution. Let $P(\lambda)$ be chosen as follows:

$$P(\lambda) = (\lambda - \alpha_1)\cdots(\lambda - \alpha_n),$$

where its (either real or complex conjugate pairs) roots $\alpha_i \in \mathbb{C}, i \in \{1, \dots, n\}$, satisfy $\Re(\alpha_i) < 0$. Notice that $\Re(\alpha_i) < 0$ imply that all the coefficients $\{p_1, \dots, p_{n-k}\}$ are non-negative. Since $\{\alpha_1, \dots \alpha_n\}$ are arbitrary and for any such choice each element of $p = (p_1, \dots, p_{n-k})$ is non-negative, we will seek solutions of (11) in a neighborhood of $p = 0$.

Since $\{t_1, \dots, t_k\}$ are pre-specified (and linearly independent), equation (10a) defines a set of nk linearly independent equations in the variables $a = (a_c, a_v)$, which we denote compactly as $0 = h(a_c, a_v)$, where $h : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{nk}$. Equation (10b) relates p and (a_c, a_v) by meas of a nonlinear mapping $p = g(a_c, a_v)$, where $g : \mathbb{R}^{|\mathcal{E}|} \to \mathcal{T}$ is a smooth mapping and $\mathcal{T}$ is smooth manifold in $\mathbb{R}^{n-k}$. Since $g(\cdot)$ is a multi-linear polynomial, it is immediate to verify that it admits the following decomposition:

$$g(a_c, a_v) = \frac{\partial g}{\partial a_v} \cdot a_v.$$

By denoting in compact form

$$G(a_c, a_v) := \frac{\partial g}{\partial a_v} \in \mathbb{R}^{n-k\times n-k},$$
$$H(a_c, a_v) := \frac{\partial h}{\partial a_v} \in \mathbb{R}^{nk\times n-k},$$

the system of equations (10) can be rewritten as

$$0 = H(a_c, a_v)a_v, \tag{11a}$$
$$p = G(a_c, a_v)a_v. \tag{11b}$$

As discussed above, we will now seek solutions to (11) in a neighborhood of $p = 0$. By the Inverse Function Theorem [33, Thm. 9.24], solvability of (11) in a neighborhood of $p = 0$ is guaranteed when there exists a particular point (a_c^*, a_v^*) such that $0 = H(a_c^*, a_v^*)a_v^* = G(a_c^*, a_v^*)a_v^*$ and $G(a_c^*, a_v^*)$ is invertible. To show this, we first notice that $a_v^* = 0$ is a solution of (11) for any $a_c \in \mathbb{R}^{|\mathcal{E}|-n+k}$. Thus, we are left to show that there exists a choice a_c^* such that $G(a_c^*, a_v^*)$ is invertible. Thus, we will next provide an inductive method to construct a_c^* such that $G(a_c^*, a_v^*)$ is diagonally dominant.

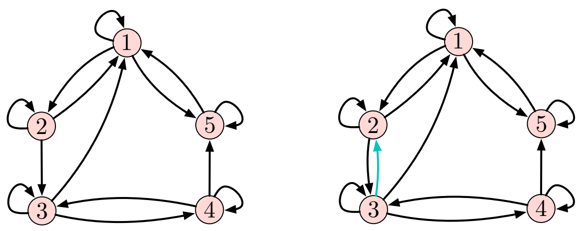
Let $a_c^{(1)} \in \mathbb{R}^{|\mathcal{E}|-n+k}$ be an arbitrary choice for a_c such that all its entries are nonzero. Notice that condition *(i)* in the statement guarantees that there exists a nonzero product in entry $(1,1)$ of $G(a_c^{(1)}, a_v^*)$, while condition *(ii)* guarantees that such product is independent of a_v^*. Thus, by letting $G^{(1)}(a_c, a_v) := G(a_c, a_v)$, the matrix $G^{(1)}(a_c^{(1)}, a_v^*)$ can be partitioned as:

$$G^{(1)}(a_c^{(1)}, a_v^*) = \begin{bmatrix} G_{11}^{(1)}(a_c^{(1)}) & G_{12}^{(1)}(a_c^{(1)}, a_v^*) \\ G_{21}^{(1)}(a_c^{(1)}, a_v^*) & G_{22}^{(1)}(a_c^{(1)}, a_v^*) \end{bmatrix},$$

