Two Roads to Koopman Operator Theory for Control: Infinite Input Sequences and Operator Families

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Abstract—**The Koopman operator, originally defined for dynamical systems without input, has inspired many applications in control. Yet, the theoretical foundations underpinning this progress in control remain underdeveloped. This paper investigates the theoretical structure and connections between two extensions of Koopman theory to control: (i) Koopman operator via infinite input sequences and (ii) the Koopman control family. Although these frameworks encode system information in fundamentally different ways, we show that under certain conditions on the function spaces they operate on, they are equivalent. The equivalence is both in terms of the actions of the Koopman-based formulations in each framework as well as the function values on the system trajectories. Our analysis provides constructive tools to translate between the frameworks, offering a unified perspective for Koopman methods in control.**

I. Introduction

The Koopman operator provides a representation of the evolution of nonlinear systems through linear operators acting on vector spaces of functions. This viewpoint enables the use of both the algebraic structure of vector spaces and the spectral properties of linear operators to analyze the behavior of nonlinear dynamics. These features have motivated extensive research on the theoretical foundations and practical applications of the Koopman operator. The original formulation was developed for systems without inputs, where both theory and applications are now well established. In contrast, for control systems, although the practical side has received significant attention, the theoretical foundation remains in its early stages. In this paper, we investigate the connections between two established theoretical frameworks that extend Koopman operator theory to control systems. Our analysis studies how each framework encodes the information about the system and how information from one framework can be translated into the other, and establishes conditions on the respective function spaces under which both frameworks are equivalent.

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Literature Review

The Koopman operator represents the dynamics of a nonlinear system through a linear operator acting on a vector space of functions [2], [3]. The linearity of both the function space and the operator lead to well-structured algebraic properties that can be leveraged for the systematic study of complex nonlinear systems, particularly in situations where geometric methods are difficult to apply [4]. These properties have sparked significant research in analyzing complex systems with a myriad of applications, including stability analysis [5]–[8], signal processing [9], fluid dynamics [10], [11], power networks [12]–[14], biological systems [8], [15], and hybrid systems [16].

While the Koopman operator was initially formulated for systems without input, the literature has adapted Koopman-inspired methods to control applications. Many Koopman-inspired works in control do not propose a formal extension of the Koopman theory to systems with input, and instead rely on the idea of *"lifting"* to higher dimensions inspired by finite-dimensional linear forms associated with the Koopman operator for systems without input. These lifting techniques, combined with methods from classical system theory and differential geometry, have proven effective in practice. Among them, *lifted linear* models [17] are the simplest and most widely used, as they allow to leverage highly efficient linear control techniques such as linear quadratic regulators (LQR) and linear model predictive control (MPC). While lifted linear models lack the structural richness needed to capture cross terms between inputs and states or input nonlinearities, see e.g., [18] for a discussion, they remain effective in many applications, since feedback loops or MPC schemes can often compensate for the resulting model mismatch. The work [19] addresses some of the limitations of lifted linear models via a two-stage learning scheme using orthogonal and oblique projections and [20] provides a method to find lifted linear models based on the physical system structure. For the special case of control-affine systems, one can formulate an operator-theoretic framework using a family of Koopman generators that are affine in the input, see e.g., [21] for a discussion. This perspective has motivated the development of finite-dimensional lifted bilinear models [22]. The work [23] establishes error bounds for bilinear models and [24] provides controllers with closed-loop stability guarantees. We refer the reader to the recent overview [18] and the references therein for a comprehensive discussion of bilinear models and their properties. A different and effective way of modeling control systems that are not necessarily control-affine via the

Koopman operator is to fix finitely many constant values of the input and control the system by switching between the constant-input systems. This idea has been explored in different forms in the literature. The works [25], [26] have used variation of pulse-based control for monotone systems. The work [27] uses finite-dimensional approximations of the Koopman operators associated with systems created by setting the input to be constant and turns the control problem into a switched linear form. The work [28] takes a similar approach and solves the resulting optimal control problem for switched linear systems via a Markov Chain Monte Carlo (MCMC) method. The aforementioned lifted linear, bilinear, and switch linear models have found their way into many applications in controls, including optimal control [26], [29], [30], feedback linearization [31], safety and reachability analysis [32], [33], MPC [34], [35], control design based on control Lyapunov functions [36] and a wide range of robotics applications [20], [32], [37], [38].

Despite the widespread use of finite-dimensional Koopman-inspired lifted models in control and robotics, the theoretical basis of Koopman extensions for control has remained relatively unexplored. The work [34] formally extends Koopman operator theory to control systems by considering the all behaviors generated under all possible infinite input sequences. Specifically, by augmenting the state space with the space of infinite input sequences, one constructs a dynamical system (without input) that encapsulates all trajectories the original control system can produce. A Koopman operator is then naturally associated with this extended dynamical system. Importantly, this representation does not impose restrictive assumptions on the function space, nor does it depend on particular structural properties of the system (such as control-affine dynamics). The work [39] provides a different extension of Koopman theory to general (not necessarily input affine) control systems, termed Koopman Control Family (KCF). This framework characterizes the system behavior through the collection of Koopman operators associated with the systems without input obtained by fixing the control input to a constant value, for all possible values. KCF framework also provides a finite-dimensional form termed "input-state separable" model which captures the lifted linear, bilinear, and switched linear models mentioned above as special cases. KCF also provides methods to compute the optimal approximation of input-state separable forms as well as error bounds for the prediction of all functions in the subspace via the notion of invariance proximity [40], [41]. Finally, the work [42] provides an alternative operator-theoretic approach to encode the system behavior. This framework relies on a product space to distinguish between the effect of states and inputs, thus providing an effective way to capture and predict the system's behavior accompanied by data-driven methods.

Statement of Contributions

We study formal extensions of Koopman operator theory to discrete-time control systems that are not necessarily in control-affine form. Our exposition begins by studying two intuitive approaches to extending Koopman theory to control systems and showing that they are unable to capture the system's evolution beyond a single time step. This observations indicates that, contrary to common belief, extending the Koopman theory to control systems is far from trivial and requires rigorous theoretical analysis. We then turn our attention to two formal extensions of Koopman theory to control systems: (i) the Koopman operator via infinite input sequences [34] and (ii) the Koopman control family (KCF) [39]. These extensions encode system information in fundamentally different ways: the former employs a single operator but requires infinite input sequences, whereas the latter consists of infinitely many operators defined for constant inputs. To connect these frameworks, we utilize a parameterization of KCF using a single operator. Since different paradigms are based on substantially different function spaces, we establish connections between them through linear composition operators. With these connections at hand, we derive precise algebraic relationships between the frameworks, enabling the translation of information across them. Notably, under mild conditions on function spaces, we show that the frameworks are equivalent and capture the same information about the systems through linear operators. Moreover, we introduce a different type of equivalence, concerning the evolution of function *values* along system trajectories. Our analysis shows that the frameworks, albeit structurally different, are two sides of the same coin in terms of representing the control system. Moreover, the algebraic tools we present allow to translate information from one framework to the other, thus unifying them and providing a consistent Koopman-based representation for control systems.

Notation

We use $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ to represent natural, non-negative integer, real, and complex numbers. Given sets A and B, $A \subseteq (\subset) B$ means that A is a (proper) subset of B. Moreover, $A \cup B$ and $A \cap B$ are the union and intersection of A and B. In addition, $A \times B := \{(a,b) \mid a \in A,\ b \in B\}$ denotes the Cartesian product of A and B. We also denote the cardinality of A by $\mathrm{card}(A)$. Given a function $f : A \to B$ and the set $S \subseteq A$, we denote by $f\restriction_S : S \to B$, the function created by restricting the domain[1] of f to S. The image of set $E \subseteq A$ under function $f : A \to B$ is $f(E) := \{f(y) \mid y \in E\}$. Given that we study functions with various domains and co-domains throughout the paper, when convenient we use the domain and co-domain as sub- and super-scripts, respectively, e.g., for $f : A \to B$, we would use f_A^B. Given functions f and g with appropriate domains and co-domains, we denote their composition with $f \circ g$. We define the canonical projections $\pi_{A\times B}^A : A \times B \to A$ and $\pi_{A\times B}^B : A \times B \to B$ by maps $(a,b) \mapsto a$ and $(a,b) \mapsto b$, respectively. Given a set A, $\mathrm{id}_A : A \to A$ denotes the identity map.

[1] We also use a similar notation for restricting the domain of functions from product spaces $A \times B$ to one of the sets in the pair (e.g., A or B). In such cases the notation is explained where it is used.

