Safe Control of Second-Order Systems with Linear Constraints

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Abstract— **Control barrier functions (CBFs) offer a powerful tool for enforcing safety specifications in control synthesis. This paper deals with the problem of constructing valid CBFs. Given a second-order system and any desired safety set with linear boundaries in the position space, we construct a provably control-invariant subset of this desired safety set. The constructed subset does not sacrifice any positions allowed by the desired safety set, which can be nonconvex. We show how our construction can also meet safety specification on the velocity. We then demonstrate that if the system satisfies standard Euler-Lagrange systems properties then our construction can also handle constraints on the allowable control inputs. We finally show the efficacy of the proposed method in a numerical example of keeping a 2D robot arm safe from collision.**

I. Introduction

Control barrier functions (CBF) provide a flexible framework in safety-critical control to certify the forward invariance of a desired set with respect to system trajectories and design feedback controllers that ensure it. Because of their versatility, CBFs have made their way into numerous applications in robotics, transportation, power systems, and beyond. By definition, every boundary point of the CBF's 0-superlevel set admits a control value that holds the system's trajectory from instantaneously leaving it. This point-wise condition is known as the CBF condition. Finding CBFs is a challeging task: it amounts to finding a set whose states can be made safe, i.e., for which control actions ensuring safety can be identified. This is not trivial given the complexity of the dynamics and limitations on the control inputs. After clearing this challenge, one must still figure out whether a well-behaved control law can synthesize out of all the point-wise safe control actions. In this work, we construct valid CBFs that enforce any positional safety requirements with linear boundaries for second-order dynamics and provide an associated continuous safe controller.

Literature review: Whether from Nagumo's theorem [1] or from comparison results in the theory of differential equations [2], the CBF condition was first derived for smooth functions [3]–[5]. This condition was extended to non-smooth functions in multiple works, such as [6]–[8]. Many approaches have been proposed to construct a CBF or verify whether a given function is a CBF. One approach applies learning methods to construct CBFs [9]–[12]. Another uses reachability analysis to construct the maximal invariant set and use it in safe control design [13]. Another class uses backstepping to design CBFs for cascaded systems [14]. Still, another group of works, most closely related to the treatment here, utilizes properties of specific systems to construct suitable CBFs. For instance, [15], [16] constructs CBFs for polynomial systems using sum-of-square optimization. The work [17] constructs non-smooth CBFs for fully actuated Euler-Lagrange systems, with constraints on position, velocity, and inputs given by hypercubes. The work [18] proposes a method to construct a safe subset of a hypersphere in the position space, assuming no input constraints. Both these works consider convex constraint sets. In contrast, our treatment here is valid for a highly expressive class of positional (potentially nonconvex) constraints defined by linear boundaries. Once a valid CBF is constructed, safe feedback controllers are usually synthesized via state-parameterized optimization programs [4], [19] due to their flexibility, convenience, and computational lightness. This motivated the study of the regularity properties of such controllers, see e.g. [20]–[22]. Our recent work [23] synthesizes a provably feasible optimization-based safe feedback controller for safe sets given by arbitrarily nested unions and intersections of superlevel sets of differentiable functions.

Statement of Contributions: We consider[1] second-order system dynamics and positional constraints specified by nested unions and intersections of half-spaces. We construct a control-invariant subset of the full state space which contains all positions allowed by the original positional constraints. We derive a general condition which, if satisfied, proves that our constructed set is safe for general, possibly not fully actuated, second-order systems and provide an associated QP safeguarding controller. We then show that this safety condition is always satisfied for fully actuated systems. We further show that a compact allowable controls set suffices for safety when the desired safe set is bounded. We show how our method can be utilized to incorporate velocity and input constraints when the dynamics satisfy standard Euler-Lagrange system properties. Finally, we apply our method to design safe controls for a 2D robotic arm.

II. Problem Statement

Consider the second-order dynamics

$$\dot{x} = f(x) + G(x)u \tag{1}$$

where $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(x) = (x_2, f_2(x))$, $G(x) = (\mathbf{0}_{n\times m}, G_2(x))$, with $f_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$

[1]We use the following notation. We let $\mathbf{1_{n\times m}}$ and $\mathbf{0_{n\times m}}$ denote the $n\times m$ matrices of ones and zeroes, resp. Likewise, $\mathbf{1_n}$ and $\mathbf{0_n}$ denote the n vectors of ones and zeroes, resp. The boundary, interior, and convex hull of $\mathcal{S}$ are $\partial\mathcal{S}$, $\mathrm{int}(\mathcal{S})$, and $\mathrm{co}(\mathcal{S})$, resp. The projection of $\mathcal{C} \subset \mathbb{R}^n$ on the first n' components is denoted $\mathrm{Proj}_{n'}(\mathcal{C})$. The vectors of the standard basis of $\mathbb{R}^n$ are denoted $\{e_\ell\}_{\ell=1}^n$. We denote the 2-norm of a vector x by $\|x\|$. The norm of a matrix A is the induced 2-norm, $\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$. We refer to the 0-superlevel set of a function simply as superlevel set.

