

# Continuous Approximations of Projected Dynamical Systems via Control Barrier Functions

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**Abstract**— *Projected Dynamical Systems* (PDSs) form a class of discontinuous constrained dynamical systems, and have been used widely to solve optimization problems and variational inequalities. Recently, they have also gained significant attention for control purposes, such as high-performance integrators, saturated control and feedback optimization. In this work, we establish that locally Lipschitz continuous dynamics, involving *Control Barrier Functions* (CBFs), namely *CBF-based dynamics*, approximate PDSs. Specifically, we prove that trajectories of CBF-based dynamics *uniformly converge* to trajectories of PDSs, as a CBF-parameter is taken to infinity. Towards this, we also prove that CBF-based dynamics are perturbations of PDSs, with quantitative bounds on the perturbation. Our results pave the way to implement discontinuous PDS-based controllers in a continuous fashion, employing CBFs. Moreover, they can be employed to numerically simulate PDSs, overcoming disadvantages of existing discretization schemes, such as computing projections to possibly non-convex sets. Finally, this bridge between CBFs and PDSs may yield other potential benefits, including novel insights on stability.

## I. INTRODUCTION

*Projected Dynamical Systems* (PDSs) form a class of discontinuous dynamical systems, with trajectories constrained in some set  $\mathcal{S}$  (see [1]). In particular, while the state of the system lies in the interior of  $\mathcal{S}$ , it evolves according to some nominal vector-field, but when it reaches the boundary of  $\mathcal{S}$ , the vector field is modified in a discontinuous manner, such that it keeps the trajectory inside  $\mathcal{S}$ . PDSs have been extensively employed for analyzing constrained optimization problems and variational inequalities [1]–[4], with applications in price markets, traffic networks, power grids, etc. Recently, they have also attracted considerable interest for control purposes, such as high-performance hybrid integrators called HIGS [5], [6], passivity-based control [7], saturated control [8], [9] and feedback optimization<sup>1</sup> [10]–[12]. In all these control settings, a dynamic controller with PDS-like dynamics is interconnected to a dynamical system.

*Control Barrier Functions* (CBFs) refer to a set of methods to control systems such that their trajectories remain in a safe set  $\mathcal{S}$  [13]. In contrast to PDSs, here the nominal dynamics is modified proactively, even before reaching the boundary of  $\mathcal{S}$ . This way, in contrast to PDSs, CBF-based control results in a locally Lipschitz continuous closed-loop vector field. The “safe” dynamics is specified as the solution to a Quadratic Program (QP), with its constraint (the *CBF constraint*) preventing the system to leave  $\mathcal{S}$ . The CBF constraint involves a tunable parameter, which, loosely, determines how far from the boundary of  $\mathcal{S}$  the nominal vector-field starts being modified. CBFs have had numerous successful applications in safety-critical scenarios, such as robotics and automotive [13]–[17].

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<sup>1</sup>Driving the system’s state to the solution of an optimization problem.

In this work, we reveal an intimate relationship between PDSs and CBFs: we show that PDSs can be approximated by (locally Lipschitz) continuous dynamics, employing CBFs. Specifically, we present results formally establishing the fact that system interconnections containing CBF-based continuous dynamic controllers (hereby termed *CBF-based dynamics*) approximate interconnections with discontinuous PDSs (hereby termed *PDS-based dynamics*). Our results pave the way for implementing the PDS-based feedback controllers mentioned above in a continuous fashion, employing CBFs, with all the advantages that may come with continuity, such as increased robustness and easier sampled-data implementations.

Moreover, our work has implications for purposes other than PDS-based control as well. First, one may employ locally Lipschitz CBF-based dynamics to numerically simulate discontinuous PDSs. While simulating Lipschitz CBF-based dynamics is computationally easy, standard PDS discretization schemes (see e.g. [1], [18]) involve computing a projection to  $\mathcal{S}$ , which is computationally expensive, especially when  $\mathcal{S}$  is non-convex. This may be particularly useful in online scenarios, such as model predictive control and online optimization. Further, our results build bridges between PDS theory and CBF theory. E.g., in our preliminary work [19], we demonstrated how, in simple scenarios, one may infer stability of CBF-based control from the associated PDS, addressing the notorious undesired equilibrium problem of CBFs (cf. [16])<sup>2</sup>.

## Contributions

We prove that trajectories of CBF-based dynamics uniformly converge to trajectories of PDS-based dynamics, as the tunable parameter  $\alpha$  in the CBF constraint is taken to  $\infty$ . Thus, as  $\alpha$  becomes larger, CBF-based dynamics approximate more accurately PDS-based dynamics. This supports the use of CBF-based dynamics to implement PDS-based dynamics, as, for large  $\alpha$ , their trajectories are arbitrarily close. Towards proving convergence of trajectories, we also show that CBF-based dynamics are perturbations of PDSs, in the sense that the CBF-based vector field belongs in a set of perturbations of the set-valued map of the differential inclusion that describes the PDS. We obtain quantitative bounds on the perturbation, that depend on and vanish with  $\alpha$ . Finally, we demonstrate our results on a feedback-optimization scenario, showing how PDS-based controllers can be implemented in a continuous fashion, employing CBFs.

## Related work

Our preliminary work [19] explored the relationship between PDSs and CBFs. Focusing on autonomous dynamics, it proved that CBF-based dynamics are perturbed versions of PDSs [19, Thm. III.1]. Here, we extend [19, Thm. III.1] in two ways: a) we consider the

<sup>2</sup>Note, however, that, in our scenario, there is no restriction on modifying the dynamics, as it is the controller’s dynamics that is modified, whereas in the standard CBF-literature there are restrictions on modifying the dynamics. In standard CBF literature, the goal is designing a controller  $u(x)$ , such that trajectories of a system  $\dot{x} = f(x) + g(x)u$  remain in a set  $\mathcal{S}$ . In contrast, we focus on freely shaping the dynamics of a dynamic controller; this reduces to the above scenario, if the dynamics is simplified to  $\dot{x} = f(x) + u$ .

more general non-autonomous case of interconnections of systems with dynamics that are either PDS-like or CBF-based, and b) we provide bounds on the perturbed set-valued map, establishing local boundedness thereof, which is necessary for proving convergence of trajectories. What is more, convergence of trajectories - the present article's main theme - is not studied in [19].

The works [20]–[22] also employ CBF-based dynamics instead of PDS-based ones for constrained optimization problems, variational inequalities and feedback optimization. Mainly, they focus on the relationship between the *equilibria* of CBF-based and PDS-based dynamics, and provide asymptotic stability and contractivity results on CBF-based dynamics. Further, in [20], it is proven that the vector-fields of PDSs and CBF-based dynamics coincide at  $\alpha = \infty$ . However, neither convergence of trajectories nor the fact that CBF-based dynamics are perturbations of PDS-based ones, with quantitative bounds on the perturbation, depending on  $\alpha$ , is established. Overall, our work can be viewed as *complementary* to [20]–[22].

Finally, [10] proposed a type of continuous approximations of PDSs, based on antiwindup control, called Anti-Windup Approximations (AWA). Some steps on establishing convergence of trajectories in our work are inspired by [10] (e.g. proving that CBF-based dynamics are perturbations of PDS-based ones). Nonetheless, as CBF-based dynamics is different than AWA, proving the corresponding results is also significantly different. Further, trajectories of AWA, contrary to CBF-based dynamics, stay in an inflation of  $\mathcal{S}$ . If one needs to force AWA to stay in  $\mathcal{S}$ , modifications have to be made, which may result in undesired behaviors, such as discarding useful equilibria (e.g. extrema of feedback optimization) or shrinking regions of attraction. In addition, AWA involve calculating projections to  $\mathcal{S}$ , which can be computationally expensive. On the other hand, when  $\mathcal{S}$  represents limitations of the physical system (e.g., input saturation), and said projections are physically enforced by the system, implementation of AWA does not even require knowledge of  $\mathcal{S}$ , in contrast to CBF-based dynamics. Apart from the above, both our work and [10] establish interesting, previously unknown, connections between PDSs and other popular classes of dynamics, and might even suggest connections between antiwindup control and CBFs.