II. Koopman Operator Viewpoint of Dynamical Systems

Here, we introduce the Koopman operator viewpoint following [43] for dynamical systems when no inputs are present. Consider the discrete-time system

$$x^+ = T(x), \quad x \in \mathcal{X}, \tag{1}$$

where $T : \mathcal{X} \to \mathcal{X}$ is a function describing the system's behavior and $\mathcal{X}$ is the state space. The Koopman operator provides an alternative viewpoint for describing the behavior of system (1) based on examining the evolution of functions defined over the state space. Let $\mathcal{F}$ be a vector space (over field $\mathbb{C}$) of complex-valued functions with domain $\mathcal{X}$. Assume $\mathcal{F}$ is closed under function composition with the map T. This means that, for all functions $f : \mathcal{X} \to \mathbb{C}$ in $\mathcal{F}$, one has $f \circ T \in \mathcal{F}$. Then, one defines the Koopman operator $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ associated with (1) as

$$\mathcal{K} f = f \circ T, \quad \forall f \in \mathcal{F}. \tag{2}$$

The action of the Koopman operator $\mathcal{K}$ can be viewed as evolving the value of each function f in $\mathcal{F}$, one timestep forward across all trajectories of (1), and encoding the outcome in the new function $\mathcal{K} f \in \mathcal{F}$,

$$[\mathcal{K} f](x) = f \circ T(x) = f(x^+), \quad \forall f \in \mathcal{F}, \, \forall x \in \mathcal{X}. \tag{3}$$

An important property of the Koopman operator is its linearity, i.e., for all $f_1, f_2 \in \mathcal{F}$ and $c_1, c_2 \in \mathbb{C}$,

$$\mathcal{K}(c_1 f_1 + c_2 f_2) = c_1 \mathcal{K} f_1 + c_2 \mathcal{K} f_2.$$

The linearity of the Koopman operator paves the way to many interesting applications. We do not deal with them here, but we mention the most important one: spectral analysis for nonlinear systems based on a carefully chosen function space $\mathcal{F}$ equipped with additional structure (e.g., norm or inner product). We refer the reader to e.g., [4], [44] for more information.

In this paper, our goal is to explore different frameworks to extend the Koopman operator viewpoint to control systems.

III. A Cautionary Tale of Naive Koopman Extensions to Control Systems

In this section we analyze two naive approaches to build operator viewpoints for control systems and show their limitations. This sheds light on the theoretical roadblocks one faces in extending the Koopman operator to control systems.

Consider the control system

$$x^+ = \mathcal{T}(x, u), \quad x \in \mathcal{X}, \quad u \in \mathcal{U}, \tag{4}$$

where $\mathcal{T} : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is the function defining the system's behavior, $\mathcal{X}$ is the state space, and $\mathcal{U}$ is the input space. We do not assume any special form (e.g., control affine, etc) on system (4). We also do not assume any structure on the sets $\mathcal{X}$ and $\mathcal{U}$ (not even being subsets of Euclidean space). As long as $\mathcal{T}$ with its given domain and codomain is a well-defined function, for each input, system (4) has a unique solution starting from each initial condition defined for all time.

A. Naive Approach 1: Simple Composition Operators

One might be tempted to extend the Koopman operator theory to control systems by defining a composition operator akin to (2) in the following sense: let $\mathcal{S}_1$ and $\mathcal{S}_2$ be vector spaces over $\mathbb{C}$, where $\mathcal{S}_1$ is comprised of complex-valued functions with domain $\mathcal{X}$ while $\mathcal{S}_2$ is comprised of complex-valued functions with domain $\mathcal{X} \times \mathcal{U}$. Moreover, let $f \circ \mathcal{T} \in \mathcal{S}_2$ for all $f \in \mathcal{S}_1$. Then one can define a linear composition operator $\mathcal{K}^{\text{naive}} : \mathcal{S}_1 \to \mathcal{S}_2$ such that

$$\mathcal{K}^{\text{naive}} f = f \circ \mathcal{T}, \quad \forall f \in \mathcal{S}_1. \tag{5}$$

Note that $\mathcal{K}^{\text{naive}}$ is a linear operator and, similarly to (3), pushes the value of the function f one step forward in time according to the trajectory of system (4), i.e.,

$$[\mathcal{K}^{\text{naive}} f](x, u) = f(\mathcal{T}(x, u)) = f(x^+), \quad \forall x \in \mathcal{X}, \, \forall u \in \mathcal{U}.$$

Therefore, $\mathcal{K}^{\text{naive}}$ has the two central properties of Koopman operator: linearity and evolving the function values on system trajectories. However, it suffers from the "*curse of domain mismatch*", which invalidates its use for multi-step prediction over the system trajectories: the domain and co-domain of $\mathcal{K}^{\text{naive}}$ are different. Hence, if $(f : \mathcal{X} \to \mathbb{C}) \in \mathcal{S}_1$, then $(\mathcal{K}^{\text{naive}} f : \mathcal{X} \times \mathcal{U} \to \mathbb{C}) \in \mathcal{S}_2$. Therefore, one *cannot* apply $\mathcal{K}^{\text{naive}}$ on $\mathcal{K}^{\text{naive}} f$ to move forward in time for a second timestep, since the function $\mathcal{K}^{\text{naive}} f$ does not belong to the domain of $\mathcal{K}^{\text{naive}}$.

The curse of domain mismatch is a major drawback of the operator $\mathcal{K}^{\text{naive}}$, since it does not allow for multi-step prediction. It is worth noting that, when u in (4) represents a disturbance or process noise rather than a control input, and follows a probability distribution (which renders the system to be stochastic), one can resolve the domain mismatch issue by taking the expectation over u, see [45][2]. However, since here we focus on open-loop control systems, we do not explore this direction further and refer the interested reader to [45, Section 4] for a detailed treatment of this approach.

B. Naive Approach 2: Treating the Input as a State

The second naive approach to extend the Koopman operator theory to control systems treats the input on the same grounds as the system state. One might be tempted to address the curse of domain mismatch by treating the input as an augmented state and then defining a Koopman operator with action similar to (2). To make this clear, consider a vector space (over $\mathbb{C}$) of functions $\mathcal{S}$ comprised of complex-valued functions with domain $\mathcal{X} \times \mathcal{U}$ and define a naive Koopman operator for (4) denoted by $\mathcal{K}^{\text{input-aug}} : \mathcal{S} \to \mathcal{S}$, whose action on each function is defined similarly to (3) as

$$\begin{aligned} &[\mathcal{K}^{\text{input-aug}} f](x, u) = f(x^+, u^+) \\ &\quad = f(\mathcal{T}(x, u), u^+), \quad \forall f \in \mathcal{S}, \, \forall x \in \mathcal{X}, \, \forall u, u^+ \in \mathcal{U}. \end{aligned} \tag{6}$$

One already can see an issue with this equation: for an open-loop system, the input in future timesteps u^+ is arbitrary and

[2] One can also envision a similar argument for applications in probabilistic reachability analysis.

does not depend on the state x or the input u at current timestep. For example, given state and input pair $(x,u) \in \mathcal{X} \times \mathcal{U}$ at current time, the state and input pair at the next timestep can be $(\mathcal{T}(x,u), u_1)$ or $(\mathcal{T}(x,u), u_2)$ for *different* $u_1, u_2 \in \mathcal{U}$ (if $\mathcal{U}$ contains more than one point). Therefore, it is not even clear whether the operator $\mathcal{K}^{\text{input-aug}}$ is well defined. The next result establishes a necessary condition for $\mathcal{K}^{\text{input-aug}}$ to be well defined, which turns out to be quite restrictive.

Lemma 3.1: (Necessary Condition for Well-defined $\mathcal{K}^{\text{input-aug}}$): Let the operator $\mathcal{K}^{\text{input-aug}} : \mathcal{S} \to \mathcal{S}$ associated with open-loop system (4) be well defined. Define the range of the dynamic map $\mathcal{T}$ in (4) as follows

$$\mathcal{R}(\mathcal{T}) := \{y \in \mathcal{X} \mid \exists x \in \mathcal{X}, \exists u \in \mathcal{U} \text{ s.t. } y = \mathcal{T}(x,u)\}. \quad (7)$$

Then, at least one of the following hold:

(a) the set $\mathcal{U}$ is a singleton;

(b) For each $f \in \mathcal{S}$, its restriction to $\mathcal{R}(\mathcal{T}) \times \mathcal{U}$ is independent of the second argument.