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and $G_2 : \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ Lipschitz functions. Consider also the half-spaces parameterized by $i \in \mathcal{I} \coloneqq \{1, \dots, r\}$

$$\mathcal{C}_i \coloneqq \{x \in \mathbb{R}^{2n} \mid h_i(x) \coloneqq a_i^\top x_1 + b_i \geq 0\}, \tag{2}$$

with $\|a_i\| \neq 0 \neq b_i$. We require that if $i \neq i'$ then the augmented vectors (a_i, b_i) and $(a_{i'}, b_{i'})$ are linearly independent. Our objective is to design a control law for (1) that keeps invariant a desired set $\mathcal{C}$ given as a union of intersections of the half-spaces $\mathcal{C}_i$'s. That is,

$$\mathcal{C} = \bigcup_{\ell\in\mathcal{L}} \bigcap_{i\in\mathcal{I}^\ell} \mathcal{C}_i, \tag{3}$$

where $\mathcal{L} \subset \mathbb{N}$ and the sets $\mathcal{I}^\ell \subset \mathcal{I}$ are sets of indices. This set corresponds to the superlevel set of the function

$$h(x) = \max_{\ell\in\mathcal{L}} \min_{i\in\mathcal{I}^\ell} h_i(x). \tag{4}$$

Note that the safety set $\mathcal{C}$ constrains only the states x_1 corresponding to the generalized position. This setup is common in many problems, such as collision avoidance [24], where the main concern is to avoid the physical locations occupied by obstacles. This form for $\mathcal{C}$ is flexible enough to capture the safety requirement of staying in any set of positions with linear boundaries while simultaneously avoiding obstacles of linear boundaries. This is applicable in a wide range of situations for autonomous robotic systems from geofencing to protect a human collaborator to avoiding collisions in human environments.

We impose the following structural assumptions on the safety requirement. Let $\mathcal{S}_\cap$ be the collection of sets of indices whose corresponding half-spaces intersect in $\mathcal{C}$: that is, if $I \subseteq \mathcal{I}$ is such that $(\cap_{i\in I}\mathcal{C}_i) \cap \mathcal{C} \neq \emptyset$, then $I \in \mathcal{S}_\cap$.

Assumption 1. The set $\text{Proj}_n(\mathcal{C})$ is compact. Furthermore, for any $I \in \mathcal{S}_\cap$, one point $y_I \in \mathcal{C}$ can be chosen to satisfy $h_i(y_I) > 0$ for all $i \in I$. •

Assumption 1 is reasonable. Assuming the compactness of $\text{Proj}_n(\mathcal{C})$ is the same as saying that the set of safe positions x_1 is compact. A sufficient condition for this is the boundedness of $\cap_{i\in\mathcal{I}^\ell}\mathcal{C}_i$ for every ℓ. The last part of the assumption only requires that any nonempty intersection $(\cap_{i\in I}\mathcal{C}_i)\cap\mathcal{C}$ has a non-empty interior. We discuss the impact of the specific choice of points $\{y_I\}$ in Remark III.2.

Since our system (1) is second order, the control is only available in the order of the generalized acceleration $\dot{x}_2$. Thus, $\mathcal{C}$, which only constrains x_1, is generally not control-invariant due to the lack of constraints on the velocity: e.g., initial conditions starting exactly at the boundary of $\mathcal{C}$ with a velocity heading outwards are unsafe, with no possible control value to counter it. Hence, there is a need to identify a control-invariant set $\mathcal{C}^s \subseteq \mathcal{C}$ that constrains the velocity. This construction should contain as much of $\text{Proj}_n(\mathcal{C})$ as possible. We thus formulate our problem as follows.

Problem 1. Given the second-order dynamics (1) and the set $\mathcal{C}$ defined by (3) that satisfies Assumption 1, construct a set $\mathcal{C}^s$ such that:

(i) $\mathcal{C}^s \subseteq \mathcal{C}$,
(ii) $\text{Proj}_n(\mathcal{C}^s) = \text{Proj}_n(\mathcal{C})$,
(iii) $\mathcal{C}^s$ is control-invariant,

and design a continuous controller that renders $\mathcal{C}^s$ invariant.

III. Construction of Control-Invariant Set

In this section, we solve Problem 1 by constructing a function B of the form

$$B(x) = \max_{\ell\in\bar{\mathcal{L}}} \min_{i\in\bar{\mathcal{I}}^\ell} B_i(x),$$

and then proving that its superlevel set $\mathcal{C}^s$ satisfies the requirements in Problem 1. Constructing B amounts to defining the functions B_i and the index sets $\bar{\mathcal{L}}$ and $\bar{\mathcal{I}}^\ell$.

Definition of functions: For each $1 \leq i \leq r$, define $B_i(x) = h_i(x)$ and

$$B_{i+r}(x) = a_i^\top x_2 + \gamma(a_i^\top x_1 + b_i) - \epsilon, \tag{5}$$

where ϵ and γ are two positive design parameters, each of which will have a special role in customizing the design and proving its safety. We denote by $\mathcal{C}_i^s$ the superlevel set of B_i.

Definition of sets: Let $\bar{\mathcal{L}} = \mathcal{L}$ and $\bar{\mathcal{I}}^\ell = \mathcal{I}^\ell \cup (\{r\} + \mathcal{I}^\ell)$.

The choice of B follows this logic: for $i \in \{1, \dots, r\}$, $\mathcal{C}_i$ is the superlevel set of B_i; the function B_{i+r} is then chosen such that its superlevel set contains the points on the boundary of $\mathcal{C}_i$ only if the system drift f points towards the interior of the safe set at those boundary points. Thus, the system is safe without requiring any inputs at the boundary points defined by $B_i = 0$ with $i \in \{1, \dots, r\}$ if these boundary points are in the superlevel set of B_{i+r}. This choice of B_{i+r} is inspired by the concept of high-order control barrier function [25].