where $G_{11}^{(1)} \in \mathbb{R}$, $G_{12}^{(1)} \in \mathbb{R}^{1\times n-k-1}$, $G_{21}^{(1)} \in \mathbb{R}^{n-k-1\times 1}$, $G_{22}^{(1)} \in \mathbb{R}^{n-k-1\times n-k-1}$. By condition *(ii)* and since all entries of $a_c^{(1)}$ are nonzero, we have $G_{11}^{(1)}(a_c^{(1)}) \neq 0$. Moreover, either no element of a_v^* appears in any 1-long cycle family, in which case we have $G_{12}^{(1)}(a_c^{(1)}, a_v^*) = 0$ or, otherwise, by condition *(iii)*, each entry in $G_{12}^{(1)}(a_c^{(1)}, a_v^*)$ is described by a product that contains at least one scalar variable in $a_c^{(1)}$ that does not appear in $G_{11}^{(1)}(a_c^{(1)})$. Denote such scalar variable by $\tilde{a}$ and notice that, by choosing $\tilde{a}$ sufficiently small, the first row of $G^{(1)}$ can be made diagonally dominant. Thus, we update $a_c^{(1)}$ as follows: $a_c^{(2)} = \min\{\tilde{a}, a_c^{(i)}\}$ (where the minimum is taken entrywise).

For the inductive step i, notice that $G^{(i)}(a_c^{(i)}, a_v^*)$ is diagonally dominant if $G_{22}^{(i)}(a_c^{(i)}, a_v^*)$ is diagonally dominant. Thus, by defining $G^{(i+1)}(\cdot,\cdot) = G_{22}^{(i)}(\cdot,\cdot)$, $i \in \{1, \dots, n-1\}$, by letting $a_c^{(i+1)} = \min\{\tilde{a}, a_c^{(i)}\}$ (entrywise minimum), and by iterating the argument, we conclude that $G(a_c^{(n-k)}, a_v^*)$ is diagonally dominant. Invertibility of $G(a_c^*, a_v^*)$ thus follows by letting $a_c^* = a_c^{(n-k)}$, which concludes the proof. Ξ

Theorem 4.5 characterizes a class of digraphs that admit agreement protocols on arbitrary weights. Precisely, it iden-



(a) (b)

Fig. 4. (a) Example of a graph that admits a 2-dimensional agreement protocol on arbitrary weights. (b) Graph obtained by adding green edges to (a); this graph admits a 3-dimensional agreement protocol on arbitrary weights. See Example 4.7.

tifies cycle families as a basic structural property that ensures the existence of an agreement protocol. Intuitively, $|\mathcal{E}| \geq nk + n - k$ ensures that there is a sufficiently-large number of free parameters in a to enforce the desired eigenstructure, and the conditions *(i)-(iii)* guarantees the existence of such a.

The applicability of Theorem 4.5 depends largely on the problem of determining a partitioning of a into the two sets a_v and a_c. An algorithm to determine whether such partitioning exists can be constructed by using ideas similar to [35], where a_v and a_c are derived from a directed spanning tree of $\mathcal{G}$.

We conclude this section by discussing how Theorem 4.5 modifies under edge addition, and by demonstrating its applicability through an example.

Remark 4.6 ***(Agreement reachability under edge addition)*** It is important noting that cycle families do not vanish under edge addition; thus, if *(i)-(iii)* hold for a certain graph $\mathcal{G}$, they continue to hold for any other graph obtained by edge addition. To see this, denote by $\mathcal{C}_\ell(\mathcal{G})$ the set of ℓ-long cycle families of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose that $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is any graph such that $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E} \subset \mathcal{E}'$. Since no edge has been removed, the set of ℓ-long cycle families of $\mathcal{G}'$ satisfies $\mathcal{C}_\ell(\mathcal{G}) \subseteq \mathcal{C}'_\ell(\mathcal{G})$. It follows that, if the agents are k-agreement reachable on arbitrary weights when interacting through $\mathcal{G}$, then they are also k-agreement reachable on arbitrary weights when the interaction graph is any graph obtained by adding edges to $\mathcal{G}$. □