## II. PRELIMINARIES

### A. Notation

Given a closed set  $\mathcal{S} \subseteq \mathbb{R}^n$ , denote by  $\partial\mathcal{S}$  and  $\text{Int}(\mathcal{S})$  its boundary and interior, respectively. Given  $x \in \mathbb{R}^n$ , its Euclidean distance to  $\mathcal{S}$  is  $d(x, \mathcal{S}) := \min_{y \in \mathcal{S}} \|x - y\|$ , where  $\|\cdot\|$  denotes the Euclidean norm. Its projection to  $\mathcal{S}$  is  $\text{proj}_{\mathcal{S}}(x) := \arg \min_{y \in \mathcal{S}} \|x - y\|$ . Given a function  $f : X \rightarrow X'$ ,  $\text{dom} f$  denotes its domain.

Denote the set of positive-definite symmetric matrices in  $\mathbb{R}^{n \times n}$  by  $\mathbb{S}_+^n$ . Given  $P \in \mathbb{S}_+^n$ , denote its minimum and maximum eigenvalues by  $\underline{\lambda}(P)$  and  $\bar{\lambda}(P)$ , respectively. Given  $P \in \mathbb{S}_+^n$  and  $x \in \mathbb{R}^n$ , denote  $\|x\|_P = \sqrt{x^\top P x}$ . Given a function  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a differentiable function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , denote  $\mathcal{L}_f h(z, x) := \nabla h^\top(x) \cdot f(z, x)$ , where  $(z, x) \in \mathbb{R}^m \times \mathbb{R}^n$ . A continuous function  $\gamma : [0, \infty) \rightarrow \mathbb{R}$  is said to belong to  $\mathcal{K}_\infty$ , if  $\gamma(0) = 0$ ,  $\gamma$  is strictly increasing, and  $\lim_{a \rightarrow \infty} \gamma(a) = +\infty$ . Finally, denote the closed unit ball of appropriate dimensions by  $\mathbb{B}$ .

### B. Variational Analysis

Here, we recall some basic concepts from variational analysis. For more detail, the reader is referred to [23].

**Definition II.1** (Tangent Cone [23, Def. 6.1]). *Given a closed set  $\mathcal{S} \subseteq \mathbb{R}^n$ , the tangent cone to  $\mathcal{S}$  at  $x \in \mathcal{S}$ , denoted by  $T_{\mathcal{S}}(x)$ , is the set of all vectors  $w \in \mathbb{R}^n$  for which there exist sequences  $\{x_i\}_{i \in \mathbb{N}} \in \mathcal{S}$*

*and  $\{t_i\}_{i \in \mathbb{N}}$ ,  $t_i > 0$ , with  $x_i \rightarrow x$  and  $t_i \rightarrow 0$ , such that  $w = \lim_{i \rightarrow \infty} \frac{x_i - x}{t_i}$ .*

Intuitively,  $T_{\mathcal{S}}(x)$  is the set of all vectors to the direction of which if we move infinitesimally from  $x$ , we still remain in  $\mathcal{S}$ .

**Definition II.2** (Clarke regularity [23, Cor. 6.29]). *Given a closed set  $\mathcal{S} \subseteq \mathbb{R}^n$ ,  $\mathcal{S}$  is Clarke regular if and only if the set-valued map  $x \mapsto T_{\mathcal{S}}(x)$  is inner semicontinuous<sup>3</sup>.*

As, in this paper, we work with sets that are at least Clarke regular, we employ the following definition of *normal cone* and need not distinguish between different notions thereof:

**Definition II.3** (Normal Cone [23, Prop. 6.5]). *Given a closed set  $\mathcal{S} \subseteq \mathbb{R}^n$ , define the normal cone of  $\mathcal{S}$  at  $x \in \mathcal{S}$  as  $N_{\mathcal{S}}(x) := \{\eta \in \mathbb{R}^n \mid \eta^\top v \leq 0, \forall v \in T_{\mathcal{S}}(x)\}$ .*

**Definition II.4** (Prox-regularity [24, Def. 2.1]). *A Clarke regular set  $\mathcal{S}$  is prox-regular, if there exists  $\gamma > 0$  such that, for any  $x \in \mathcal{S}$ :  $\eta^\top(y - x) \leq \gamma \|\eta\| \|y - x\|^2$ , for all  $\eta \in N_{\mathcal{S}}(x)$  and  $y \in \mathcal{S}$ .*

Sets of the form

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\} \quad (1)$$

where  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable,  $x \mapsto \nabla h(x)$  is locally Lipschitz, and  $\nabla h(x) \neq 0$  for all  $x \in \partial\mathcal{S}$ , are prox-regular and the expressions for their tangent and normal cones are obtained as follows (see [4, Example 6.8]), for any  $x \in \mathcal{S}$ :

$$\begin{aligned} T_{\mathcal{S}}(x) &= \begin{cases} \mathbb{R}^n, & x \in \text{Int}(\mathcal{S}) \\ \{v \in \mathbb{R}^n \mid \nabla^\top h(x) \cdot v \geq 0\}, & x \in \partial\mathcal{S} \end{cases} \\ N_{\mathcal{S}}(x) &= \begin{cases} \{0\}, & x \in \text{Int}(\mathcal{S}) \\ \{\eta \in \mathbb{R}^n \mid \eta = \lambda \nabla h(x), \lambda \leq 0\}, & x \in \partial\mathcal{S} \end{cases} \end{aligned} \quad (2)$$

### C. Differential Inclusions

Given  $\mathcal{S} \subseteq \mathbb{R}^n$ , consider the constrained *differential inclusion* (DI)

$$\dot{\xi} \in F(\xi), \quad \xi \in \mathcal{S} \quad (3)$$

where  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  and we have omitted the dependence of  $\xi$  on  $t$ , for conciseness. When it is clear from the context that the DI's constraint set is  $\mathbb{R}^n$ , we might omit it. Given an initial condition  $x_0 \in \mathcal{S}$  and a  $T \in [0, \infty)$ , a function  $\phi : [0, T] \rightarrow \mathbb{R}^n$  is a (Carathéodory) solution to DI (3), if it is absolutely continuous,  $\phi(0) = x_0$ ,  $\phi(t) \in \mathcal{S}$  and  $\dot{\phi}(t) \in F(\phi(t))$  for almost all  $t \in [0, T]$ . We adopt the same definition for solutions of differential equations, as well.

**Definition II.5** ( $\sigma$ -perturbation [25, Def. 6.27 simplified]). *The  $\sigma$ -perturbation of DI (3) is defined as*

$$\dot{\xi} \in F_\sigma(\xi), \quad \xi \in \mathcal{S}_\sigma$$

where  $F_\sigma(x) = \overline{\text{con}F}((x + \sigma\mathbb{B}) \cap \mathcal{S}) + \sigma\mathbb{B}$ ,  $\overline{\text{con}}$  denotes convex closure and  $\mathcal{S}_\sigma = \mathcal{S} + \sigma\mathbb{B}$ .

## III. BACKGROUND ON PROJECTED DYNAMICAL SYSTEMS AND CONTROL BARRIER FUNCTIONS

Both PDSs and CBF-based control methods start from an unconstrained nominal system

$$\dot{\xi} = f(\xi) \quad (4)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and end up with one that is constrained in some set  $\mathcal{S} \subseteq \mathbb{R}^n$ . As commonly done in the literature [13]–[15], we

<sup>3</sup>For the definition of inner semicontinuity, see [23, Def. 5.4].

consider sets  $\mathcal{S}$  of the form (1). For most of the results stated in this article, the following set of assumptions on  $\mathcal{S}$  and  $h$  is employed:

**Assumption 1.**  $\mathcal{S}$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfy the following:

- 1)  $\mathcal{S}$  is nonempty and compact.
- 2)  $h$  is continuously differentiable,  $x \mapsto \nabla h(x)$  is locally Lipschitz, and its Lipschitz constant on  $\mathcal{S}$  is  $L_{\nabla h}$ .
- 3) For all  $x \in \mathbb{R}^n$  such that  $h(x) = 0$ , we have  $\nabla h(x) \neq 0$ .
- 4) There exists  $\gamma \in \mathcal{K}_\infty$ , such that, for all  $x \in \mathcal{S}$ :  $d(x, \partial\mathcal{S}) \leq \gamma(h(x))$ .