The following result shows that this construction satisfies requirements (i) and (ii) in Problem 1.

Lemma III.1 (No Positions Lost)**.** *Let $\mathcal{C}^s$ be the superlevel set of B. Then $\mathcal{C}^s \subset \mathcal{C}$. Under Assumption 1, define*

$$\delta \coloneqq \min_{I\in\mathcal{S}_\cap,\, i\in I} h_i(y_I) > 0, \tag{6}$$

and let γ, ϵ with $\gamma\delta > \epsilon$. Then, $\text{Proj}_n(\mathcal{C}^s) = \text{Proj}_n(\mathcal{C})$. •

Proof. That $\mathcal{C}^s$ is a subset of $\mathcal{C}$ follows directly from the definition of these sets and the fact that $\mathcal{C}_i^s = \mathcal{C}_i$ for $i \in \{1, \dots, r\}$. This also implies that $\text{Proj}_n(\mathcal{C}^s) \subset \text{Proj}_n(\mathcal{C})$. Note that Assumption 1 ensures that $\delta > 0$. Since by assumption $\gamma\delta > \epsilon$, there exists $0 < \sigma < 1$ such that $\gamma\sigma\delta > \epsilon$. Given $(x_1, x_2) \in \mathcal{C} = \cup_{\ell\in\mathcal{L}} \cap_{i\in\mathcal{I}^\ell} \mathcal{C}_i$, there is an $\ell' \in \mathcal{L}$ such that $(x_1, x_2) \in \mathcal{C}_i$, for all $i \in \mathcal{I}^{\ell'}$, i.e., $h_i(x) = B_i(x) \geq 0$, for all $i \in \mathcal{I}^{\ell'}$. By Assumption 1, there exists $y_{\mathcal{I}^{\ell'}} \in \mathcal{C}$ satisfying $h_i(y_{\mathcal{I}^{\ell'}}) > 0$ for all $i \in \mathcal{I}^{\ell'}$. The choice $x' = (x_1, x_2')$, where $x_2' = -\gamma\sigma(x_1 - y_{\mathcal{I}^{\ell'}})$ gives

$$\begin{aligned} B_{i+r}(x') &= a_i^\top x_2' + \gamma B_i(x) - \epsilon \\ &= \gamma(1-\sigma)B_i(x) + \gamma\sigma B_i(y_{\mathcal{I}^{\ell'}}) - \epsilon \\ &\geq \gamma(1-\sigma)B_i(x) + \gamma\sigma\delta - \epsilon \\ &> \gamma(1-\sigma)B_i(x) \geq 0. \end{aligned}$$

Therefore, $B_i(x') \geq 0$ for all $i \in \bar{\mathcal{I}}^{\ell'}$, implying that $x' \in \cap_{i\in\bar{\mathcal{I}}^{\ell'}}\mathcal{C}_i^s \subseteq \mathcal{C}^s$. Therefore, $\text{Proj}_n(\mathcal{C}) \subset \text{Proj}_n(\mathcal{C}^s)$. □

Remark III.2 (Maximizing Flexibility of Safe Set Design)**.** Lemma III.1 requires that the parameters γ and ϵ satisfy $\gamma\delta > \epsilon$. Since δ is dependent on the choice of points $\{y_I\}$, whose existence is assumed in Assumption 1, the choice of γ and ϵ is dependent on the set of points $\{y_I\}$. The greater δ, the more flexibility for choosing γ and ϵ. Therefore, for a set $I \in \mathcal{S}_\cap$, the choice of y_I that promotes the most design flexibility is $y_I := \arg\max_{x \in \mathcal{C}} \min_{i \in I} h_i(x)$. It is important to note, however, that the specific choice of points $\{y_I\}$, while important for design flexibility, is not crucial for the results of this paper: any choice makes the condition $\gamma\delta > \epsilon$ satisfiable and thus allows for the design of a safe $\mathcal{C}^s$ satisfying $\text{Proj}_n(\mathcal{C}^s) = \text{Proj}_n(\mathcal{C})$. •

To address requirement (iii) in Problem 1, we need to identify a control action at each state of $\mathcal{C}^s$ that makes $\mathcal{C}^s$ forward-invariant. We introduce the following condition.

Condition 1 (General Safety Condition)**.** For each $x \in \partial\mathcal{C}^s$, there exists $u_x \in \mathcal{U}$ such that

$$\dot{B}_i(x) = \nabla B_i(x)^\top (f(x) + G(x)u_x) > 0$$

for all $i \in \tilde{\mathcal{I}}(x) := \{i \in \{1, \ldots, 2r\} \mid \exists \ell \in \mathcal{L}_i : \ B(x) = B^\ell(x) = B_i(x)\}$, with $B^\ell(x) = \min_{i \in \bar{\mathcal{I}}^\ell} B_i(x)$ and $\mathcal{L}_i := \{\ell \in \bar{\mathcal{L}} \mid i \in \bar{\mathcal{I}}^\ell\}$. •

Condition 1 requires that there exists a control input that steers the system to the interior of the safe set at all boundary points where the drift of the system is not guaranteed to to do that on its own.