Proof: At an arbitrary timestep k, let $x \in \mathcal{X}$ and $u \in \mathcal{U}$ be the state and input of the system. Since operator $\mathcal{K}^{\text{input-aug}}$ is well defined, it maps every function $f \in \mathcal{S}$ to a *unique* function $\mathcal{K}^{\text{input-aug}} f \in \mathcal{S}$. Moreover, since system (4) is open loop, the input at a future timestep u^+ is arbitrary. Therefore, one can choose the following scenarios: take $u^+ = u_1$ or $u^+ = u_2$ for arbitrary $u_1, u_2 \in \mathcal{U}$. Since equation (6) holds for all $x \in \mathcal{X}$, all $u, u^+ \in \mathcal{U}$, and all $f \in \mathcal{S}$, one can write

$$f(\mathcal{T}(x,u), u_1) = [\mathcal{K}^{\text{input-aug}} f](x,u) = f(\mathcal{T}(x,u), u_2), \quad \forall f \in \mathcal{S}, \forall x \in \mathcal{X}, \forall u, u_1, u_2 \in \mathcal{U}.$$

Noting that $\mathcal{R}(\mathcal{T})$ consists of all $\mathcal{T}(x,u)$ for all $(x,u) \in \mathcal{X} \times \mathcal{U}$, the previous equation is equivalent to

$$f\restriction_{\mathcal{R}(\mathcal{T}) \times \mathcal{U}} (y, u_1) = f\restriction_{\mathcal{R}(\mathcal{T}) \times \mathcal{U}} (y, u_2), \quad \forall f \in \mathcal{S},\ \forall y \in \mathcal{R}(\mathcal{T}),\ \forall u_1, u_2 \in \mathcal{U}.$$

Since $f\restriction_{\mathcal{R}(\mathcal{T}) \times \mathcal{U}}$ is a function, the equation above holds under one of the following scenarios:

(a) for all $u_1, u_2 \in \mathcal{U}$, $u_1 = u_2$;

(b) for all $f \in \mathcal{S}$, $f\restriction_{\mathcal{R}(\mathcal{T}) \times \mathcal{U}}$ does not depend on the second variable (the input at next timestep).

If the former holds, it follows that $\mathcal{U}$ contains only one element and the proof is complete. Otherwise, the latter must hold for all $f \in \mathcal{S}$, concluding the proof. ■

Lemma 3.1 reveals a serious limitation of the operator defined in (6). Lemma 3.1(a) essentially means that (4) is not a control system, since u is a constant and can be viewed as a parameter and not as a control input. Lemma 3.1(b) means that $\mathcal{K}^{\text{input-aug}}$ cannot encode multi-step behavior in open-loop systems, as we explain next. One should note that by definition $\mathcal{R}(\mathcal{T}) \subseteq \mathcal{X}$ is forward invariant (given all possible inputs) and all the trajectories of the system will end up in $\mathcal{R}(\mathcal{T})$ after at most one timestep. Therefore, the effect of functions $f \in \mathcal{S}$ on the system trajectories after the first timestep can be *completely* captured by their restriction to $\mathcal{R}(\mathcal{T}) \times \mathcal{U}$. However, based on Lemma 3.1(b), all these restrictions are independent of the input and *cannot* encode its effect. This implies the operator $\mathcal{K}^{\text{input-aug}}$ in (6) is unable to capture the system's behavior for longer than one timestep.

Note the parallelism between this discussion and the curse of domain mismatch for $\mathcal{K}^{\text{naive}}$ in Section III-A, implying that both operators $\mathcal{K}^{\text{naive}}$ in (5) and $\mathcal{K}^{\text{input-aug}}$ in (6) fail to encode multistep trajectories. Next, we remark that despite its limitation, the operator (6) can be useful for the analysis of the closed-loop behavior.

Remark 3.2: (Usefulness of $\mathcal{K}^{\text{input-aug}}$ for Closed-Loop Systems): Although $\mathcal{K}^{\text{input-aug}}$ in (6) is of limited utility when it comes to open-loop control systems, this operator can be useful if the system is closed loop, the input sequence is fixed in advance, or it complies with a predetermined dynamics which determines the input *uniquely*. In these cases, $\mathcal{K}^{\text{input-aug}}$ can be viewed as Koopman operator associated with the system created by fixing the input structure (which means the resulting system does not admit a *control* input), see [46]. □

The naive approaches (5)-(6) to extend the Koopman operator theory to open-loop control systems reveal a major difficulty arising from the fact that the input has a fundamentally different role compared to the state of a control system: state follows a prescribed dynamic map while input in an open loop system is arbitrary and can change the behavior of the dynamic map itself.

Remark 3.3: (What Went Wrong? Input $\neq$ State): The difference between the roles of state and input highlights a subtle but crucial point in extending Koopman operator theory to control. The Koopman operator in (2) is simply an alternative representation of the dynamical map T, cf. (1), in an appropriate function space. Without the underlying map T, the Koopman operator has no meaning. Unlike the state, which evolves according to a map, the input in an open-loop control system does **not** follow any predefined evolution. Hence, there is no map that can be directly represented as a linear operator. Consequently, one must develop other approaches to properly capture the effect of input, which is what we discuss next. □

IV. Formal Extensions of Koopman Operator Theory to Control Systems

The difficulty of extending the Koopman operator theory to control systems lies in the fact that input to an open-loop system is arbitrary. Changing the input sequences may drastically alter the system behavior. Therefore, in extending the Koopman operator theory to control systems, one should take into account all possible behaviors arising from different input sequences.

To the best of our knowledge, there are two general extensions[3] of the Koopman operator theory to control (not necessarily control-affine) systems and both turn the control system

[3]By a general extension, we mean extensions based on general operator-theoretic descriptions for general (not necessarily control affine) nonlinear systems. There exist methods based on finite-dimensional lifting approaches (e.g., super-linearization [17] or bilinearization [22]) and operator theoretic methods based on specific assumptions on dynamic maps (e.g., control affine [21]), which we do not discuss here.

into systems without input. The first approach [34] achieves this by considering all possible infinite input sequences, while the second approach [39] considers all possible systems that one can build by setting the input in (4) to be a constant. We describe both next.

A. Koopman Operator via Infinite Input Sequences

Following [34], consider the space $\ell(\mathcal{U})$ comprised of all infinite sequences $\mathbf{u} := (u_n)_{n=0}^{\infty}$, where $u_n \in \mathcal{U}$ for all $n \in \mathbb{N}_0$. Then, one can define a system without input on the set $\mathcal{X} \times \ell(\mathcal{U})$ as

$$\begin{bmatrix} x \\ \mathbf{u} \end{bmatrix}^+ = \begin{bmatrix} \mathcal{T}(x, \mathbf{u}(0)) \\ S_{\text{left}}\mathbf{u} \end{bmatrix}, \tag{8}$$

where $S_{\text{left}} : \ell(\mathcal{U}) \to \ell(\mathcal{U})$ is the left shift operator defined by the mapping $\big(u(0), u(1), \dots\big) \mapsto \big(u(1), u(2), \dots\big)$.

For convenience, we denote the system in (8) by a tuple notation

$$(x, \mathbf{u})^+ = \mathcal{T}^\infty(x, \mathbf{u}) := (\mathcal{T}(x, \mathbf{u}(0)), S_{\text{left}}\mathbf{u}), \quad \forall (x, \mathbf{u}) \in \mathcal{X} \times \ell(\mathcal{U}). \tag{9}$$

Note that the system defined by $\mathcal{T}^\infty : \mathcal{X} \times \ell(\mathcal{U}) \to \mathcal{X} \times \ell(\mathcal{U})$ is now a system without input and admits a Koopman operator similarly to (2). Let $\mathcal{F}^\infty$ be a vector space (over $\mathbb{C}$) of complex-valued functions with domain $\mathcal{X} \times \ell(\mathcal{U})$ that is closed under composition with $\mathcal{T}^\infty$. Then, we define the following Koopman operator $\mathcal{K}^\infty : \mathcal{F}^\infty \to \mathcal{F}^\infty$ as

$$\mathcal{K}^\infty f = f \circ \mathcal{T}^\infty, \quad \forall f \in \mathcal{F}^\infty. \tag{10}$$

Unlike the approaches in Section III, the operator $\mathcal{K}^\infty$ is always well-defined as long as the function space $\mathcal{F}^\infty$ is closed under composition with $\mathcal{T}_\infty$. Note that the operator $\mathcal{K}^\infty$ is the Koopman operator associated with the system (9), which is not the control system (4). Therefore, we need to show that it can capture the information about the control system (4). To achieve this goal we rely on the following notion.