Items 2 and 3, as well as item 1 without the compactness assumption, are standard in the literature of CBFs. Compactness of  $\mathcal{S}$  is needed for several bounds in the proof of Prop. IV.1. Nonetheless, in our context, it is not a restrictive assumption, since, later on,  $\mathcal{S}$  is the set where the controller's state is confined, which is not expected to be unbounded in practice. In fact, as  $\mathcal{S}$  can be taken arbitrarily large, our results are semiglobal. Further, item 4 holds a-priori if  $h$  is a real-analytic function, by the *Łojasiewicz inequality* (see e.g. [26]). Also, as aforementioned, items 2 and 3 imply that  $\mathcal{S}$  is prox-regular and its tangent and normal cones are expressed as in (2). Finally, the following standard assumption on  $f$  is considered:

**Assumption 2.**  $f$  is locally Lipschitz.

### A. Projected Dynamical Systems

Given a matrix  $P \in \mathbb{S}_+^n$ , for any two vectors  $x, v \in \mathbb{R}^n$ , we define the projection operator  $\Pi_S^P : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  by

$$\Pi_S^P(x, v) = \arg \min_{\mu \in T_S(x)} \|\mu - v\|_P^2$$

Now, consider the following DI:

$$\dot{\xi} \in \Pi_S^P(\xi, f(\xi)) = \arg \min_{\mu \in T_S(\xi)} \|\mu - f(\xi)\|_P^2 \quad (5)$$

Systems of the form (5) are called *Projected Dynamical Systems* (PDSs) [1]. When  $\xi \in \text{Int}(\mathcal{S})$ , we have  $T_S(\xi) = \mathbb{R}^n$ , and thus  $\Pi_S^P(\xi, f(\xi)) = f(\xi)$ , i.e. the PDS (5) evolves according to the unconstrained dynamics (4). However, when  $\xi \in \partial\mathcal{S}$ , a vector  $\mu \in T_S(\xi)$  is chosen (minimizing  $\|\mu - f(\xi)\|_P^2$ ), so that the trajectory remains in  $\mathcal{S}$ . Finally, when  $\mathcal{S}$  is prox-regular, the right-hand side of (5) is a singleton and (5) becomes a discontinuous ODE.

Given a set  $\mathcal{S}$ , a matrix  $P \in \mathbb{S}_+^n$ , two vectors  $x, v \in \mathbb{R}^n$  and a scalar  $d \in [0, \infty]$  define the set-valued map  $N_S^P : \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty] \rightrightarrows \mathbb{R}^n$  by

$$N_S^P(x, v, d) := v - \left( P^{-1} N_S(x) \cap d\mathbb{B} \right)$$

Given a function  $d : \mathbb{R}^n \rightarrow [0, \infty]$ , under Assumptions 1 and 2, and further assumptions on  $d$ , solutions of the PDS (5) are equivalent to the solutions of the following DI (see Thm. III.1, below)

$$\dot{\xi} \in N_S^P(\xi, f(\xi), d(\xi)), \quad \xi \in \mathcal{S} \quad (6)$$

The interpretation of (6) is: when  $\xi(t)$  is about to leave  $\mathcal{S}$  (i.e.  $\xi(t) \in \partial\mathcal{S}$ ), an element  $\eta$  from the truncated normal cone  $P^{-1}N_S(\xi) \cap d(\xi)\mathbb{B}$  is chosen, such that  $f(\xi) - \eta$  points inwards  $\mathcal{S}$ , thereby keeping  $\xi(t) \in \mathcal{S}$ . The following is a collection of results on solutions of PDSs from [4], [10]:

**Theorem III.1** ([4], [10]). *The following hold:*

- 1) Let  $\mathcal{S}$  be Clarke regular and  $f$  be continuous. Then, (5) admits a solution for every initial condition in  $\mathcal{S}$ .
- 2) Let  $\mathcal{S}$  be prox-regular and  $f$  be locally Lipschitz. Then, (5) admits a unique solution for every initial condition in  $\mathcal{S}$ .

- 3) Let  $\mathcal{S}$  be Clarke regular and  $f$  be continuous. Consider  $d : \mathbb{R}^n \rightarrow [0, \infty]$ , such that  $d(x) \geq \sqrt{\frac{\lambda(P)}{\lambda(P)}} \|f(x)\|$  for all  $x \in \mathcal{S}$ . Then, for every initial condition in  $\mathcal{S}$ , (5) and (6) admit the same solutions.

Notice that prox-regularity implies Clarke regularity and that Assumption 1 implies prox-regularity of  $\mathcal{S}$ . Thus, in our context, (5) and (6) share the same unique solution, for every initial condition.

Finally, of particular interest to us are cases where PDSs are interconnected with other systems:

$$\dot{\zeta} = g(\zeta, \xi), \quad \dot{\xi} \in \Pi_S^P(\xi, f(\zeta, \xi))$$

where  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ . We interpret the above setup as:  $\zeta(t)$  is the state of some system controlled by a dynamic controller  $\xi(t)$ , with PDS dynamics. Such interconnections have been proposed e.g. for saturated control [8], [9] and feedback optimization [4]<sup>4</sup>. The above interconnection can be written as:

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\xi} \end{bmatrix} \in \Pi_{\mathbb{R}^m \times \mathcal{S}}^{\tilde{P}} \left( \begin{bmatrix} g(\zeta, \xi) \\ f(\zeta, \xi) \end{bmatrix} \right) \quad (7)$$

where  $\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$ . The normal-cone DI associated to (7) is:

$$\begin{bmatrix} \dot{\zeta} \\ \dot{\xi} \end{bmatrix} \in \left[ N_S^P \left( \xi, f(\zeta, \xi), d(\zeta, \xi) \right) \right], \quad (\zeta, \xi) \in \mathbb{R}^m \times \mathcal{S} \quad (8)$$

where  $d : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [0, \infty]$  is such that  $d(z, x) \geq \sqrt{\frac{\lambda(P)}{\lambda(P)}} \|f(z, x)\|$  for all  $(z, x) \in \mathbb{R}^m \times \mathcal{S}$ , and again by Thm. III.1, under Assumption 1 and local Lipschitz assumptions on  $f, g$ , it shares the same solutions with the PDS.

### B. Control-Barrier-Function-based dynamics

Given  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , a tunable parameter  $\alpha > 0$  and  $P \in \mathbb{S}_+^n$ , consider

$$\dot{\xi} = f_{\text{cbf}, \alpha}^P(\xi) := \begin{cases} \arg \min_{\mu} \|\mu - f(\xi)\|_P^2 \\ \text{s.t. : } \mathcal{L}_\mu h(\xi) + \alpha h(\xi) \geq 0 \end{cases} \quad (9)$$

We refer to (9) as the *CBF-based dynamics*. The term comes from the CBF literature [13], which refers to  $h$  as the *control barrier function* (CBF). Under Assumption 1, the Quadratic Program (QP) above has a unique solution, for any  $\xi$ , which can be written in closed form as follows (see [17, Thm. 1]):

$$f_{\text{cbf}, \alpha}^P(\xi) = f(\xi) - \min \left( 0, \mathcal{L}_f h(\xi) + \alpha h(\xi) \right) \frac{P^{-1} \nabla h(\xi)}{\|\nabla h(\xi)\|_{P^{-1}}^2}$$

We call  $f_{\text{cbf}, \alpha}^P$  the *CBF-based vector field*. In contrast to PDSs (5), here the nominal dynamics is modified even when  $\xi \in \text{Int}\mathcal{S}$ . The nominal vector field  $f$  is modified whenever  $\mathcal{L}_f h(\xi) + \alpha h(\xi) \leq 0$ , i.e. whenever the value of  $h$  decreases along the trajectories of the nominal system (4) faster than a state-dependent threshold  $-\alpha h(\xi)$ . The larger  $\alpha$  is, the less invasive the modification becomes, as it allows for even more negative decreases  $\mathcal{L}_f h(\xi)$ . Typically, this allows trajectories to approach closer to the boundary, before the nominal vector-field is modified.

Vector fields like (9) arise in control systems of the form  $\dot{x} = f(x) + u(x)$ , where the controller  $u(x)$  is designed via CBF-based

<sup>4</sup>A generalization of (7), termed *extended PDSs*, has also been employed in [5], [6] for high-performance integral control and [7] for passivity-based control. Extension of our results to ePDSs is considered future work.

methods. From standard CBF theory [13], it follows that trajectories of (9) starting in  $\mathcal{S}$  stay in  $\mathcal{S}$ . Under Assumptions 1 and 2,  $f_{\text{cbf},\alpha}^P$  is locally Lipschitz, implying existence and uniqueness of solutions of (9).