Example 4.7 ***(Illustration of the conditions in Theorem 4.5)*** Consider the communication graph in Fig. 4(a). The corresponding agreement protocol is:

$$\mathbf{A}_{\mathcal{G}}(a) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

By Theorem 4.1, a necessary condition for agreement is

$$k \leq \left\lfloor \frac{|\mathcal{E}|}{n} \right\rfloor = \left\lfloor \frac{14}{5} \right\rfloor = 2.$$

Thus, we will fix $k = 2$. To illustrate the conditions of Theorem 4.5, for simplicity, we let $a_{22} = a_{33} = a_{44} = a_{55} = 0$ (according to Remark 4.6, if the graph without self-cycles has an independent set of cycle families, then the graph obtained by adding these self-cycles will retain the same set of decompositions). With this choice, the set of all ℓ-long cycle families, $\ell \in \{1, \ldots, n-k\}$, is:

$$\begin{aligned} \mathcal{C}_1 &= \{\{a_{11}\}\}, \\ \mathcal{C}_2 &= \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}\}, \\ \mathcal{C}_3 &= \{\{a_{11}, a_{34}, a_{43}\}, \{a_{13}, a_{21}, a_{32}\}\}. \end{aligned} \tag{12}$$

By selecting a_v and a_c as follows

$$\begin{aligned} a_v &= \{a_{11}, a_{12}, a_{13}\}, \\ a_c &= \{a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\}, \end{aligned}$$

it follows that a set of ℓ-long cycle families that satisfies the conditions in Theorem 4.5 is:

$$\mathcal{C}_1^* = \{a_{11}\}, \quad \mathcal{C}_2^* = \{a_{12}, a_{21}\}, \quad \mathcal{C}_3^* = \{a_{13}, a_{21}, a_{32}\}.$$

Indeed, with this choice, the set of equations (8b) reads as:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -a_{21} & 0 \\ a_{34}a_{43} & 0 & -a_{21}a_{32} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix},$$

where $\gamma = a_{34}a_{43} + a_{15}a_{51}$, which is generically solvable for any $(p_1, p_2, p_3) \in \mathbb{R}^3$. Any choice of weights such that $a_{21} > 0$ and $|a_{21}a_{32}| > |a_{34}a_{43}|$ guarantees that the above matrix is invertible and thus the set of equations is solvable.

To achieve agreements on subspaces of dimension $k = 3$, consider the graph in Fig. 4(b), obtained by adding edges to the graph of Fig. 4(a). The necessary condition (6) yields

$$k \leq \left\lfloor \frac{|\mathcal{E}|}{n} \right\rfloor = \left\lfloor \frac{15}{5} \right\rfloor = 3,$$

which is satisfied. The set of cycle families (12) shall be modified to:

$$\begin{aligned} \mathcal{C}_1 &= \{\{a_{11}\}\}, \\ \mathcal{C}_2 &= \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}, \{a_{23}, a_{32}\}\}. \end{aligned}$$

By selecting a_v and a_c as follows

$$\begin{aligned} a_v &= \{a_{11}, a_{12}\}, \\ a_c &= \{a_{13}, a_{23}, a_{45}, a_{35}, a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\}, \end{aligned}$$

a set of cycle families that satisfies Theorem 4.5 is:

$$\mathcal{C}_1^* = \{a_{11}\}, \qquad \mathcal{C}_2^* = \{a_{12}, a_{21}\},$$

thus showing that the sufficient conditions also hold. □

5 Extensions to tracking dynamics for agreement

In analogy with consensus protocols [22], agreement algorithms can be modified to track the oblique projection of a time-varying forcing signal $u(t)$ (in place of $x(0)$ as in (3)). Given a dgraph $\mathcal{G}$, consider the network process:

$$\dot{x} = Ax + \dot{u}, \qquad x(0) = u(0), \tag{13}$$

where A is chosen so that (3) holds and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a continuously-differentiable function. In this framework, the i-th entry of $\dot{u}$ is known only by agent i, and the goal is to construct an algorithm with state $x(t)$ that tracks $Wu(t)$, asymptotically. The protocol (13) can be interpreted as a generalization of the dynamic average consensus algorithm [22], where the communication matrix is an agreement matrix instead than a Laplacian. The following result characterizes the transient behavior of (13).