Definition 4.1: (Control-Independent Functions in $\mathcal{F}^\infty$ and State Component): The function $f : \mathcal{X} \times \ell(\mathcal{U}) \to \mathbb{C}$ in $\mathcal{F}^\infty$ is *control-independent* if $f(x, \mathbf{u_1}) = f(x, \mathbf{u_2})$ for all $x \in \mathcal{X}$ and all $\mathbf{u_1}, \mathbf{u_2} \in \ell(\mathcal{U})$. Alternatively, f can be decomposed as

$$f(x, \mathbf{u}) = f_{\mathcal{X}}(x) 1_{\ell(\mathcal{U})}(\mathbf{u}), \quad \forall (x, \mathbf{u}) \in \mathcal{X} \times \ell(\mathcal{U}),$$

where $1_{\ell(\mathcal{U})} : \ell(\mathcal{U}) \to \mathbb{C}$ is the constant function equal to 1. We call $f_{\mathcal{X}} : \mathcal{X} \to \mathbb{C}$ the *state component* of f. We denote by $\mathcal{F}^\infty_{\text{CI}}$ the set of all control-independent functions in $\mathcal{F}^\infty$. □

Although the domain of control-independent functions is $\mathcal{X} \times \ell(\mathcal{U})$, they only depend on the part corresponding to the state of the original control system (4) (which is $\mathcal{X}$) and discard the information from $\ell(\mathcal{U})$. Therefore, control-independent functions can be completely captured via their state component.

The next result shows that control-independent functions recover the information of the trajectories of original control system (4) through their state components.

Lemma 4.2: (Encoding Information of Control System (4) *via the Action of $\mathcal{K}^\infty$ on Control-Independent Functions):* Let $f \in \mathcal{F}^\infty_{\text{CI}}$ and let $\{x_k\}_{k \in \mathbb{N}_0}$ be the trajectory of (4) from initial condition x_0 with input sequence $\mathbf{u} = (u_0, u_1, \dots)$. Then,

$$[(\mathcal{K}^\infty)^k f](x_0, \mathbf{u}) = f_{\mathcal{X}}(x_k), \quad \forall k \in \mathbb{N}_0,$$

where $(\mathcal{K}^\infty)^k$ is the composition, k times, of $\mathcal{K}^\infty$ with itself. □

The proof of Lemma 4.2, omitted for space reasons, trivially follows from an inductive process on $\mathcal{K}^\infty$ and applying the decomposition in Definition 4.1. Lemma 4.2 shows that one can use control-independent functions in conjunction with the operator $\mathcal{K}^\infty$ to encode information from the trajectories of (4). A simple example of control-independent functions are state observables: given $\mathcal{X} \subseteq \mathbb{R}^n$, let $\mathcal{F}^\infty$ contain the function h_i, $h_i(x, \mathbf{u}) = x^{(i)}$, where $x^{(i)}$ is the ith element of the state of (4), $i \in \{1, \dots, n\}$. Then, one can use Lemma 4.2 to extract the ith component of the control system's trajectories for all time using the operator $\mathcal{K}^\infty$.

The state space of system (9) is $\mathcal{X} \times \ell(\mathcal{U})$; therefore, to evaluate the dynamics, one requires a point in $\mathcal{X}$, the state space of (4), and an *infinite* sequence of inputs. Given that the domain of functions in $\mathcal{F}^\infty$ is also $\mathcal{X} \times \ell(\mathcal{U})$, the same requirements apply to evaluate them. Hence, it is not possible to truncate the input sequence for function evaluation. The infinite-input sequences are the reason why the operator $\mathcal{K}^\infty$ can encode complete information about system (4) and does not suffer from the same issues as the approaches in Section III.

Remark 4.3: (Finite-Dimensional Representations for Infinite-Input Sequences Framework): Applying Koopman-based techniques on digital computers usually involves approximations on finite-dimensional spaces often achieved through projections, which lead to information loss. Therefore, for such finite-dimensional forms, it is critical to provide error bounds on the prediction of all functions in the subspace and derive methods to learn such models from data. In the case of control systems, deriving finite-dimensional forms is considerably involved. Many works draw inspiration from the idea of *lifting* to obtain such forms. Lifted linear and bilinear models are common choices, motivated by their simplicity and ease of use. However, the connection of these models to the Koopman-based description is not always clear, and in some cases may be absent altogether[4]. Given that the functions in $\mathcal{F}^\infty$ fuse the information of states and infinite input sequences together, directly finding finite-dimensional forms for this framework is more difficult and, to the best of our knowledge, has not been done in the literature for general (not necessarily control-affine) control systems. □

[4]It is worth noting that for *continuous-time control-affine* systems, the lifted bilinear form (while still an approximation) can be used effectively for finite-horizon prediction in model predictive control schemes and related feedback designs. For detailed error bounds, and closed-loop guarantees, we refer the reader to [18] and references therein.

B. Koopman Control Family (KCF)

The second approach also turns the control system (4) into systems without input, so that Koopman operators akin to (2) can be used. Following [39], this construction is done by considering all possible systems that one can build by fixing the input to be constant in (4). This leads to

$$x^+ = \mathcal{T}_{u^*}(x) := \mathcal{T}(x, u \equiv u^*), \quad u^* \in \mathcal{U}, \tag{11}$$

where the set $\{\mathcal{T}_{u^*} : \mathcal{X} \to \mathcal{X}\}_{u^* \in \mathcal{U}}$ forms a family of systems without input. Note that the domain and co-domain of this parametric family match, since they are no longer control systems. Therefore, we can assign to each map a Koopman operator similarly to (2). Let $\mathcal{F}$ be a vector space of functions (over $\mathbb{C}$) of complex-valued functions that are closed under composition with the members of family $\{\mathcal{T}_{u^*} : \mathcal{X} \to \mathcal{X}\}_{u^* \in \mathcal{U}}$. Then, we define the Koopman control family (KCF) as $\{\mathcal{K}_{u^*} : \mathcal{F} \to \mathcal{F}\}_{u^* \in \mathcal{U}}$, where

$$\mathcal{K}_{u^*} f = f \circ \mathcal{T}_{u^*}, \quad \forall f \in \mathcal{F}, \quad \forall u^* \in \mathcal{U}. \tag{12}$$

The following result shows how the KCF encodes the trajectories of (4).

Lemma 4.4: (Encoding Information of Control System (4) *via KCF):* Let $g \in \mathcal{F}$ and let $\{x_k\}_{k \in \mathbb{N}_0}$ be the trajectory of (4) from initial condition x_0 with input sequence $\mathbf{u} = (u_0, u_1, \ldots)$. Then,

$$[\mathcal{K}_{u_0} \mathcal{K}_{u_1} \ldots \mathcal{K}_{u_{k-1}} g](x_0) = g(x_k), \quad \forall k \in \mathbb{N}_0. \qquad \square$$

The proof of Lemma 4.4 trivially follows from the definition of KCF and is omitted. One can use Lemma 4.4 to fully extract the system trajectories via state observables if they belong to $\mathcal{F}$: given $\mathcal{X} \subseteq \mathbb{R}^n$, let $\mathcal{F}$ contain the function o_i, $o_i(x) = x^{(i)}$, where $x^{(i)}$ is the ith component of x, $i \in \{1, \ldots, n\}$. Then, one can use Lemma 4.4 to fully recover the ith component of the trajectory of system (4) for all time.