Consider now the feedback controller u^* defined by the following quadratic program:

$$\begin{aligned} &(u^*(x), \alpha^*(x), M^*(x)) := \\ &\underset{\alpha, M, u \in \mathcal{U}}{\arg\min}\ u^\top Q(x) u + q(x)^\top u + q_\alpha \alpha^2 + q_M M^2 \qquad (7) \\ &\text{s.t. } M \geq c_M,\ \alpha \geq c_\alpha \\ &\quad \nabla B_i(x)^\top (f(x) + G(x)u) + \alpha B_i(x) \\ &\quad + M\left(B(x) - B^\ell(x)\right) \geq 0,\ \forall \ell \in \bar{\mathcal{L}},\ \forall i \in \bar{\mathcal{I}}^\ell. \end{aligned}$$

Here, $Q : \mathcal{X} \to \mathbb{R}^{m \times m}$ is a Lipschitz function which takes values in the set of positive-definite matrices, $q : \mathcal{X} \to \mathbb{R}^m$ is a Lipschitz function, and q_α, q_M, c_M, c_α are positive design constants. The following result summarizes the properties of u^* and, in particular, that it makes $\mathcal{C}^s$ control-invariant. under Condition 1.

Theorem III.3. *(Safe Controller [23, Thm. 4.12]): Let $\mathcal{C}^s$ be compact. Under Condition 1, there exists a neighborhood of $\mathcal{C}^s$ where program (7) is feasible and u^* is single-valued, continuous, and renders $\mathcal{C}^s$ control-invariant.* •

Our approach to establish (iii) in Problem 1 is then to verify the hypotheses of Theorem III.3. The following result shows that $\mathcal{C}^s$ is compact if $\text{Proj}_n(\mathcal{C})$ is compact, which is readily ensured by Assumption 1.

Proposition III.4 (Compactness of Safe Set)**.** *Under Assumption 1, $\mathcal{C}^s$ is compact.* •

Proof. We reason by contradiction. Suppose $\mathcal{C}^s = \cup_{\ell \in \bar{\mathcal{L}}} \cap_{i \in \bar{\mathcal{I}}^\ell} \mathcal{C}_i^s$ is not compact. Then, for some $\ell' \in \mathcal{L}$, the convex closed set $\cap_{i \in \bar{\mathcal{I}}^{\ell'}} \mathcal{C}_i^s$ is unbounded and, thus, contains a ray [26, Result 2.5.1]. That is there is a point $x_0 = (x_{0,1}, x_{0,2})$ and a direction $\zeta = (\zeta_1, \zeta_2) \neq \mathbf{0}$ such that $B_i(x_0 + t\zeta) \geq 0$ for all $t \geq 0$ and all $i \in \bar{\mathcal{I}}^{\ell'}$. We distinguish two cases: (a) $\zeta_1 \neq \mathbf{0}$ and (b) $\zeta_1 = \mathbf{0}$. In case (a), $B_i(x_0 + t\zeta) = h_i(x_0 + t\zeta) \geq 0$ for all $i \in \mathcal{I}^{\ell'}$, so $\text{Proj}_n(\mathcal{C})$ is not compact, which contradicts Assumption 1. In case (b), $\zeta_1 = \mathbf{0} \neq \zeta_2$. So, $B_{i+r}(x_0 + t\zeta) = t a_i^\top \zeta_2 + a_i^\top x_{0,2} + \gamma(a_i^\top x_{0,1} + b_i) - \epsilon \geq 0$, for all $i \in \mathcal{I}^{\ell'}$ and all $t \geq 0$. Since all terms in the inequality are constants except for the first one, we deduce that $t a_i^\top \zeta_2 \geq 0$ for t large enough, which implies $a_i^\top \zeta_2 \geq 0$, for all $i \in \mathcal{I}^{\ell'}$. Therefore, $h_i(x_0 + t(\zeta_2, \mathbf{0})) = t a_i^\top \zeta_2 + a_i^\top x_{0,1} + b_i = t a_i^\top \zeta_2 + h_i(x_0) \geq 0$ for all $t \geq 0$ and all $i \in \mathcal{I}^{\ell'}$, which implies that $\text{Proj}_n(\mathcal{C})$ is unbounded, contradicting Assumption 1. □

Next, we focus on the satisfaction of Condition 1. The following result particularizes this condition to our context.

Lemma III.5 (General Invariance Condition)**.** *If for all $x \in \partial\mathcal{C}^s$ and $i' + r \in \tilde{\mathcal{I}}(x)$ for which $B_{i'+r}(x) = 0$, there exists $u_x \in \mathcal{U}$ such that*

$$a_{i'}^\top (\gamma x_2 + f_2(x) + G_2(x)u_x) > 0, \qquad (8)$$

then Condition 1 is satisfied. •

Proof. If $x \in \partial\mathcal{C}^s$, then $B(x) = 0$ and thus $B_i(x) = 0$ for $i \in \tilde{\mathcal{I}}(x)$. For any such $i \in \tilde{\mathcal{I}}(x)$, it follows that $B_{i'}(x) \geq 0$ for any $\ell \in \mathcal{L}_i$ and $i' \in \bar{\mathcal{I}}^\ell$. Let us now verify the inequality of Condition 1 for $i \in \tilde{\mathcal{I}}(x)$. We distinguish two cases: (a) $i \leq r$ or (b) $i > r$. In case (a), $B_i(x) = a_i^\top x_1 + b_i = 0$ and $i' = i + r \in \bar{\mathcal{I}}^\ell$. Thus,

$$0 \leq B_{i'}(x) = a_i^\top x_2 + \gamma B_i(x) - \epsilon = a_i^\top x_2 - \epsilon. \qquad (9)$$