Finally, as with PDSs, we also consider the interconnection

$$\begin{aligned} \dot{\zeta} &= g(\zeta, \xi) \\ \dot{\xi} &= f_{\text{cbf},\alpha}^P(\zeta, \xi) := \begin{cases} \arg \min_{\mu} \|\mu - f(\zeta, \xi)\|_P^2 \\ \text{s.t. : } \mathcal{L}_{\mu}h(\xi) + \alpha h(\xi) \geq 0 \end{cases} \quad (10) \\ &= f(\zeta, \xi) - \min \left( 0, \mathcal{L}_f h(\zeta, \xi) + \alpha h(\xi) \right) \frac{P^{-1}\nabla h(\xi)}{\|\nabla h(\xi)\|_{P^{-1}}^2} \end{aligned}$$

where  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The article's main objective is to show that the feedback interconnection with a CBF-based controller (10) is a legitimate continuous approximation of the feedback interconnection with a PDS-based controller (7), in the sense of uniform convergence of trajectories. Briefly, as  $\alpha \rightarrow \infty$ , trajectories of (10) become identical to trajectories of (7). Towards this, we also derive another highly relevant result: that (10) is a perturbation of (7), with quantitative bounds on the perturbation.

**Remark 1.** Note that all our results relating (trajectories of) (10) with (trajectories of) (7), also hold for (9) and (5), as (9) and (5) are special cases of (10) and (7), respectively. ■

#### IV. THE RELATIONSHIP BETWEEN PDSs AND CBF-BASED DYNAMICS

In this section, we delve into the relationship between PDS-based dynamics (8) and CBF-based dynamics (10). First, we prove that CBF-based dynamics are perturbations of PDSs. Then, building on this, we prove that trajectories of CBF-based dynamics uniformly converge to trajectories of PDSs as  $\alpha \rightarrow \infty$ . The proofs to our results can be found in Section VII, unless otherwise stated.

##### A. CBF-based dynamics as perturbations of PDSs

Here, we show how CBF-based dynamics can be viewed as perturbed versions of PDSs, in the sense of Def. II.5. First, as an extension to our preliminary work [19, Thm. III.1], we show that  $f_{\text{cbf},\alpha}^P(z, x)$  belongs to a set of perturbations of the set-valued map  $f(z, x) - P^{-1}N_{\mathcal{S}}(x)$ .

**Proposition IV.1.** Consider a vector field  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , a compact set  $\mathcal{Z} \subset \mathbb{R}^m$  and a set  $\mathcal{S} \subset \mathbb{R}^n$  defined by (1). Let Assumptions 1 and 2 hold. Denote the Lipschitz constant of  $f$  on  $\mathcal{Z} \times \mathcal{S}$  by  $L_f$ . For  $\epsilon$  such that  $0 < \epsilon < \min_{z \in \partial \mathcal{S}} \|\nabla h(z)\|$ , define

$$\alpha_* = \frac{\max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} |\mathcal{L}_f h(z, x)|}{\gamma^{-1} \left( \frac{\min_{z \in \partial \mathcal{S}} \|\nabla h(z)\| - \epsilon}{L_{\nabla h}} \right)}$$

Moreover, denote

$$\begin{aligned} M_1 &:= \min_{x \in \partial \mathcal{S}} \|\nabla h(x)\|, \quad M_2 := \max_{x \in \partial \mathcal{S}} \|\nabla h(x)\| \\ M_3 &:= M_2 + L_{\nabla h} \gamma \left( \frac{1}{\alpha_*} \max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} |\mathcal{L}_f h(z, x)| \right) \\ L_1 &:= \frac{\bar{\lambda}(P)}{\underline{\lambda}(P)\epsilon^2} L_{\nabla h} \left[ 1 + \frac{M_2 \bar{\lambda}(P) (M_2 + M_3)}{\underline{\lambda}(P) M_1^2} \right] \end{aligned}$$

Finally, define

$$\begin{aligned} \sigma(\alpha) &:= \max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} \max \left\{ \gamma \left( \frac{1}{\alpha} |\mathcal{L}_f h(z, x)| \right), \right. \\ &\quad \left. \left( L_f + L_1 |\mathcal{L}_f h(z, x)| \right) \gamma \left( \frac{1}{\alpha} |\mathcal{L}_f h(z, x)| \right) \right\} \end{aligned}$$

and

$$\delta := \max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} \left( 1 + \frac{L_{\nabla h} \gamma \left( \frac{1}{\alpha_*} |\mathcal{L}_f h(z, x)| \right)}{M_1} \right) \frac{\bar{\lambda}(P)}{\underline{\lambda}(P)} \|f(z, x)\|$$

Then, for any  $\alpha \geq \alpha_*$ , it holds that, for all  $(z, x) \in \mathcal{Z} \times \mathcal{S}$ :

$$f_{\text{cbf},\alpha}^P(z, x) \in K_{\sigma(\alpha)} \left( N_{\mathcal{S}}^P(x, f(z, x), \delta) \right)$$

where  $f_{\text{cbf},\alpha}^P$  is defined in (10) and

$$\begin{aligned} K_{\sigma(\alpha)} \left( N_{\mathcal{S}}^P(x, f(z, x), \delta) \right) &:= \\ \left\{ N_{\mathcal{S}}^P(y, f(z, y), \delta) + \sigma(\alpha) \mathbb{B} \mid y \in (x + \sigma(\alpha) \mathbb{B}) \cap \mathcal{S} \right\} \end{aligned}$$

**Remark 2.** Prop. IV.1 extends [19, Thm. III.1] in two ways. First, towards addressing interconnections (8) and (10), it considers  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , in contrast to [19, Thm. III.1], which only considers  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Second, it gives bounds  $\delta$  on the normal cone elements in the right-hand side. This is crucial for proving convergence of trajectories later on, as it enables results from [25] for *well-posed* DIs, requiring local boundedness of the set-valued map. ■

Observe that, due to Assumption 1 item 3,  $M_1 > 0$ , and, hence, there is always an  $\epsilon$  such that  $0 < \epsilon < \min_{x \in \partial \mathcal{S}} \|\nabla h(x)\|$ . Moreover, both  $\sigma(\alpha)$  and  $\delta$  are well-defined, as they are maxima of continuous locally bounded functions over compact sets. Prop. IV.1 gives rise to the two following results:

**Corollary IV.1.** Given  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and any  $\alpha \geq \alpha_*$ , under the assumptions of Prop. IV.1, consider the following DI:

$$\begin{aligned} \begin{bmatrix} \dot{\zeta} \\ \dot{\xi} \end{bmatrix} &\in \left[ K_{\sigma(\alpha)} \left( N_{\mathcal{S}}^P(\xi, f(\zeta, \xi), \delta) \right) \right], \quad (11) \\ (\zeta, \xi) &\in \mathbb{R}^m \times (\mathcal{S} + \sigma(\alpha) \mathbb{B}) \end{aligned}$$

Given an initial condition  $(z_0, x_0) \in \mathcal{Z} \times \mathcal{S}$  and any  $T \in [0, \infty)$ , consider  $\phi : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ , such that  $\phi(0) = (z_0, x_0)$ . Assume that  $\phi(t) \in \mathcal{Z} \times \mathcal{S}$  for all  $t \in [0, T]$ . Then, if  $\phi$  is a solution to the CBF-based dynamics (10), it is a solution to DI (11).

*Proof.* Follows readily from Prop. IV.1. □

**Corollary IV.2.** Given  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , under the assumptions of Prop. IV.1, for any  $\alpha \geq \alpha_*$ , DI (11) is a  $\sigma(\alpha)$ -perturbation of the following DI:

$$\begin{aligned} \begin{bmatrix} \dot{\zeta} \\ \dot{\xi} \end{bmatrix} &\in \left[ N_{\mathcal{S}}^P(\xi, f(\zeta, \xi), \delta) \right], \quad (\zeta, \xi) \in \mathbb{R}^m \times \mathcal{S} \quad (12) \end{aligned}$$

*Proof.* Follows directly by combining Prop. IV.1 and Def. II.5. □

Moreover, DI (12), under our assumptions, shares the same bounded solutions with PDS (7):

**Corollary IV.3.** Given  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , under the assumptions of Prop. IV.1, for any  $\alpha \geq \alpha_*$ , consider PDS (7) and DI (12). Given an initial condition  $(z_0, x_0) \in \mathcal{Z} \times \mathcal{S}$  and any  $T \in [0, \infty)$ , consider a function  $\phi : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ , such that  $\phi(0) = (z_0, x_0)$ . Assume that  $\phi(t) \in \mathcal{Z} \times \mathcal{S}$  for all  $t \in [0, T]$ . The following hold:

- 1) Let  $g$  be continuous. Then,  $\phi$  is a solution to PDS (7) if and only if it is a solution to DI (12).
- 2) Further, let  $g$  be locally Lipschitz. Then, if  $\phi$  is a solution to PDS (7), and thus to DI (12), it is the only solution.