Remark 4.5: (KCF and Switch-Based Koopman Control): Although not as formal extensions of Koopman operator theory to control systems, the idea of controlling nonlinear systems by considering finitely many possible constant inputs and switching between finite-dimensional approximations of the associated Koopman operators was already explored in the literature (see, e.g., [27]) prior to the introduction of KCF [39]. Moreover, the works [25], [26] also considered piece-wise constant input control for special types of systems. One can view KCF as a formal generalization of such approaches to the case of abstract vector spaces, uncountable input sets, and arbitrary control systems. We refer the reader to [39] for a detailed discussion of differences and how the finite-dimensional form derived from KCF generalizes lifted linear, lifted bilinear, and lifted switched-linear forms in the literature. $\square$

Remark 4.6: (Finite-Dimensional Representations for KCF): Given a finite-dimensional subspace $\mathcal{S} \in \mathcal{F}$, let $\Psi : \mathcal{X} \to \mathbb{C}^{\dim(\mathcal{S})}$ be a vector-valued function whose elements span $\mathcal{S}$. Then the finite-dimensional from for KCF, termed "input-state separable" model, is as follows

$$\Psi(x^+) = \Psi \circ \mathcal{T}(x, u) \approx \mathcal{A}(u)\Psi(x), \quad \forall x \in \mathcal{X}, u \in \mathcal{U}, \tag{13}$$

where $\mathcal{A} : \mathcal{U} \to \mathbb{C}^{\dim(\mathcal{S}) \times \dim(\mathcal{S})}$ is a matrix-valued function. The form in (13) is a result of a tight (necessary and sufficient) condition which cannot be relaxed [39, Theorem 4.3]. The input-state separable model is linear in $\Psi(x)$ (often referred to as lifted state) but generally nonlinear in input u. The reason for this is as follows: Koopman-based methods represent nonlinear dynamic maps via linear operators; however, unlike the state x, which is governed by the dynamic map (4), the free input u is arbitrary an is not governed by a map; therefore, one cannot represent the evolution of input via a linear operator. This distinction is the reason why the naive approaches in Section III fail, cf. Remark 3.3. Interestingly, the commonly used lifted linear, bilinear, and switched linear models mentioned above are all special cases of the input-state separable form. We refer the interested reader to [39] for the detailed derivation of input-state separable forms, error bounds for function prediction using the notion of invariance proximity, as well as data-driven learning with accuracy guarantees. $\square$

V. Motivation and Problem Statement

The question we address in this paper is how the frameworks described in Sections IV-A and IV-B are related to each other. Based on our discussion, one would expect them to be equivalent. However, this is not always the case and the answer depends on the choice of function spaces, as the following simple example shows.

Example 5.1: (Dependence on Function spaces): Let $\mathcal{F}^\infty = \text{span}(1_{\mathcal{X} \times \ell(\mathcal{U})})$ be the space of constant functions on $\mathcal{X} \times \ell(\mathcal{U})$ and $\mathcal{F} = B(\mathcal{X})$ be the space of bounded functions on $\mathcal{X}$ (both on the field $\mathbb{C}$). One can readily check that the operators in both frameworks are well defined. However, in this case, the operator $\mathcal{K}^\infty$ is trivial and does not capture any information about the system since constant functions do not change under composition with any map. On the other hand, $\mathcal{F}$ is richer and can capture some information about the system dynamics. One could switch the construction of spaces to $\mathcal{F}^\infty = B(\mathcal{X} \times \ell(\mathcal{U}))$ and $\mathcal{F} = \text{span}(1_{\mathcal{X}})$, in which case, KCF will not capture any information, while $\mathcal{K}^\infty$ captures some. $\square$

Example 5.1 makes it clear that the function spaces $\mathcal{F}^\infty$ and $\mathcal{F}$ should satisfy certain conditions for the frameworks to capture complete information about the system, as a necessary step to establish their equivalence.

Problem 1: (Equivalence of Formal Extensions of Koopman Theory to Control Systems): Given the frameworks of the Koopman operator via infinite input sequences ($\mathcal{K}^\infty$) and the Koopman control family (KCF):

(a) provide conditions on the function spaces $\mathcal{F}$ and $\mathcal{F}^\infty$ such that $\mathcal{K}^\infty$ and KCF can be converted to each other only via compositions with linear operators between the functions spaces;

(b) given access to $\mathcal{K}^\infty$ or KCF, provide constructive recipes to predict the action of the operators on functions in the other framework. This task should be performed at the highest level of generality with no reliance on specific structure of function spaces (e.g., topology, metric, norm, inner product, etc) or specific structures on the dynamics (e.g., being control affine). □

Beyond the mathematical relevance of establishing the equivalence between the two different frameworks, there are also important practical implications. Each framework handles system information in fundamentally different ways. Depending on the application, one framework may therefore be more convenient than the other. In practice, function spaces are often equipped with additional structures –such as norms or inner products– or are subject to constraints, for example through the use of tensor product spaces. For a given application, one can select the framework best suited to the task at hand. This choice is dictated directly by the nature of the application. For applications concerned with the general study of control systems under all possible inputs – such as the analysis of invariant quantities, the characterization of unreachable sets, or theoretical investigations into the spectral properties of operators – the infinite-input sequence framework is a natural choice, since it relies on a single operator. By contrast, in applications involving finite-dimensional representations or finite-time trajectories, such as finite-horizon prediction in model predictive control (MPC) or data-driven learning via trajectory data, the fact that the KCF framework avoids the need to work with infinite-input sequences when evaluating functions in the function space together with the availability of the input-state separable form (13) makes it more attractive.

VI. Connections Between Dynamical Systems in Different Frameworks

Both approaches in Section IV use infinitely many objects to provide Koopman-based descriptions: one relies on infinite input sequences while the other utilizes infinitely many constant-input systems. This infinite cardinality leads to practical difficulties. To address this, here we introduce a different, easier-to-work-with system and show it indirectly captures all the information needed to reconstruct the original system trajectories. We then utilize this system to study and unify the aforementioned extensions of Koopman operator theory to control systems.

A. Augmented System and its Associated Koopman Operator

We utilize the control system (4) to synthesize a new system without input, cf. [39],

$$\begin{bmatrix} x \\ u \end{bmatrix}^+ = \begin{bmatrix} \mathcal{T}(x,u) \\ u \end{bmatrix}, \quad x \in \mathcal{X}, \quad u \in \mathcal{U}.$$

Note that in this system, u is part of the state and does not evolve in time. For convenience, we use the following notation for the augmented system

$$(x,u)^+ = (\mathcal{T}(x,u), u) =: \mathcal{T}^{\text{aug}}(x,u), \tag{14}$$

where $\mathcal{T}^{\text{aug}} : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \times \mathcal{U}$ is the function defining the augmented system. Note that (14) does not have a control input; hence, given a suitable function space, admits a well-defined Koopman operator similarly to (2). Let $\mathcal{F}^{\text{aug}}$ be a vector space (over $\mathbb{C}$) of complex-valued functions with domain $\mathcal{X} \times \mathcal{U}$ that is closed under composition with $\mathcal{T}^{\text{aug}}$. Then, we define the augmented Koopman operator $\mathcal{K}^{\text{aug}} : \mathcal{F}^{\text{aug}} \to \mathcal{F}^{\text{aug}}$ as

$$\mathcal{K}^{\text{aug}} f = f \circ \mathcal{T}^{\text{aug}}, \quad \forall f \in \mathcal{F}^{\text{aug}}. \tag{15}$$

Unlike the system with infinite input sequences (9) or the family of constant input systems (11), the augmented system (14) does not rely on any infinite objects and therefore is easier to study and evaluate; hence, it serves as an effective intermediary between the two frameworks.

Similarly to Definition 4.1, we define a set of control-independent functions in $\mathcal{F}^{\text{aug}}$ to connect the action of $\mathcal{K}^{\text{aug}}$ to the trajectories of control system (4).

Definition 6.1: (Control-Independent Functions in $\mathcal{F}^{\text{aug}}$ and State Component): The function $f : \mathcal{X} \times \mathcal{U} \to \mathbb{C}$ in $\mathcal{F}^{\text{aug}}$ is *control-independent* if $f(x,u_1) = f(x,u_2)$ for all $x \in \mathcal{X}$ and all $u_1, u_2 \in \mathcal{U}$. Alternatively, f can be decomposed as

$$f(x,u) = f_{\mathcal{X}}(x) 1_{\mathcal{U}}(u), \quad \forall (x,u) \in \mathcal{X} \times \mathcal{U},$$

where $1_{\mathcal{U}} : \mathcal{U} \to \mathbb{C}$ is a constant function equal to 1. We call $f_{\mathcal{X}} : \mathcal{X} \to \mathbb{C}$ the *state component* of f. We denote the set of all control-independent functions in $\mathcal{F}^{\text{aug}}$ by $\mathcal{F}^{\text{aug}}_{\text{CI}}$. □

The next result shows that one can use the control-independent functions in conjunction with $\mathcal{K}^{\text{aug}}$ to extract information about the trajectories of (4).