So $\dot{B}_i(x) = a_i^\top x_2 \geq \epsilon > 0$ by (9). In case (b), for $i' = i - r$, $\dot{B}_i(x) = \dot{B}_{i'+r}(x)$ equals the left-hand side of (8), which verifies Condition 1 by assumption. □

We use the inequality (8) to establish safety under specific structural assumptions on our dynamics. The inequality can fail to hold in two ways. The first is when there is no $u_x \in \mathbb{R}^m$ that satisfies it. The second is when such a $u_x \in \mathbb{R}^m$ exists, but it does not belong to the allowable input set $\mathcal{U}$. The distinction between the two possibilities is useful since, in practice, they can be overcome by different tools. If the failure is of the second type, then a practical solution might be to expand $\mathcal{U}$ (e.g., by employing a more powerful actuator). If the failure is of the first type, such a solution is not possible, but rather the set $\mathcal{C}^s$ cannot be made safe given the dynamics, and a better design must be sought. We exploit this distinction in what follows.

The following result shows that Condition 1 is satisfied by fully actuated systems.

Proposition III.6 (Control-Invariance with Full Actuation)**.** *Assume that, for all $x \in \partial\mathcal{C}^s$, $G_2(x)$ is right invertible and $\mathcal{U} = \mathbb{R}^m$. Then, under Assumption 1, choosing γ and ϵ such that $\gamma\delta > \epsilon$ ensures that Condition 1 is satisfied.* •

Proof. Given $x = (x_1, x_2) \in \partial\mathcal{C}^s$, $B(x) = 0$. Define $I_x := \{i \in \{1, \dots, r\} \mid i + r \in \tilde{\mathcal{I}}(x), B_{i+r}(x) = 0\}$. If I_x is empty then the premise of Lemma III.5 is satisfied by default. Otherwise, by the definitions of $\tilde{\mathcal{I}}(x)$ and $\bar{\mathcal{I}}^\ell$'s, $B_i(x) \geq 0$ for all $i \in I_x$. Therefore, by Assumption 1, there exists $y_{I_x} \in \mathcal{C}$ satisfying $h_i(y_{I_x}) > 0$ for all $i \in I_x$. Then, for all $i \in I_x$

$$\begin{aligned} &B_{i+r}(x) = a_i^\top x_2 + \gamma(a_i^\top x_1 + b_i) - \epsilon = 0 \\ \implies &a_i^\top(-x_2 - \gamma x_1 + \gamma y_{I_x}) = \gamma(a_i^\top y_{I_x} + b_i) - \epsilon \\ &\qquad\qquad\qquad\qquad \geq \gamma\delta - \epsilon > 0. \end{aligned} \tag{10}$$

Now, because of (10), one can choose $u_x = \rho G_2^\dagger(x) y_x$, where $G_2^\dagger$ denotes the right inverse of G_2 and $y_x = -x_2 - \gamma x_1 + \gamma y_{I_x}$, with ρ large enough, so that the inequality (8) is satisfied. The result then follows from Lemma III.5. □

The combination of Proposition III.4 and Proposition III.6 allows us to invoke Theorem III.3 to establish that u^*, as defined in (7), renders $\mathcal{C}^s \subseteq \mathcal{C}$ forward-invariant. This, together with Lemma III.1, means that $\mathcal{C}^s$ solves Problem 1 for the case of full actuation.

The assumption of full actuation is not uncommon in the control of second-order systems, whether when studying safety [17], [18] or stability [27]. The following result shows that the result of Proposition III.6 still holds for sufficiently large compact input sets $\mathcal{U}$.

Corollary III.7 (Compact Input Set Suffices for Control-Invariance)**.** *Assume that, for all $x \in \partial\mathcal{C}^s$, $G_2(x)$ is right invertible. Then, under Assumption 1, choosing γ and ϵ such that $\gamma\delta > \epsilon$ ensures that Condition 1 is satisfied for some compact input set $\mathcal{U} \subsetneq \mathbb{R}^m$.* •

Proof. Since, by Proposition III.4, $\partial\mathcal{C}^s$ is compact and the left-hand side of (8) is continuous in x, $a_i^\top(\gamma x_2 + f_2(x))$ is bounded in $\partial\mathcal{C}^s$. But $a_i^\top G_2(x) G_2(x)^\dagger y_x = a_i^\top y_x \geq \gamma\delta - \epsilon > 0$. So there is a finite ρ that validates (8) for all x with $u_x = \rho G_2(x)^\dagger y_x$. Noting the boundedness of y_x and $G_2(x)^\dagger$ in $\partial\mathcal{C}^s$ completes the proof. □

This result does not specify how large the input set should be. This is the task we tackle in the forthcoming section.

IV. Determining Input Magnitude for Control Invariance of Euler-Lagrange Systems

In this section we consider the class of Euler-Lagrange systems [27] and study how large the input set should be to render $\mathcal{C}^s$ control-invariant.