*Proof.* Observe that, for all  $(z, x) \in \mathcal{Z} \times \mathcal{S}$ , we have  $\delta \geq \sqrt{\frac{\bar{\lambda}(P)}{\underline{\lambda}(P)}} \|f(z, x)\|$ . The result then follows by Theorem III.1. □



Let us explain how Cors. IV.1, IV.2 and IV.3 are integral to our work. First, note that solutions of the CBF-based dynamics (10) are solutions of perturbed DI (11) (from Cor. IV.1). Also, notice that  $\sigma$  is continuous, strictly decreasing on  $\alpha$ , and satisfies  $\sigma(\alpha) \geq 0$  and  $\lim_{\alpha \rightarrow \infty} \sigma(\alpha) = 0$ . Thus, from Cor. IV.2, as  $\alpha \rightarrow \infty$ , the perturbed DI (11), and thus the CBF-based dynamics (10), tends to the *limiting* DI (12); and, the larger  $\alpha$  is picked, the closer is perturbed DI (11), and thus the CBF-based dynamics (10), to the limiting DI (12). Finally, from Cor. IV.3, we know that the limiting DI (12) shares the same solutions with the PDS (7). Thus, CBF-based dynamics (10) are perturbed versions of PDS-based ones (7), and  $\sigma(\alpha)$  is a bound on that perturbation, that vanishes as  $\alpha \rightarrow \infty$ . In the coming section, we employ all the above, to prove that trajectories of CBF-based dynamics (10) uniformly converge to trajectories of PDS-based dynamics (7).

**Remark 3.** [20, Prop. 4.4] proves that the right-hand sides of (7) and (10) coincide at  $\alpha \rightarrow \infty$ . Nonetheless, this is generally not enough to infer convergence of solutions. In contrast, here we provide quantitative bounds, depending on  $\alpha$ , on the perturbation that (10) is to (7). As becomes evident from the following section, this is what is indeed needed to prove convergence of solutions. ■

**Remark 4.** Notice that we only study solutions that are constrained in the compact set  $\mathcal{Z} \times \mathcal{S}$ . On the other hand,  $\mathcal{Z} \times \mathcal{S}$  can be chosen arbitrarily large. In this sense, our results are semiglobal. ■

### B. Trajectories of CBF-based dynamics converge to trajectories of PDSs

Here, we employ Cors. IV.1, IV.2 and IV.3, to show that trajectories of CBF-based dynamics (10) uniformly converge to bounded trajectories of PDSs (7), (8), as  $\alpha \rightarrow \infty$ . This establishes that CBF-based dynamics are legitimate approximations of PDSs.

Specifically, we first show that any sequence of bounded solutions to the CBF-based dynamics, corresponding to an increasing sequence  $\{\alpha_i\}_{i \in \mathbb{N}}$  of parameters  $\alpha_i$  with  $\lim_{i \rightarrow \infty} \alpha_i = \infty$ , has a subsequence which converges to a solution of the PDS (7). Moreover, the whole sequence of bounded solutions to the CBF-based dynamics converges to a solution of the PDS (7), when this solution is unique.

**Theorem IV.1.** Consider vector fields  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ , a compact set  $\mathcal{Z} \subset \mathbb{R}^m$  and a set  $\mathcal{S} \subset \mathbb{R}^n$  defined by (1). Let Assumptions 1 and 2 hold. Consider an increasing sequence  $\{\alpha_i\}_{i \in \mathbb{N}}$ , such that  $\alpha_i \geq \alpha_*$  and  $\lim_{i \rightarrow \infty} \alpha_i = \infty$ , where  $\alpha_*$  is defined in Prop. IV.1. Given an initial condition  $(z_0, x_0) \in \mathcal{Z} \times \mathcal{S}$  and any  $T \in [0, \infty)$ , let the functions  $\phi_i : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ , with  $\phi_i(0) = (z_0, x_0)$ , be the solutions to CBF-based dynamics (10) with  $\alpha = \alpha_i$ . Assume that  $\phi_i(t) \in \mathcal{Z} \times \mathcal{S}$  for all  $t \in [0, T]$  and for all  $i$ . The following hold:

- 1) Let  $g$  be continuous and locally bounded. Then, there is a subsequence  $\{\phi_{i_k}\}_{k \in \mathbb{N}}$  of  $\{\phi_i\}_{i \in \mathbb{N}}$  that converges uniformly, on any compact subinterval of  $[0, T)$ , to a solution  $\phi$  of PDS (7), with  $\text{dom}\phi = [0, T]$  and  $\phi(0) = (z_0, x_0)$ .
- 2) Further, let  $g$  be locally Lipschitz. Then, the sequence  $\{\phi_i\}_{i \in \mathbb{N}}$  converges uniformly, on any compact subinterval of  $[0, T)$ , to the unique solution  $\phi$  of PDS (7), with  $\text{dom}\phi = [0, T]$  and  $\phi(0) = (z_0, x_0)$ .

**Remark 5.** Once Props. IV.1 and VII.1 (See Section VII), and Cors. IV.1, IV.2 and IV.3 are given, the proof of Thm. IV.1 follows steps similar to [10, Section 4]. ■

While Thm. IV.1 establishes uniform convergence of solutions of CBF-based dynamics (10) to solutions of PDSs (7), it assumes that solutions of CBF-based dynamics stay in  $\mathcal{Z} \times \mathcal{S}$ . And, while it is

guaranteed that the  $\xi$ -part of the solution is always in  $\mathcal{S}$ , this is not always the case for the  $\zeta$ -part. Nevertheless, if PDS (7) obeys a stability property, then, as the following result shows, we can guarantee that solutions to CBF-based dynamics stay in  $\mathcal{Z} \times \mathcal{S}$ , for sufficiently large  $\alpha$ , and thus they converge to solutions of PDS (7).

**Theorem IV.2.** Consider vector fields  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ , a compact set  $\mathcal{Z} \subset \mathbb{R}^m$  and a set  $\mathcal{S} \subset \mathbb{R}^n$  defined by (1). Let Assumptions 1 and 2 hold. Further, let  $g$  be continuous and locally bounded. Given  $\gamma > 0$ , consider compact sets  $\mathcal{Z}_0, \mathcal{Z}'$ , such that  $\mathcal{Z}_0 \subseteq \mathcal{Z}'$  and  $\mathcal{Z}' + \gamma\mathbb{B} \subseteq \mathcal{Z}$ . Given any initial condition  $(z_0, x_0) \in \mathcal{Z}_0 \times \mathcal{S}$  and a  $T > 0$ , assume any solution  $\phi : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$  to PDS (7), with  $\phi(0) = (z_0, x_0)$ , satisfies  $\phi(t) \in \mathcal{Z}' \times \mathcal{S}$  for all  $t \in [0, T]$ . Then, there exists  $\alpha' \geq \alpha_*$ , such that for any  $\alpha \geq \alpha'$ , the solution  $\psi : [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$  to the CBF-based dynamics (10), with  $\psi(0) = (z_0, x_0)$ , satisfies  $\psi(t) \in \mathcal{Z} \times \mathcal{S}$  for all  $t \in [0, T]$ . Therefore, for any increasing sequence  $\{\alpha_i\}_{i \in \mathbb{N}}$ , such that  $\alpha_i \geq \alpha'$  and  $\lim_{i \rightarrow \infty} \alpha_i = \infty$ , and respective sequence of solutions  $\{\psi_i\}_{i \in \mathbb{N}}$  to CBF-based dynamics (10), statements 1 and 2 of Thm. IV.1 hold.