Lemma 6.2: (Encoding Single-Step Information of Control System (4) *via the Action of $\mathcal{K}^{\text{aug}}$ on Control-Independent Functions):* For $f \in \mathcal{F}^{\text{aug}}_{\text{CI}}$, we have

$$[\mathcal{K}^{\text{aug}} f](x,u) = f_{\mathcal{X}}(x^+), \quad \forall (x,u) \in \mathcal{X} \times \mathcal{U}. \qquad \square$$

The proof of Lemma 6.2 directly follows from the definitions. It is crucial to note that unlike Lemma 4.2, the prediction in Lemma 6.2 only holds for one time step. Hence, $\mathcal{K}^{\text{aug}}$ is not a Koopman operator associated with control system (4). In fact, one can easily show that, in general, $\mathcal{K}^{\text{aug}}$ cannot directly capture the action of (4) for longer than a single timestep and suffers from similar issues as the naive approaches in Section III. However, $\mathcal{K}^{\text{aug}}$ has several important properties in *indirectly* parameterizing the KCF and deriving finite-dimensional forms for Koopman-based approaches, cf. [39].

B. All Introduced Input-Free Systems Capture the Behavior of the Control System

The next result shows that one can completely, but indirectly, recover the original system's behavior from the input-free dynamical systems introduced so far.

Proposition 6.3: (Recovering the Original System's Behavior from Different Input-Free Systems): Let $\{x_k\}_{k \in \mathbb{N}_0}$ be the trajectory of the original system (4) with initial condition x_0 for the input sequence $\mathbf{u} = (u_0, u_1, \ldots)$. Then, the trajectory can be fully recovered from the system with infinite input-sequences (9), the family of constant-input systems (11), and the augmented system (14) as follows: for all $k \in \mathbb{N}$,

(a) $x_k = [\pi^{\mathcal{X}}_{\mathcal{X} \times \ell(\mathcal{U})} \circ \big(\mathcal{T}_\infty\big)^k](x_0, \mathbf{u})$;
(b) $x_k = \mathcal{T}_{u_k} \circ \cdots \circ \mathcal{T}_{u_0}(x_0)$;
(c) $x_k = \pi^{\mathcal{X}}_{\mathcal{X} \times \mathcal{U}} \circ \mathcal{T}^{\text{aug}}(x_{k-1}, u_{k-1})$. □

The proof of Proposition 6.3 can be done via direct calculation. One should note the difference in Proposition 6.3(c) with the other parts, which is rooted in the fact that $\mathcal{T}^{\text{aug}}$ encodes the information of the system (4) only for one timestep. Proposition 6.3 shows that all the introduced frameworks can indirectly capture complete information about the original system. Therefore, by properly choosing the function spaces, one should be able to connect their associated Koopman-based structures. This is what we tackle in the next sections.

VII. Connecting the Function Spaces

Here, we take a step towards connecting the different operator-theoretic descriptions by first connecting their associated function spaces. As summarized in Table I, the domains of the each set of functions are the state space of the corresponding dynamics and are therefore different. We then need operations to change the domain of functions and move between function spaces. To achieve this goal, we use $\mathcal{F}^{\text{aug}}$ as an intermediary to connect $\mathcal{F}^\infty$ and $\mathcal{F}$.

Framework	Dynamics	State Space	Function Space
Infinite Sequences	$\mathcal{T}^\infty$	$\mathcal{X} \times \ell(\mathcal{U})$	$\mathcal{F}^\infty$
KCF	$\{\mathcal{T}_{u^*}\}_{u^* \in \mathcal{U}}$	$\mathcal{X}$	$\mathcal{F}$
Augmented	$\mathcal{T}^{\text{aug}}$	$\mathcal{X} \times \mathcal{U}$	$\mathcal{F}^{\text{aug}}$

TABLE I: Comparison of dynamics, state space, and function spaces in different frameworks.

A. Connecting $\mathcal{F}^\infty$ and $\mathcal{F}^{\text{aug}}$

To connect $\mathcal{F}^\infty$ and $\mathcal{F}^{\text{aug}}$, we provide two operations to switch the domain of functions between $\mathcal{X} \times \ell(\mathcal{U})$ and $\mathcal{X} \times \mathcal{U}$.

Definition 7.1: (Domain Restriction and Extension between $\mathcal{X} \times \ell(\mathcal{U})$ and $\mathcal{X} \times \mathcal{U}$): Let $f : \mathcal{X} \times \ell(\mathcal{U}) \to \mathbb{C}$ be a function in $\mathcal{F}^\infty$ and $g : \mathcal{X} \times \mathcal{U} \to \mathbb{C}$ a function in $\mathcal{F}$. Then,

(a) the *domain restriction* of f to $\mathcal{X} \times \mathcal{U}$, denoted $f\restriction_{\mathcal{X}\times\mathcal{U}}$, is

$$f\restriction_{\mathcal{X}\times\mathcal{U}} (x, u) := f(x, (u, u, \ldots)), \quad \forall (x, u) \in \mathcal{X} \times \mathcal{U}.$$

(b) the *domain extension* of g to $\mathcal{X} \times \ell(\mathcal{U})$, denoted g^∞, is

$$g^\infty(x, \mathbf{u}) := g(x, \mathbf{u}(0)), \quad \forall (x, \mathbf{u}) \in \mathcal{X} \times \ell(\mathcal{U}). \quad \square$$

Definition 7.1 provides an intuitive way to connect the function spaces. The following provides an equivalent description based on mappings, which later will be more convenient to reason with the operator-theoretic descriptions.

Lemma 7.2: (Mapping-based Connection between $\mathcal{X}\times\ell(\mathcal{U})$ and $\mathcal{X} \times \mathcal{U}$): Define the mappings[5] $R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})} : \mathcal{X} \times \ell(\mathcal{U}) \to \mathcal{X} \times \mathcal{U}$ and $E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}} : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \times \ell(\mathcal{U})$ as

$$R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})}(x, \mathbf{u}) = (x, \mathbf{u}(0)), \quad \forall (x, \mathbf{u}) \in \mathcal{X} \times \ell(\mathcal{U}), \quad (16a)$$

$$E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}}(x, u) = (x, (u, u, \ldots)), \quad \forall (x, u) \in \mathcal{X} \times \mathcal{U}. \quad (16b)$$

Then,

(a) $f\restriction_{\mathcal{X}\times\mathcal{U}} = f \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}}$, for all $f \in \mathcal{F}^\infty$;

(b) $g^\infty = g \circ R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})}$, for all $g \in \mathcal{F}^{\text{aug}}$;

(c) $R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})} \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}} = \text{id}_{\mathcal{X}\times\mathcal{U}}$. $\square$

We omit the proof of Lemma 7.2, which follows from the definitions. This result allows us to move between the functions spaces $\mathcal{F}^\infty$ and $\mathcal{F}^{\text{aug}}$ via linear composition operators, as we explain next.

Proposition 7.3: (Linear Operator Connection Between $\mathcal{F}^\infty$ and $\mathcal{F}^{\text{aug}}$): Assume $\mathcal{F}^{\text{aug}}$ and $\mathcal{F}^\infty$ satisfy:

(Ci) $f\restriction_{\mathcal{X}\times\mathcal{U}} \in \mathcal{F}^{\text{aug}}$, for all $f \in \mathcal{F}^\infty$;

(Cii) $g^\infty \in \mathcal{F}^\infty$, for all $g \in \mathcal{F}^{\text{aug}}$.

Define the operators $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} : \mathcal{F}^\infty \to \mathcal{F}^{\text{aug}}$ and $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} : \mathcal{F}^{\text{aug}} \to \mathcal{F}^\infty$ as

$$\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} f = f \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}}, \qquad \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g = g \circ R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})}.$$

Then,

(a) $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty}$ and $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}$ are well-defined and linear;

(b) $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} f = f\restriction_{\mathcal{X}\times\mathcal{U}}$, for all $f \in \mathcal{F}^\infty$;

(c) $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g = g^\infty$, for all $g \in \mathcal{F}^{\text{aug}}$;

(d) $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} = \text{id}_{\mathcal{F}^{\text{aug}}}$;

(e) $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}}) = \mathcal{F}^\infty_{\text{CI}}$;

(f) $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty}(\mathcal{F}^\infty_{\text{CI}}) = \mathcal{F}^{\text{aug}}_{\text{CI}}$.

Proof: (a) For $f \in \mathcal{F}^\infty$, using Lemma 7.2, we have $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} f = f \restriction_{\mathcal{X}\times\mathcal{U}}$. Therefore, $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} f \in \mathcal{F}^{\text{aug}}$ and the operator is well-defined. A similar argument holds for $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}$. Linearity of the operators directly follows from their definition.

(b)-(c) The proof follows from Lemma 7.2.