Assumption 2 (Input Set Structure and Euler-Lagrange Systems Properties)**.** Let $\mathcal{U} = \{u \in \mathbb{R}^m \mid \|u\| \leq d\}$ for some $d > 0$. Further, assume the dynamics (1) is such that:

(a) the matrix function $G_2(x)$ is only dependent on x_1 and has right inverse $G_2^\dagger(x_1)$ defined for all $x \in \mathcal{C}$; and
(b) the function $f_2(x) = f_2^1(x_1) + f_2^2(x)$, with $\|f_2^2(x)\| \leq k_2\|x_2\|$. •

Assumption 2(a) can be interpreted as having the inertia matrix $G_2^\dagger(x)$ solely depend on the system's positions (and not on the system's velocity). Assumption 2(b) splits the forcing on the system into a component generated by potential fields, such as gravity, $f_2^1(x_1)$, and a component containing other forces, such as damping, $f_2^2(x)$. It also asks that the magnitude of the latter to be at most proportional to the system's velocity $\|x_2\|$. Define constants k_1, k_G by

$$k_1 := \max_{x\in\mathcal{C}} \|f_2^1(x_1)\| \quad \text{and} \quad k_G := \max_{x\in\mathcal{C}} \|G_2^\dagger(x_1)\|. \tag{11}$$

The existence of these constants is ensured by the continuity of f_2 and $G_2^\dagger$ and the compactness of $\mathrm{Proj}_n(\mathcal{C})$, cf. [28, Thm. 4.16]. Our approach to establishing control-invariance under Assumption 2 has two steps. First, we show that the velocity magnitude $\|x_2\|$ in the safe set $\mathcal{C}^s$ can be arbitrarily bounded by the design parameter γ, cf. Lemma IV.1. Second, we show that $\mathcal{U}$ in Assumption 2 is sufficient for control-invariance when $\|x_2\|$ is forced to be small enough through suitable design of γ, cf. Theorem IV.2.

Lemma IV.1 (Bound on Velocity Magnitude in Safe Set)**.** *Under Assumption 1, there exists a constant c that depends only on $\{a_i, b_i\}_{i\in\mathcal{I}}$ defining $\{h_i\}_{i\in\mathcal{I}}$ such that, for all γ and ϵ satisfying $\gamma\delta > \epsilon$, $\|x_2\| \leq \gamma c$ for all $x = (x_1, x_2) \in \mathcal{C}^s$.* •

Proof. By Proposition III.4, $\mathcal{C}^s$ is bounded. Therefore, each of its (finitely many) components $\cap_{i\in\bar{\mathcal{I}}^\ell}\mathcal{C}_i^s$ is bounded too. Since each $\mathcal{C}_i^s$ is a half-space, $\cap_{i\in\bar{\mathcal{I}}^\ell}\mathcal{C}_i^s$ is a bounded polytope. Given $\ell \in \bar{\mathcal{L}}$, consider the n programs

$$\begin{aligned} x_j^* = \arg\max &\,|e_j^\top x_2| \\ \text{s.t. } &x \in \cap_{i\in\bar{\mathcal{I}}^\ell}\mathcal{C}_i^s, \end{aligned}$$

where $j \in \{1, \dots, n\}$. Let $j_\ell^* := \arg\max_{1\leq j\leq n}\{|e_j^\top x_{j,2}^*|\}$, where $x_{j,2}^*$ denotes the last n components of x_j^*, and $x^{*,\ell} := x_{j_\ell^*}^*$. Since $x^{*,\ell} \in \mathbb{R}^{2n}$ is a solution to a linear program over a polytope, it is a vertex of the polytope [29, Thm. 2.4]. By [30, Thm. 10.4], there are $2n$ indices $\mathcal{I}_v \subseteq \bar{\mathcal{I}}^\ell$ such that $B_i(x^{*,\ell}) = 0$, for all $i \in \mathcal{I}_v$. Direct evaluation yields

$$\begin{aligned} a_i^\top \begin{bmatrix} I & \mathbf{0}_{n\times n} \end{bmatrix} x^{*,\ell} &= -b_i, \text{ if } i \in \mathcal{I}_v \cap \mathcal{I}, \\ a_i^\top \begin{bmatrix} \gamma I & I \end{bmatrix} x^{*,\ell} &= -\gamma b_i + \epsilon, \text{ if } i + r \in \mathcal{I}_v. \end{aligned}$$

Stacking the above equations into one matrix equation gives

$$\begin{bmatrix} A_1 & \mathbf{0} \\ \gamma A_2 & A_2 \end{bmatrix} x^{*,\ell} = \begin{bmatrix} b_1 \\ \gamma b_2 + \epsilon\mathbf{1_n} \end{bmatrix}, \tag{12}$$

with appropriate matrices A_1, A_2, b_1 and b_2. Since $x^{*,\ell}$ is a unique solution as it is a vertex, the coefficient matrix in (12) is invertible, and thus is of rank $2n$. The matrices A_1 and A_2 have n columns, so their ranks are at most n. Since the rank of the block $\begin{bmatrix} \gamma A_2 & A_2 \end{bmatrix}$ is equal to the rank of A_2, then it must be that both $\mathrm{rank}(A_1) = \mathrm{rank}(A_2) = n$, and hence and both A_1 and A_2 are invertible. Solving for $x_2^{*,\ell}$, which is the vector comprising the last n components of $x^{*,\ell}$ gives $x_2^{*,\ell} = \gamma(A_2^{-1}b_2 - A_1^{-1}b_1) + \epsilon A_2^{-1}\mathbf{1_n}$. By definition of $x^{*,\ell}$ and j_ℓ^* and the triangle inequality, for all $j \in \{1, \dots, n\}$ and all $x = (x_1, x_2) \in \cap_{i\in\bar{\mathcal{I}}^\ell}\mathcal{C}_i^s$,