Proposition 5.1 ***(Convergence of dynamic agreement protocol)*** *Consider the update* (13) *and let A be such that* (3) *holds. Then, for all $t \geq 0$:*

$$\|x(t) - Wu(t)\| \leq e^{-\hat{\lambda}t}\|x(0) - Wu(0)\| + \frac{1}{\hat{\lambda}} \sup_{0\leq\tau\leq t} \|\dot{u}(\tau)\|, \tag{14}$$

where $\hat{\lambda} = \lambda_{\max}\left(\frac{A+A^\mathsf{T}}{2}\right)$. □

PROOF. The proof is inspired from [22, Thm. 2] and extends the result to non Laplacian-based protocols and non weight-balanced digraphs. Let W be decomposed as in (7), and consider the following decompositions for T_W and T_W^{-1}:

$$T_W = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \qquad (T_W^{-1})^\mathsf{T} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \tag{15}$$

where $T_1, U_1 \in \mathbb{R}^{n\times k}$ and $T_2, U_2 \in \mathbb{R}^{n\times n-k}$. Let $e = x - Wu$ denote the tracking error, and consider the change of variables $\bar{e} = T_W^{-1}e$. In the new variables:

$$\begin{aligned}\dot{\bar{e}} &= T_W^{-1}(\dot{x} - W\dot{u})\\ &= T_W^{-1}AT_W\bar{e} + T_W^{-1}AWu + T_W^{-1}\dot{u} - T_W^{-1}W\dot{u},\\ &= T_W^{-1}AT_W\bar{e} + T_W^{-1}\dot{u} - T_W^{-1}W\dot{u},\end{aligned}$$

where the last identity follows by using (7), which implies $AW = 0$. By substituting (15) and by noting that $T^{-1}W = [U_1 \ 0]^\mathsf{T}$:

$$\begin{aligned}\dot{\bar{e}} &= \begin{bmatrix} U_1^\mathsf{T}AT_1 & U_1^\mathsf{T}AT_2 \\ U_2^\mathsf{T}AT_1 & U_2^\mathsf{T}AT_2 \end{bmatrix}\bar{e} + \begin{bmatrix} 0 \\ U_2^\mathsf{T} \end{bmatrix}\dot{u}\\ &= \begin{bmatrix} 0 \\ U_2^\mathsf{T}AT_2 \end{bmatrix}\bar{e} + \begin{bmatrix} 0 \\ U_2^\mathsf{T} \end{bmatrix}\dot{u},\end{aligned} \tag{16}$$

where the last inequality follows by noting that $0 = U_1^\mathsf{T}AT_1 = U_1^\mathsf{T} = AT_1$ according to [5, Thm. 5.3 - cond. (i)].

Next, decompose $e = (e_1, e_2)$ and $\bar{e} = (\bar{e}_1, \bar{e}_2)$, where $e_1, \bar{e}_1 \in \mathbb{R}^k$ and $e_2, \bar{e}_2 \in \mathbb{R}^{n-k}$, and notice that the following identities hold:

$$\bar{e}_2 = U_2^\mathsf{T}e, \qquad e = T_2\bar{e}_2. \tag{17}$$

The first identity follows immediately from (15), while the second follows from (15) and $\bar{e}_1(t) = 0$ at all times. To see that $\bar{e}_1(t) = 0 \, \forall t \geq 0$, notice that $\bar{e}_1(0) = U_1^\mathsf{T}(x(0) - u(0)) = 0$ thanks to the initialization (13), and $\dot{\bar{e}}_1 = 0$ according to (16). By using (17), we conclude that $\dot{e} = Ae + \dot{u}$, from which (14) follows by noting that

$$e(t) = \exp(At)\cdot e(0) + \int_0^t \exp(A(t-\tau))B\dot{u}(\tau)d\tau,$$

and by using $\|\exp(At)\| \leq \exp\left(-\lambda_{\max}\left(\frac{\mathbf{A}_{\mathcal{G}}(a)+\mathbf{A}_{\mathcal{G}}(a)^\mathsf{T}}{2}\right)t\right)$.