(d) For $g \in \mathcal{F}^{\text{aug}}$, using Lemma 7.2(c), one can write $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty} \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g = g \circ R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})} \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}} = g \circ \text{id}_{\mathcal{X}\times\mathcal{U}} = g$.

(e) We first prove $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}}) \subseteq \mathcal{F}^\infty_{\text{CI}}$. Let $g \in \mathcal{F}^{\text{aug}}_{\text{CI}}$. Then, by Definition 6.1, $g(x, u_1) = g(x, u_2)$, for all $x \in \mathcal{X}$, $u_1, u_2 \in \mathcal{U}$. Hence, we can write $[\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g](x, \mathbf{u_1}) = g(x, \mathbf{u_1}(0)) = g(x, \mathbf{u_2}(0)) = [\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g](x, \mathbf{u_2})$ for all $x \in \mathcal{X}$ and all $\mathbf{u_1}, \mathbf{u_2} \in \ell(\mathcal{U})$. Hence $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g \in \mathcal{F}^\infty_{\text{CI}}$.

Next, we prove $\mathcal{F}^\infty_{\text{CI}} \subseteq \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}})$. Given $f \in \mathcal{F}^\infty_{\text{CI}}$, let $g_f = f \restriction_{\mathcal{X}\times\mathcal{U}}$. Since f is control-independent, then $g_f = f \restriction_{\mathcal{X}\times\mathcal{U}}$ is also control-independent. Hence, we only need to show $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g_f = f$. For $(x, \mathbf{u}) \in \mathcal{X} \times \ell(\mathcal{U})$,

$$\begin{aligned}[\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}} g_f](x, \mathbf{u}) = f\restriction_{\mathcal{X}\times\mathcal{U}} (x, \mathbf{u}(0)) &= f(x, (\mathbf{u}(0), \mathbf{u}(0), \ldots)) \\ &= f(x, \mathbf{u}),\end{aligned}$$

where in the last equality we use that f is control-independent. This proves $\mathcal{F}^\infty_{\text{CI}} \subseteq \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}})$, and we conclude $\mathcal{F}^\infty_{\text{CI}} = \mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}})$.

(f) Directly follows from (d) and (e). ■

Proposition 7.3 provide conditions under which we can move between the function spaces $\mathcal{F}^\infty$ and $\mathcal{F}^{\text{aug}}$ via linear operators. These conditions on function spaces will later be essential in establishing an equivalence between the frameworks. We note that the restriction operator $\mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^\infty}$ is built with the extension map $E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}}$ and the extension operator $\mathcal{E}^{\mathcal{F}^\infty}_{\mathcal{F}^{\text{aug}}}$ is built with the restriction map $R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})}$. This is because the operators act from the left and the maps are composed from the right.

[5]The letters R and E in the name of mappings (16) refer to restriction and extension, respectively. We use similar terminology for maps and operators throughout the paper.

B. Connecting $\mathcal{F}$ and $\mathcal{F}^{\text{aug}}$

Here, we connect the function spaces $\mathcal{F}$ and $\mathcal{F}^{\text{aug}}$. This connection is more complicated than connecting $\mathcal{F}^{\infty}$ and $\mathcal{F}^{\text{aug}}$ because the KCF is generally comprised of uncountably many operators. In our previous work [39], we have analyzed the connection between KCF and the augmented operator, but the perspective here is slightly different, with milder conditions suited for the problem at hand. We start by providing suitable notions of domain restriction and extension between $\mathcal{X} \times \mathcal{U}$ and $\mathcal{X}$.

Definition 7.4: (Domain Restriction and Extension between $\mathcal{X} \times \mathcal{U}$ and $\mathcal{X}$): Let $f : \mathcal{X} \times \mathcal{U} \to \mathbb{C}$ be a function in $\mathcal{F}^{\text{aug}}$ and $g : \mathcal{X} \to \mathbb{C}$ a function in $\mathcal{F}$. Then,

(a) the family of *constant-input domain restrictions* of f to $\mathcal{X}$, denoted $\{f\restriction_{\mathcal{X},u\equiv u^*}\}_{u^* \in \mathcal{U}}$, is, for each $u^* \in \mathcal{U}$,

$$f\restriction_{\mathcal{X},u\equiv u^*}(x) := f(x,u^*), \quad \forall x \in \mathcal{X}.$$

(b) the *input-independent domain extension* of g to $\mathcal{X} \times \mathcal{U}$, denoted g_e, is

$$g_e(x,u) := g(x), \quad \forall (x,u) \in \mathcal{X} \times \mathcal{U}. \qquad \square$$

We note that there is a family of restrictions in Definition 7.4(a), instead of the one restriction in Definition 7.1(a). The next result, analogous to Lemma 7.2, provides mapping-based descriptions of the domain restriction and extension.

Lemma 7.5: (Mapping-based Connection Between $\mathcal{X} \times \mathcal{U}$ and $\mathcal{X}$): Define the mapping $R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}} : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and the family of mappings $\{E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*} : \mathcal{X} \to \mathcal{X} \times \mathcal{U}\}_{u^* \in \mathcal{U}}$ as

$$R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}}(x,u) = x, \quad \forall (x,u) \in \mathcal{X} \times \mathcal{U}, \tag{17a}$$
$$E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*}(x) = (x,u^*), \quad \forall x \in \mathcal{X}, \quad \forall u^* \in \mathcal{U}. \tag{17b}$$

Then,

(a) $f\restriction_{\mathcal{X},u\equiv u^*} = f \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*}$, for all $f \in \mathcal{F}^{\text{aug}}$ and $u^* \in \mathcal{U}$;
(b) $g_e = g \circ R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}}$, for all $g \in \mathcal{F}$;
(c) $R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}} \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*} = \text{id}_{\mathcal{X}}$, for all $u^* \in \mathcal{U}$. $\square$

The proof follows from the definitions. We rely on Lemma 7.5 to define appropriate linear operators to move between the function spaces $\mathcal{F}$ and $\mathcal{F}^{\text{aug}}$.

Proposition 7.6: (Linear Operator Connection Between $\mathcal{F}$ and $\mathcal{F}^{\text{aug}}$): Assume $\mathcal{F}$ and $\mathcal{F}^{\text{aug}}$ satisfy:

($\mathfrak{C}$i) $f\restriction_{\mathcal{X},u\equiv u^*} \in \mathcal{F}$, for all $f \in \mathcal{F}^{\text{aug}}$ and all $u^* \in \mathcal{U}$;
($\mathfrak{C}$ii) $g_e \in \mathcal{F}^{\text{aug}}$, for all $g \in \mathcal{F}$.

Define the family of operators $\{\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} : \mathcal{F}^{\text{aug}} \to \mathcal{F}\}_{u^* \in \mathcal{U}}$ and the operator $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\text{aug}}$ as

$$\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} f = f \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*}, \qquad \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} g = g \circ R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}},$$

for each $u^* \in \mathcal{U}$. Then,

(a) $\{\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}\}_{u^* \in \mathcal{U}}$ and $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}$ are well-defined and linear;
(b) $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} f = f\restriction_{\mathcal{X},u\equiv u^*}$, for all $f \in \mathcal{F}^{\text{aug}}$ and $u^* \in \mathcal{U}$;
(c) $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} g = g_e$, for all $g \in \mathcal{F}$;
(d) $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} = \text{id}_{\mathcal{F}}$, for all $u^* \in \mathcal{U}$;
(e) $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(\mathcal{F}) = \mathcal{F}^{\text{aug}}_{\text{CI}}$;
(f) $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}) = \mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}}) = \mathcal{F}$, for all $u^* \in \mathcal{U}$.

Proof: (a) Given any $u^* \in \mathcal{U}$, for $f \in \mathcal{F}^{\text{aug}}$, using Lemma 7.5, we have $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} f = f \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*} = f\restriction_{\mathcal{X},u\equiv u^*}$. Therefore, $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} f \in \mathcal{F}$ and the operator is well-defined. A similar argument holds for $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}$. Linearity of the operators directly follows from their definition.

(b)-(c) The proof follows from Lemma 7.5.

(d) Given $u^* \in \mathcal{U}$, for $g \in \mathcal{F}$, using Lemma 7.5(c), one can write $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}} \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} g = g \circ R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}} \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*} = g \circ \text{id}_{\mathcal{X}} = g$.

(e) We first prove $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(\mathcal{F}) \subseteq \mathcal{F}^{\text{aug}}_{\text{CI}}$. Let $g \in \mathcal{F}$. By part (c), $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} g = g_e \in \mathcal{F}^{\text{aug}}$. Moreover, by definition, g_e is control-independent. Hence, $\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}} g \in \mathcal{F}^{\text{aug}}_{\text{CI}}$.