$$|e_j^\top x_2| \leq |e_{j_\ell^*}^\top x_2^{*,\ell}| \leq \gamma c_1' + \epsilon c_2' \leq \gamma(c_1' + \delta c_2'),$$

where $c_1' = |e_{j_\ell^*}^\top(A_1^{-1}b_1 - A_2^{-1}b_2)|$ and $c_2' = \|A_2^{-1}\mathbf{1_n}\|$. Since this holds for each $\ell \in \bar{\mathcal{L}}$ with the appropriate matrices A_1, A_2, b_1 and b_2, every component of $x_2 \in \mathcal{C}^s =$

$\cup_{j\in\bar{\mathcal{L}}}\cap_{i\in\bar{\mathcal{I}}^\ell}\mathcal{C}_i^s$ is bounded by $\gamma c'^*$, where c'^* is the greatest of the finite constants $c'_1+\delta c'_2$ for the different ℓ's. Therefore, $\|x_2\|\leq\gamma(\sqrt{n}c'^*)$. □

An interesting byproduct of Lemma IV.1 is that the design parameter γ can be used to ensure safety constraints on the velocity too. In fact, one can leverage the result to meet any safety specification on the magnitude of x_2 by taking a sufficiently small γ. The smaller the parameter γ is, the slower the system will move, and therefore control objectives other than safety may be hindered. Notice also that, no matter how small γ is taken to be, with a suitably small ϵ, none of the originally allowed positions in $\mathcal{C}$ will be lost in $\mathcal{C}^s$, cf. Lemma III.1. We are now ready to show that Assumption 2 is enough to establish the existence of a design parameter γ that makes $\mathcal{C}^s$ control-invariant with bounded input set $\mathcal{U}=\{u\in\mathbb{R}^m \mid \|u\|\leq d\}$.

Theorem IV.2 (Control-Invariance with Input Constraints)**.** *Under Assumptions 1 and 2, if $d-k_Gk_1>0$, cf. (11), then γ and ϵ can be chosen to ensure Condition 1 is satisfied.* •

Proof. Let γ be such that $\gamma(k_2+\gamma)k_Gc<\frac{1}{2}(d-k_1k_G)$, where c is the constant in Lemma IV.1. Now, select ϵ to satisfy $\gamma\delta>\epsilon$. For all $x\in\partial\mathcal{C}^s$ and $i\in\tilde{\mathcal{I}}(x)$ with $i>r$, recall from the proof of Proposition III.6 that $a_i^\top y_x>0$, where y_x is as defined there. Let β_x be a positive constant satisfying $\beta_x\|y_x\|\leq\frac{1}{2}(d-k_1k_G)$ and choose $u_x=-G_2^\dagger(x)(f_2(x)+\gamma x_2)+\beta_x y_x$. By the triangle inequality and the definition of the matrix induced 2-norm,

$$\|u_x\|\leq\|G_2^\dagger(x)\|(\|f_2^1(x_1)\|+\|f_2^2(x)\|+\gamma\|x_2\|)+\beta_x\|y_x\|.$$

By Assumption 2, Lemma IV.1, and (11), $\|u_x\|\leq k_Gk_1+\gamma(k_2+\gamma)k_Gc+\beta_x\|y_x\|$. By our choice of γ and β_x, $\|u_x\|\leq k_Gk_1+\frac{1}{2}(d-k_Gk_1)+\frac{1}{2}(d-k_Gk_1)=d$, and thus $u_x\in\mathcal{U}$. Noting that $u_x\in\mathcal{U}$ validates (8) for all $i\in\tilde{\mathcal{I}}(x)$ with $i>r$ completes the proof. □

The condition $d-k_Gk_1>0$ in Theorem IV.2 amounts to having enough control authority to counter the force caused by the potential field. In the absence of such a force (i.e., when $k_1=0$), as in the case of a robot moving in a plane perpendicular to gravity, this condition is valid by default. For such systems, $\mathcal{C}^s$ can be made safe for any bounded $\mathcal{U}$.

V. Safe Control of Robotic Manipulator

In this section, we illustrate our results on a 2-degree-of-freedom planar elbow manipulator [31]. We build upon [17], which consider the same example with hyper-cubic safety constraints on the position, and generalize these to general polytopic constraints. The system consists of two horizontally oriented links (no gravity) hinged to each other, with the first link hinged to a fixed frame. Torque manipulation is available at each joint. The system model is

$$\underbrace{\begin{bmatrix}c_{11}+c_{12}\cos(\theta_2) & c_{13}+c_{14}\cos(\theta_2)\\ c_{22}+c_{23}\cos(\theta_2) & c_{21}\end{bmatrix}}_{M}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix}= \tag{13}$$

$$\underbrace{\begin{bmatrix}c_{15}\sin(\theta_2)\dot\theta_2 & c_{16}\sin(\theta_2)\dot\theta_2\\ c_{24}\sin(\theta_2)\dot\theta_1 & 0\end{bmatrix}}_{C}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} + \begin{bmatrix}u_1\\ c_{25}\cos(\theta_1+\theta_2)+u_2\end{bmatrix},$$

with constant coefficients c_{ij} dependent on the links' lengths and masses, cf. [31]. We require the arm to avoid collision with a wall, as shown in Figure 1a.