**Remark 6.** The stability assumption of Thm. IV.2 is not strong, for practical considerations. When PDSs are introduced as feedback controllers [5], [7]–[11], this is done in a way that guarantees stability of the closed-loop implementation (7). Thus, if the PDS-based controller is replaced by a CBF-based one, to yield the CBF-based dynamics (10), Thm. IV.2 ensures that trajectories of (10) converge to trajectories of (7), for  $\alpha \geq \alpha'$ ; that is, (10) behaves similarly to (7). ■

**Remark 7** (Stability of CBF-based dynamics). Combining Theorem IV.2 and [25, Prop. 6.34], one can derive a stability result for CBF-based dynamics: if the origin for PDS-based dynamics (7) is asymptotically stable, then it is practically stable for CBF-based dynamics (10). Asymptotic stability and contractivity results for CBF-based dynamics have also been reported in [21]. ■

## V. NUMERICAL EXAMPLE: FEEDBACK OPTIMIZATION

In *feedback optimization*, a given system has to be steered via a dynamic controller to a steady state, which is specified as the solution to an optimization problem. This type of regulation finds applications in power grids and congestion control of communication networks, among others. See [11] and references therein for a thorough exposition. As an exemplary setup, we consider the plant

$$\dot{\zeta} = A\zeta + B\xi \quad (13)$$

where  $\xi$  is the state of a dynamic controller, to be designed. As commonly done in feedback-optimization scenarios (see [11]), we assume that  $A$  is Hurwitz (i.e. the origin of the open-loop system is asymptotically stable). Thus, for every constant input  $\xi_e$ , there exists a unique steady-state equilibrium  $\zeta_e = -A^{-1}B\xi_e$ . We seek to drive the system to the solution of the following optimization problem:

$$\min_{\zeta, \xi} \Phi(\zeta), \quad \text{s.t.} \quad \zeta = -A^{-1}B\xi \quad (14)$$

while respecting the input constraint  $\xi(t) \in \mathcal{S}$  at all times. The constraint in (14) enforces exactly that the solution of the optimization problem has to comply with the fact that control system (13) behaves as the static input-output map  $\zeta = -A^{-1}B\xi$ , at the steady state.

Following [10], [11], under appropriate assumptions, the above problem can be solved by designing the dynamic controller as:

$$\dot{\xi} \in \Pi_{\mathcal{S}} \left( B^{\top} A^{-\top} \nabla \Phi(\zeta) \right) \quad (15)$$

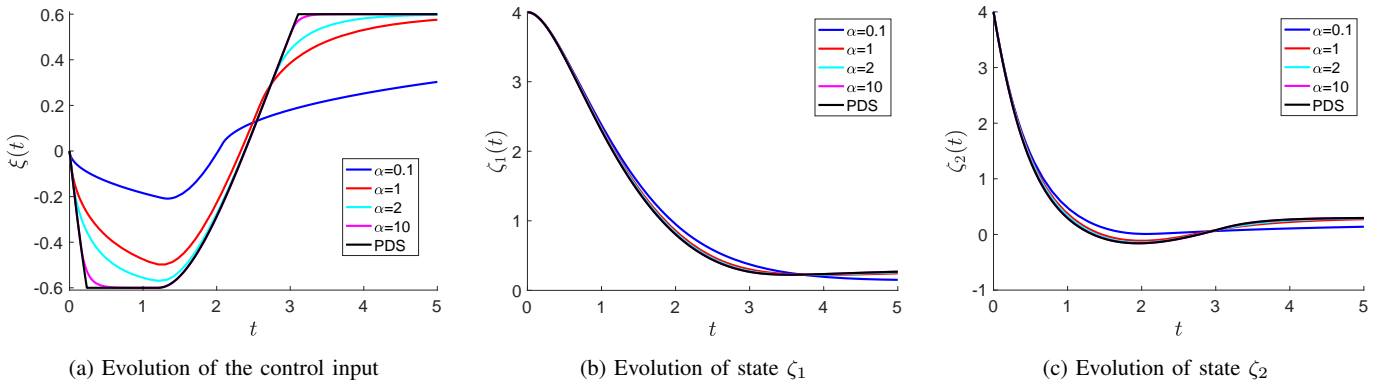


Fig. 1: Evolution of the control input  $\xi(t)$  and states  $\zeta_1(t)$  and  $\zeta_2(t)$  of system (13), when the controller has PDS dynamics and CBF-based dynamics, for different values of  $\alpha$ .

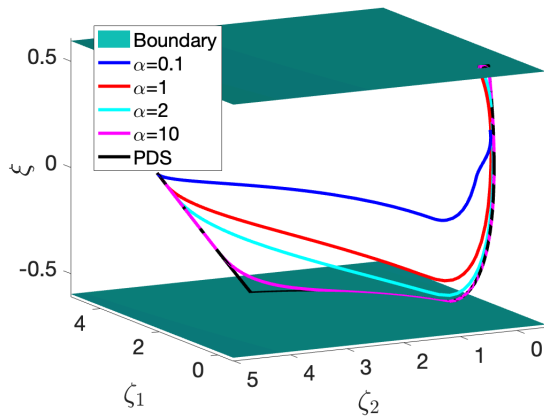


Fig. 2: Trajectories  $(\zeta(t), \xi(t))$  of system (13) coupled with dynamic controller, when the controller has PDS dynamics and CBF-based dynamics, for different values of  $\alpha$ .

which is a *projected gradient flow*. In this example, we demonstrate how one may indeed approximate the PDS-based dynamics (13)-(15), by substituting the discontinuous controller (15) with a CBF-based one, as in (10). We consider the following data:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \Phi(\zeta) = \left( \zeta - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2$$

$$\mathcal{S} = \{u \in \mathbb{R} : 0.36 - u^2 \geq 0\}$$

Observe that both Assumptions 1 and 2 are satisfied by the data. In addition, as  $A$  is Hurwitz and the controller  $\xi(t) \in \mathcal{S}$ , with  $\mathcal{S}$  compact, it follows that trajectories of the PDS-based dynamics (13)-(15) are bounded, and Thm. IV.2 applies.

We simulate trajectories of (13) coupled with a dynamic controller, which has either the PDS dynamics (15) or CBF-based approximations thereof, for different values of  $\alpha$ . Figures 1 and 2 depict the results. As expected from Thms. IV.1 and IV.2, we observe that, as  $\alpha$  becomes larger, trajectories of the CBF-based dynamics approximate more closely trajectories of PDS-based ones. Moreover, as expected from [21, Thms. 5.6 and 5.7], since  $\mathcal{S}$  and (14) are convex, both PDS-based dynamics and CBF-based ones converge to the global minimizer  $(\zeta_*, \xi_*) = (0.3, 0.3, 0.6)$ .

## VI. CONCLUSION

We have proven that trajectories of CBF-based dynamics uniformly converge to trajectories of PDS-based ones, as the tunable

parameter  $\alpha$  of the CBF-constraint is taken to  $\infty$ . Towards this, we have also proven that CBF-based dynamics are perturbations of PDSs, and quantified the perturbation. One can employ our results to implement discontinuous PDS-based controllers in a continuous manner, employing CBFs. We have demonstrated this on a numerical example emanating from feedback optimization. Similarly, our results enable novel numerical simulation schemes for PDSs, through CBF-based dynamics, overcoming disadvantages of existing schemes, such as projections to possibly non-convex sets. Finally, the established bridge between PDSs and CBFs might yield other results in the future, such as insights on stability.

## VII. TECHNICAL RESULTS AND PROOFS

### A. Proof of Prop. IV.1

To prove Prop. IV.1, we make use of Lemmas VII.1, VII.2 and VII.3 below. These lemmas have already appeared in [19], in a simpler form; here they are extended, to account for the fact that we study interconnections (8) and (10), rather than autonomous setups (6) and (9), which was the context of [19].<sup>5</sup> Moreover, in [19] the proofs of Lemmas VII.1 and VII.2 were only sketched. Here, we provide the full proofs of Lemmas VII.1 and VII.2 and recall Lemma VII.3.

The following results are stated under the assumptions of Prop. IV.1. Given any  $z \in \mathcal{Z}$ , we denote  $U_{\text{cbf},\alpha}(z) := \{x \in \mathcal{S} \mid \mathcal{L}_f h(z, x) + \alpha h(x) \leq 0\}$ .