The error bound (14) shows that the dynamics (13) are input-to-state stable [36] with respect to $\dot{u}$. It follows that, for any forcing signal $u(t)$ with bounded time-derivative, the tracking error $\|x(t) - Wu(t)\|$ is bounded at all times. As a special case, if $\lim_{t\to\infty} u(t) = u^* \in \mathbb{R}^n$ (namely, $\lim_{t\to\infty} \dot{u}(t) = 0$), then $\lim_{t\to\infty} x(t) = Wu^*$.

6 Applications to robotic formation control

We next illustrate the applicability of agreement protocols to solve formation problems [26] in multi-agent robotic networks. Consider a team of $n = 8$ planar single-integrator robots initially arranged on a unit circle (grey lines in Fig. 5(a)-(c)). By using $\mathbf{x}$- and $\mathbf{y}$-coordinates to describe the robots' positions, the network's initial state is given by: $x_0 = (\cos(0), \sin(0), \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}), \dots, \cos(\frac{7\pi}{4}), \sin(\frac{7\pi}{4})) \in \mathbb{R}^{16}$. To account for planar coordinates, the state of (2) is partitioned into $\mathbf{x}$ and $\mathbf{y}$ coordinates, and the algorithm (2) reads as:

$$\dot{x} = (A \otimes I_2)x, \qquad x(0) = x_0.$$

For our simulations, we utilized the circulant communication topology in Fig. 5(b) with $\alpha = 4$, and agreement protocols A have been constructed by solving numerically the set of equations (10b). Simulation results are shown in Fig. 5.

In Fig. 5(a) and (d), we report the state trajectories obtained by choosing $k = 1$ and $W = \frac{1}{n}\mathbb{1}\mathbb{1}^\mathsf{T}$. Notice that, in this special case, the agreement algorithm simplifies to an average consensus algorithm [28]; as expected for consensus, the robots meet at $(0, 0)$, which coincides with the average of the initial conditions. This special case corresponds to the robotic rendezvous problem [26]. In Fig.5(b) and (e), we illustrate the state trajectories obtained by choosing $k = 3$, $W = \Pi_{\mathcal{M}}$, where $\Pi_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M} = \ker(M_1)$,

$$M_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}.$$

Matrix M_1 encodes attraction and repulsion forces between the robots at convergence. Indeed, from $x(\infty) \in \ker(M_1 \otimes I_2)$, it follows that:

$$\begin{aligned}\mathbf{x}_1(\infty) + \mathbf{x}_4(\infty) &= \mathbf{x}_2(\infty) + \mathbf{x}_3(\infty),\\ \mathbf{y}_1(\infty) + \mathbf{y}_4(\infty) &= \mathbf{y}_2(\infty) + \mathbf{y}_3(\infty).\end{aligned} \tag{18}$$

Simulation results are illustrated in Fig.s 5(b) and (e). Finally, we illustrate in Fig.s 5(d) and (f) the robots' trajectories obtained by choosing the oblique projection: $W =$