In the space of angles, the positional constraints $\mathcal{C}$ correspond to the dotted hexagon in Figure 1c. Thus, y_I can be taken as the origin for all the intersections. The parameter δ can be calculated according to (6) to be $\delta=\pi/2$. We construct the control-invariant set $\mathcal{C}^s$ as described in Section III, with $\gamma=10$ and $\epsilon=0.1$, which satisfy $\gamma\delta>\epsilon$. The controller u^* is computed according to (7), with the objective function $\|u-u_{\text{nom}}(t)\|^2$. Here, u_{nom} is a nominal controller that tracks $r=(\pi\sin(t),\frac{\pi}{2}\sin(4t))$, cf. [32],

$$u_{\text{nom}}(t)=M(\ddot r-\dot e-e)+C[\dot\theta_1\ \ \dot\theta_2]^\top,$$

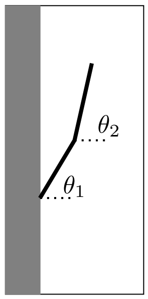
where $e=(\theta_1,\theta_2)-r$ and M and C are as defined in (13). We take $\alpha(r)=40r$. The high value 40 is not necessary for feasibility of the program (7), but we select it to allow the trajectory to come close to the boundary of the safe set.

Figure 1b shows the time evolution of the joint angles under the nominal controller and under the safety-filtered controller u^*, along with the constraint evolution as viewed from the position of the safe trajectory. Figure 1c shows a phase portrait of the evolution of the angles under the nominal and the safe controllers. Note how the safe controller renders $\mathcal{C}$ invariant, as guaranteed by Corollary III.7, while the nominal controller does not.

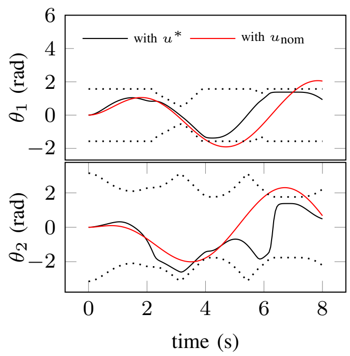
We also note that the system (13) satisfies Assumption 2 with $f_2^1(x)=M^{-1}C\begin{bmatrix}\dot\theta_1 & \dot\theta_2\end{bmatrix}^\top$, $f_2^1(x)=M^{-1}\begin{bmatrix}0 & c_{25}\cos(\theta_1+\theta_2)\end{bmatrix}^\top$ and $G_2(x_1)=M^{-1}$. Thus, by Theorem IV.2, any $\mathcal{U}=\{u\in\mathbb{R}^2 \mid \|u\|\leq d\}$ such that $d>k_Gk_1$ suffices for invariance, with a sufficiently small γ and a suitably chosen ϵ. This is reflected in Figure 1d, which shows the effect of the design parameter γ on the control and velocity magnitudes. As shown there, lower values of γ allow for safety with lower control magnitudes, at the expense of reducing the velocity of the execution. Finally, we note that extending this example to higher degrees of freedom adds complexity to the polytopic representation of the constraints but, once available, the computation of the control design remains the same.

VI. Conclusions

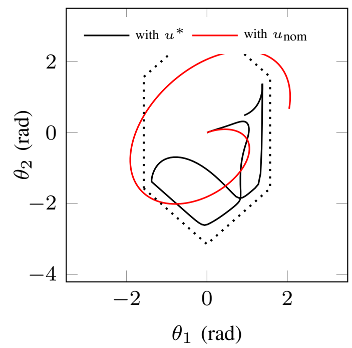
Given a second-order system and positional safety specifications described by linear boundaries, we have identified conditions that allow the explicit construction of a verifiably safe set in the full state space. We have also designed an associated QP controller that ensures this set is safe. We have shown that the identified conditions are always satisfied by fully actuated systems and, in the case of Euler-Lagrange systems, we have shown how the controller design can incorporate velocity and input constraints. We believe



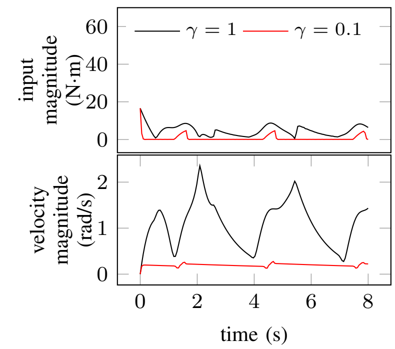
(a) A schematic of the 2-degrees-of-freedom robot arm. The gray space is the wall that should be avoided during manipulation.



(b) Time evolution of the link angles. The dotted lines are the safety constraints as seen from the safe trajectory.



(c) Trajectories on the position plane, showing the invariance enforced by the controller u^* instead of u_{nom}.



(d) Time evolution of the average velocity and average control effort on both links for different values of γ.

Fig. 1: Simulation results for safe control of a 2-link robot arm.

the approach presented here will be helpful in the design of safety-critical controllers for second-order systems with other forms of underactuation, something we plan to study in the future. Other extensions include incorporating time-varying safety considerations, robustifying the approach to handle system uncertainties, and applying a similar safe-set construction to specifications that are not necessarily linear.

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