**Lemma VII.1.** *Given  $\alpha > 0$  and  $z \in \mathcal{Z}$ , consider any  $x \in U_{\text{cbf},\alpha}(z)$  and any  $y \in \text{proj}_{\partial \mathcal{S}}(x)$ . It holds that*

$$\|x - y\| \leq \gamma\left(\frac{1}{\alpha} |\mathcal{L}_f h(z, x)|\right)$$

*Proof of Lemma VII.1.* Assumption 1 item 4 implies that:

$$\begin{aligned} \|x - y\| &= d(x, \partial \mathcal{S}) \leq \gamma(h(x)) \leq \gamma\left(-\frac{1}{\alpha} \mathcal{L}_f h(z, x)\right) \\ &= \gamma\left(\frac{1}{\alpha} |\mathcal{L}_f h(z, x)|\right) \end{aligned}$$

where in the second inequality we use that  $0 \leq h(x) \leq -\frac{1}{\alpha} \mathcal{L}_f h(z, x)$ .  $\square$

**Lemma VII.2.** *For any  $\alpha \geq \alpha_*$  and  $z \in \mathcal{Z}$ , for all  $x \in U_{\text{cbf},\alpha}(z)$ , we have*

$$0 < \epsilon \leq \|\nabla h(x)\| \leq M_3$$

<sup>5</sup>Here,  $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , that is  $f$  has not only  $\xi$  as an argument, but  $\zeta$  too; see (6) and (9).

*Proof of Lemma VII.2.* Given  $\alpha > 0$ , consider any  $x \in U_{\text{cbf},\alpha}(z)$ . From Assumption 2, we have the following, for any  $y \in \text{proj}_{\partial\mathcal{S}}(x)$ :

$$\begin{aligned} \|\nabla h(x) - \nabla h(y)\| &\leq L_{\nabla h}\|x - y\| \implies \\ \|\nabla h(x)\| &\geq \|\nabla h(y)\| - L_{\nabla h}\|x - y\| \implies \\ \|\nabla h(x)\| &\geq \|\nabla h(y)\| - L_{\nabla h}\gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right) \end{aligned}$$

where in the last step we have used Lemma VII.1. Thus, for any  $\alpha \geq \alpha_*$ , and for any  $x \in U_{\text{cbf},\alpha}(z)$ , we get:

$$\begin{aligned} \|\nabla h(x)\| &\geq M_1 - L_{\nabla h}\gamma\left(\frac{1}{\alpha} \max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} |\mathcal{L}_f h(z, x)|\right) \\ &\geq M_1 - L_{\nabla h}\gamma\left(\frac{1}{\alpha_*} \max_{(z,x) \in \mathcal{Z} \times \mathcal{S}} |\mathcal{L}_f h(z, x)|\right) = \epsilon > 0 \end{aligned}$$

Similarly, we can deduce that for any  $\alpha \geq \alpha_*$ , and for any  $x \in U_{\text{cbf},\alpha}(z)$ :

$$\|\nabla h(x) - \nabla h(y)\| \leq L_{\nabla h}\|x - y\| \implies$$

$$\begin{aligned} \|\nabla h(x)\| &\leq \|\nabla h(y)\| + L_{\nabla h}\|x - y\| \leq M_2 + L_{\nabla h}\gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right) \\ &\leq M_2 + \gamma\left(\frac{1}{\alpha_*}|\mathcal{L}_f h(z, x)|\right) \\ &\leq M_3 \end{aligned}$$

□

**Lemma VII.3.** *Given any  $\alpha \geq \alpha_*$  and  $z \in \mathcal{Z}$ , it holds that, for any  $x \in U_{\text{cbf},\alpha}(z)$  and  $y \in \text{proj}_{\partial\mathcal{S}}(x)$*

$$\left\| \frac{P^{-1}\nabla h(x)}{\|\nabla h(x)\|_{P^{-1}}^2} - \frac{P^{-1}\nabla h(y)}{\|\nabla h(y)\|_{P^{-1}}^2} \right\| \leq L_1\|x - y\|$$

$$\text{where } L_1 := \frac{\bar{\lambda}(P)}{\bar{\lambda}(P)\epsilon^2} L_{\nabla h} \left[ 1 + \frac{M_2\bar{\lambda}(P)(M_2+M_3)}{\bar{\lambda}(P)M_1^2} \right].$$

*Proof of Lemma VII.3.* See [19]. □

We are ready to prove Prop. IV.1.

*Proof of Prop. IV.1.* Observe that both  $\sigma(\alpha)$  and  $\delta$  are well-defined, as they are maxima of continuous locally bounded functions over compact sets. For  $x \in \mathcal{S}$ , we distinguish the following cases:

a) *Case 1:*  $\mathcal{L}_f h(z, x) + \alpha h(x) > 0$ . Here we have  $f_{\text{cbf},\alpha}(z, x) = f(z, x)$ , and the result holds trivially.

b) *Case 2:*  $\mathcal{L}_f h(z, x) + \alpha h(x) \leq 0$ . In this case,  $x \in U_{\text{cbf},\alpha}(z)$  and we can write

$$f_{\text{cbf},\alpha}(z, x) = f(z, x) - \frac{\mathcal{L}_f h(z, x) + \alpha h(x)}{\|\nabla h(x)\|_{P^{-1}}^2} P^{-1}\nabla h(x) \quad (16)$$

which is well-defined, as  $\nabla h(x) \neq 0$ , from Lemma VII.2.

Consider any  $y \in \text{proj}_{\partial\mathcal{S}}(x)$ . Observe that  $\eta := \left( \mathcal{L}_f h(z, x) + \alpha h(x) \right) \frac{\nabla h(y)}{\|\nabla h(y)\|_{P^{-1}}^2} \in N_{\mathcal{S}}(y)$ , since  $\frac{\mathcal{L}_f h(z, x) + \alpha h(x)}{\|\nabla h(y)\|_{P^{-1}}^2} \leq 0$ . Also,  $\nabla h(y) \neq 0$ , due to Assumption 1 item 2. We have the following:

$$\begin{aligned} &\left\| f(z, y) - P^{-1}\eta - \right. \\ &\left. f(z, x) + \frac{\mathcal{L}_f h(z, x) + \alpha h(x)}{\|\nabla h(x)\|_{P^{-1}}^2} P^{-1}\nabla h(x) \right\| \leq \\ &\quad \|f(z, y) - f(z, x)\| + \\ &\left\| \mathcal{L}_f h(z, x) + \alpha h(x) \right\| \left\| \frac{P^{-1}\nabla h(y)}{\|\nabla h(y)\|_{P^{-1}}^2} - \frac{P^{-1}\nabla h(x)}{\|\nabla h(x)\|_{P^{-1}}^2} \right\| \leq \\ &L_f\|x - y\| + |\mathcal{L}_f h(z, x) + \alpha h(x)|L_1\|x - y\| \leq \\ &L_f\|x - y\| + |\mathcal{L}_f h(z, x)|L_1\|x - y\| \leq \end{aligned}$$

$$\underbrace{\left( L_f + L_1|\mathcal{L}_f h(z, x)| \right) \gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right)}_{\sigma_1(\alpha, z, x)}$$

where in the second inequality we used Assumption 2, in the third inequality we used Lemma VII.3, in the fourth inequality we used that  $|\mathcal{L}_f h(z, x) + \alpha h(x)| \leq |\mathcal{L}_f h(z, x)|$ , and in the fifth inequality we used Lemma VII.1.

Regarding  $P^{-1}\eta$ , notice that

$$\begin{aligned} \|P^{-1}\eta\| &\leq \frac{1}{\bar{\lambda}(P)}\|\eta\| \\ &\leq \frac{1}{\bar{\lambda}(P)}|\mathcal{L}_f h(z, x) + \alpha h(x)| \left\| \frac{\nabla h(y)}{\|\nabla h(y)\|_{P^{-1}}^2} \right\| \\ &\leq \frac{1}{\bar{\lambda}(P)}|\mathcal{L}_f h(z, x)| \frac{\|\nabla h(y)\|}{\|\nabla h(y)\|_{P^{-1}}^2} \\ &\leq \frac{1}{\bar{\lambda}(P)}\|f(z, x)\| \frac{\|\nabla h(y)\|}{\|\nabla h(y)\|_{P^{-1}}^2} \\ &\leq \frac{1}{\bar{\lambda}(P)} \left( \frac{1}{\bar{\lambda}(P^{-1})} + \frac{L_{\nabla h}\|x - y\|}{\bar{\lambda}(P^{-1})\|\nabla h(y)\|} \right) \|f(z, x)\| \\ &\leq \left( 1 + \frac{L_{\nabla h}\gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right)}{M_1} \right) \frac{\bar{\lambda}(P)}{\bar{\lambda}(P)} \|f(z, x)\| \\ &\leq \delta \end{aligned}$$

where in the fifth inequality we used  $\|\nabla h(x)\| \leq \|\nabla h(y)\| + L_{\nabla h}\|x - y\|$ , as proven in the proof of Lemma VII.2, and in the last inequality we used Lemma VII.1.