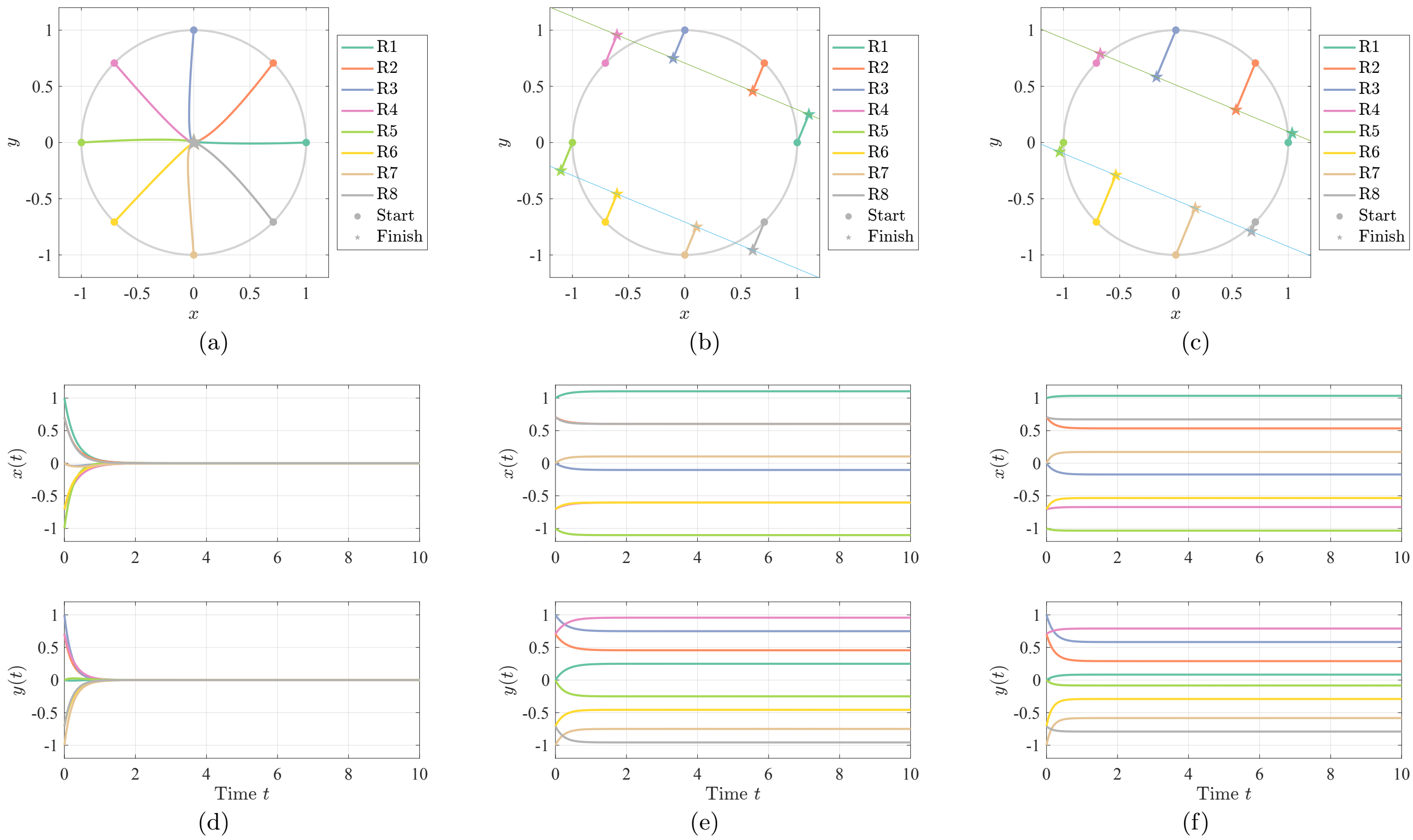


Fig. 5. (a)-(c) Time evolution of the positions of the 8 robots and (d)-(f) trajectories of the $x-$ and $y-$coordinates. (a) and (d) Consensus protocol, which allows the robots to achieve rendezvous. (b) and (e) Agreement protocol on an orthogonal projection onto $\ker(M_1)$. (c) and (f) Agreement on an oblique projection on $\ker(M_1)$ along $\mathrm{Im}(N_1)$. See Section 6.

$\Pi_{\mathcal{M},\mathcal{N}}$, $\mathcal{M} = \ker(M_1)$, $\mathcal{N} = \mathrm{Im}(N_1)$, where

$$N_1^{\mathsf{T}} = \begin{bmatrix} -1 & 5 & 5 & -1 \end{bmatrix}.$$

The use of an oblique projection can be interpreted as a non-homogeneous weighting for the vector that defines the final configuration. Indeed, as shown by the figure, in this case, the robots no longer meet "halfway", instead, robots 2 and 3 (analogously, 6 and 7) travel a longer distance as compared to robots 1 and 4 (analogously, 5 and 8).

7 Conclusions

We investigated structural properties of graphs that enable a set of agents to agree within a k-dimensional space. We showed that agreement protocols require a high network connectivity, which must scale with the dimension of the agreement space k. We identified cycle families as a basic structural property that enables agreement and, using cycle families, we characterized a class of graphs that supports agreement protocols. Although our conditions are structural and easy to check, we infer that the the class of graphs that admit agreement protocols is much larger in practice. This work opens the opportunity for several directions of future research: among them, we mention the use of nonlinear dynamics for agreement, the development of algorithms for distributed protocol synthesis, and the investigation of applications in distributed optimization. Moreover, closing the gap between the proposed necessary conditions and the sufficient conditions remains an interesting direction for future work.

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