Finally, from the above we get:

$$\begin{aligned} f_{\text{cbf},\alpha}(x) &= \\ f(z, x) - \frac{\mathcal{L}_f h(z, x) + \alpha h(x)}{\|\nabla h(x)\|_{P^{-1}}^2} P^{-1}\nabla h(x) &\in \\ f(z, y) - P^{-1}\eta + \sigma_1(\alpha, z, x)\mathbb{B} &\subseteq \\ f(z, y) - P^{-1}N_{\mathcal{S}}(y) \cap \delta\mathbb{B} + \sigma_1(\alpha, z, x)\mathbb{B} &\subseteq \\ f\left(\left(x + \gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right)\right)\mathbb{B} \cap \mathcal{S}\right) - & \\ P^{-1}N_{\mathcal{S}}\left(\left(x + \gamma\left(\frac{1}{\alpha}|\mathcal{L}_f h(z, x)|\right)\right)\mathbb{B} \cap \mathcal{S}\right) \cap \delta\mathbb{B} + & \\ \sigma_1(\alpha, z, x)\mathbb{B} &\subseteq \\ f\left(\left(x + \sigma(\alpha)\mathbb{B} \cap \mathcal{S}\right) - P^{-1}N_{\mathcal{S}}\left(\left(x + \sigma(\alpha)\mathbb{B} \cap \mathcal{S}\right) \cap \delta\mathbb{B} + \right. & \\ \left. \sigma(\alpha)\mathbb{B} = \right. & \\ \left. K_{\sigma(\alpha)}\left(N_{\mathcal{S}}^P(x, f(z, x), \delta)\right) \right) & \end{aligned}$$

□

## B. Proofs of Thms. IV.1 and IV.2

To prove Thms. IV.1 and IV.2, we employ several results from [25]. The first step to proving Thms. IV.1 and IV.2 is to establish *well-posedness*<sup>6</sup>, in the sense of [25, Def. 6.29], of limiting DI (12).

**Proposition VII.1.** *Consider DI (12). Let Assumptions 1 and 2 hold. Let  $g$  be continuous and locally bounded. Then, DI (12) is well-posed.*

*Proof.* First,  $\mathbb{R}^m \times \mathcal{S}$  is closed. Second,  $g$  is locally bounded and outer semicontinuous. Being a singleton, it is also a convex set-valued map. Further,  $N_{\mathcal{S}}^P(x, f(z, x), \delta)$  is: (a) locally bounded, as  $f$  is locally

<sup>6</sup>Well-posedness of a DI implies that solutions of its  $\sigma$ -perturbations converge to the DI's solutions, as  $\sigma \rightarrow 0$ . See [25, Def. 6.29].

Lipschitz and the normal cone is truncated by the bounded quantity  $\delta$ ; (b) convex, as a translation of the intersection of two convex sets (the normal cone and a ball); (c) outer semicontinuous, since its graph is closed (outer semicontinuity of  $N_{\mathcal{S}}(x)$  follows from [23, Prop. 6.6]). Thus, by [25, Thm. 6.30], the result follows.  $\square$

We are ready to prove Thms. IV.1 and IV.2.

*Proof of Thm. IV.1.* The proof follows steps similar to [10, Section 4]. In what follows, we prove graphical convergence<sup>7</sup> of  $\{\phi_{i_k}\}_{k \in \mathbb{N}}$  or  $\{\phi_i\}_{i \in \mathbb{N}}$  to  $\phi$ . Thus, we regularly treat  $\phi_{i_k}$ ,  $\phi_i$  and  $\phi$  as sets, through their graphs, below. From [25, Lemma 5.28], it follows that graphical convergence implies uniform convergence on compact subintervals of  $[0, T]$ . Further, note that, since  $\sigma(\cdot)$  is decreasing and  $\lim_{\alpha \rightarrow \infty} \sigma(\alpha) = 0$ , the sequence  $\sigma(\alpha_i)$  is decreasing and  $\lim_{i \rightarrow \infty} \sigma(\alpha_i) = 0$ .

**Proof of Statement 1.** From [25, Thm. 5.7],  $\{\phi_i\}_{i \in \mathbb{N}}$  has a graphically convergent subsequence, say  $\{\phi_{i_k}\}_{k \in \mathbb{N}}$ . Note that  $\phi_i$  are also solutions to perturbed DI (11) for  $\alpha = \alpha_i$  (Cor. IV.1), i.e.  $\phi_i$ , and thus  $\phi_{i_k}$ , are solutions to  $\sigma(\alpha_i)$ -perturbations, and  $\sigma(\alpha_{i_k})$ -perturbations, respectively, of the limiting DI (12) (Cor. IV.2). Thus, since the limiting DI (12) is well-posed (Prop. VII.1), and since  $\{\phi_{i_k}\}_{k \in \mathbb{N}}$  is convergent, then  $\{\phi_{i_k}\}_{k \in \mathbb{N}}$  graphically converges to a solution  $\phi$  of the limiting DI (12), from [25, Def. 6.29]. Finally, from Cor. IV.3 item 1,  $\phi$  is a solution to PDS (7).

**Proof of Statement 2.** Follows identical steps to the proof of [10, Cor. 4.6].  $\square$

*Proof of Thm. IV.2.* Let  $\gamma' > 0$  with  $\gamma' < \gamma$ . Under our assumptions, the limiting DI (12) is well-posed (Prop. VII.1) and it has the exact same solutions as the PDS (7) (from Cor. IV.3). Moreover, perturbed DI (11) is a  $\sigma(\alpha)$ -perturbation to limiting DI (12) (Cor. IV.2). Then, from [25, Prop. 6.34], it follows that there exists  $\alpha' \geq \alpha_*$ , such that for all  $\alpha \geq \alpha'$  every solution  $\chi$  to perturbed DI (11) is  $\gamma'$ -close to a solution  $\phi$  of (12), implying  $\chi(t) \in (\mathcal{Z}' + \gamma' \mathbb{B}) \times \mathcal{S}$ , for all  $t \in [0, T]$ .

Now, consider a solution  $\psi: [0, T] \rightarrow \mathbb{R}^m \times \mathbb{R}^n$  to the CBF-based dynamics (10) for  $\alpha \geq \alpha'$ , with  $\psi(0) = (z_0, x_0) \in \mathcal{Z}_0 \times \mathcal{S}$ . Note that, denoting  $\psi(t) = (\zeta(t), \xi(t))$ , from standard CBF theory, as aforementioned, it is guaranteed that  $\xi(t) \in \mathcal{S}$ , for all  $t \in [0, T]$ . We prove that  $\psi(t) \in \mathcal{Z} \times \mathcal{S}$  for all  $t \in [0, T]$ , thus proving the statement. Reasoning by contradiction, assume that there exists  $t_e \in [0, T]$  such that  $t_e = \inf\{t : \psi(t) \notin \mathcal{Z} \times \mathcal{S}\}$ . By absolute continuity of  $\psi$ , and since  $\mathcal{Z}_0 \subseteq \mathcal{Z}' + \gamma' \mathbb{B} \subset \mathcal{Z}' + \gamma \mathbb{B} \subseteq \mathcal{Z}$ , there also exists time  $t' < t_e$  such that  $\psi(t') \notin (\mathcal{Z}' + \gamma' \mathbb{B}) \times \mathcal{S}$  and  $\psi(t') \in \mathcal{Z} \times \mathcal{S}$ . Thus, from Cor. IV.1, the time-truncated solution  $\psi|_{[0, t']}$  is also a solution to the perturbed DI (11). However, from the above, this implies that  $\psi(t') \in (\mathcal{Z}' + \gamma' \mathbb{B}) \times \mathcal{S}$ , which is a contradiction.  $\square$

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<sup>7</sup>For the definition of graphical convergence, see [25, Def. 5.18].