Next, we prove $\mathcal{F}^{\text{aug}}_{\text{CI}} \subseteq \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(\mathcal{F})$. Given $f \in \mathcal{F}^{\text{aug}}_{\text{CI}}$, select an arbitrary $u^* \in \mathcal{U}$ and consider $g_f = f\restriction_{\mathcal{X},u\equiv u^*} \in \mathcal{F}$. Note

$$\begin{aligned}[\mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(g_f)](x,u) &= f\restriction_{\mathcal{X},u\equiv u^*}(x) = f(x,u^*) \\ &= f(x,u), \quad \forall x \in \mathcal{X}, \forall u, u^* \in \mathcal{U},\end{aligned}$$

where in the last equality we use that f is control-independent. This proves $\mathcal{F}^{\text{aug}}_{\text{CI}} \subseteq \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(\mathcal{F})$, and we conclude $\mathcal{F}^{\text{aug}}_{\text{CI}} = \mathcal{E}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}}(\mathcal{F})$.

(f) $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}}) = \mathcal{F}$ follows from (d) and (e). The rest follows from the fact that $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}) \subseteq \mathcal{F}$ by definition of $\mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}$ and the fact that $\mathcal{F} = \mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}}_{\text{CI}}) \subseteq \mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\text{aug}}}(\mathcal{F}^{\text{aug}})$ by virtue of $\mathcal{F}^{\text{aug}}_{\text{CI}} \subseteq \mathcal{F}^{\text{aug}}$. ■

C. Implications for Control-Independent Functions

As the discussion so far has illustrated, control-independent functions play a key role in the technical treatment. This is because the domain of functions in spaces $\mathcal{F}^{\infty}$ and $\mathcal{F}^{\text{aug}}$ are different from the state space of the control system (4) and, therefore, one has to rely on control-independent functions to connect the action of the operators $\mathcal{K}^{\infty}$ and $\mathcal{K}^{\text{aug}}$ to the trajectories of (4), cf. Lemmas 4.2 and 6.2, resp. These observations warrant a closer study of control-independent functions.

We first show that under reasonable conditions, there is a bijective relationship between $\mathcal{F}^{\text{aug}}_{\text{CI}}$ and $\mathcal{F}^{\infty}_{\text{CI}}$.

Proposition 7.7: (Isomorphism[6] Between $\mathcal{F}^{\text{aug}}_{\text{CI}}$ and $\mathcal{F}^{\infty}_{\text{CI}}$): Assume $\mathcal{F}^{\text{aug}}$ and $\mathcal{F}^{\infty}$ satisfy (Ci)-(Cii) in Proposition 7.3. Define the extension $\mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}} : \mathcal{F}^{\text{aug}}_{\text{CI}} \to \mathcal{F}^{\infty}_{\text{CI}}$ and restriction $\mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}} : \mathcal{F}^{\infty}_{\text{CI}} \to \mathcal{F}^{\text{aug}}_{\text{CI}}$ operators as

$$\mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}} f = \mathcal{E}^{\mathcal{F}^{\infty}}_{\mathcal{F}^{\text{aug}}} f, \qquad \mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}} g = \mathcal{R}^{\mathcal{F}^{\text{aug}}}_{\mathcal{F}^{\infty}} g.$$

Then,

(a) $\mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}}$ and $\mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}}$ are well-defined;
(b) $\mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}} \mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}} = \text{id}_{\mathcal{F}^{\infty}_{\text{CI}}}$;
(c) $\mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}} \mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}} = \text{id}_{\mathcal{F}^{\text{aug}}_{\text{CI}}}$;
(d) $\mathcal{E}^{\mathcal{F}^{\infty}_{\text{CI}}}_{\mathcal{F}^{\text{aug}}_{\text{CI}}}$ and $\mathcal{R}^{\mathcal{F}^{\text{aug}}_{\text{CI}}}_{\mathcal{F}^{\infty}_{\text{CI}}}$ are bijective.

[6] An isomorphism between two *abstract* vector spaces is a bijective *linear* map. Note that the linearity implies that it preserves the structure of the vector space and the two vector spaces are essentially the same under linear operations.

Proof: (a) This follows from combining Proposition 7.3(a), (e), and (f).

(b) Note that since operators have matching domain and codomain, $\mathcal{E}^{\mathcal{F}^{\infty}_{\mathrm{CI}}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}\mathcal{R}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}^{\infty}_{\mathrm{CI}}}$ is well-defined. For $g \in \mathcal{F}^{\infty}_{\mathrm{CI}}$, we have

$$\begin{aligned}\mathcal{E}^{\mathcal{F}^{\infty}_{\mathrm{CI}}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}\mathcal{R}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}^{\infty}_{\mathrm{CI}}} g(x,\mathbf{u}) &= g \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}} \circ R^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X}\times\ell(\mathcal{U})}(x,\mathbf{u}) \\ &= g \circ E^{\mathcal{X}\times\ell(\mathcal{U})}_{\mathcal{X}\times\mathcal{U}}(x,\mathbf{u}(0)) = g(x,(\mathbf{u}(0),\mathbf{u}(0),\ldots)).\end{aligned}$$

Since g is control-independent, $g(x,(\mathbf{u}(0),\mathbf{u}(0),\ldots)) = g(x,\mathbf{u})$ for all $(x,\mathbf{u}) \in \mathcal{X}\times\ell(\mathcal{U})$, and therefore $\mathcal{E}^{\mathcal{F}^{\infty}_{\mathrm{CI}}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}\mathcal{R}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}^{\infty}_{\mathrm{CI}}} g = g$.

(c) This follows from Proposition 7.3(d).

(d) This is a consequence of (b) and (c). ■

Proposition 7.7 has a major difference with respect to Proposition 7.3: when the domain and codomain of operators are restricted to control-independent functions, the restriction and extensions operators between $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ and $\mathcal{F}^{\infty}_{\mathrm{CI}}$ become bijective and are inverse of each other. This means that, for every control-independent function in $f \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$, there is a *unique* control-independent function in $g \in \mathcal{F}^{\infty}_{\mathrm{CI}}$ with $g = \mathcal{E}^{\mathcal{F}^{\infty}_{\mathrm{CI}}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} f$ such that their values on all states match,

$$f(x,u) = g(x,\mathbf{u}), \quad \forall x \in \mathcal{X},\ \forall u \in \mathcal{U},\ \forall \mathbf{u} \in \ell(\mathcal{U}). \tag{18}$$

The reverse of this statement also holds: for all $g \in \mathcal{F}^{\infty}_{\mathrm{CI}}$, there is a unique $f \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ with $f = \mathcal{R}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}^{\infty}_{\mathrm{CI}}} g$ such that (18) holds.

We are also interested in connecting the spaces $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ and $\mathcal{F}$. We first show that the family of restrictions from $\mathcal{F}^{\mathrm{aug}}$ to $\mathcal{F}$ coincide when their domain is restricted to $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$.

Lemma 7.8: (Restriction Operators from $\mathcal{F}^{\mathrm{aug}}$ to $\mathcal{F}$ Coincide on $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$): Assume $\mathcal{F}$ and $\mathcal{F}^{\mathrm{aug}}$ satisfy ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Then

$$\mathcal{R}^{\mathcal{F},u\equiv u_1}_{\mathcal{F}^{\mathrm{aug}}} f = \mathcal{R}^{\mathcal{F},u\equiv u_2}_{\mathcal{F}^{\mathrm{aug}}} f, \quad \forall f \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}},\ \forall u_1,u_2 \in \mathcal{U}.$$

Proof: This follows by noting that, for $f \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$, we have $[\mathcal{R}^{\mathcal{F},u\equiv u_1}_{\mathcal{F}^{\mathrm{aug}}} f](x) = f(x,u_1) = f(x,u_2) = [\mathcal{R}^{\mathcal{F},u\equiv u_2}_{\mathcal{F}^{\mathrm{aug}}} f](x)$, for all $x \in \mathcal{X}$, and all $u_1,u_2 \in \mathcal{U}$. ■

As a result of Lemma 7.8, we define the *restriction operator* $\mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} : \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}} \to \mathcal{F}$ as

$$\mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} f = \mathcal{R}^{\mathcal{F},u\equiv u^*}_{\mathcal{F}^{\mathrm{aug}}} f,$$

where $u^* \in \mathcal{U}$ is arbitrary. Next, we fully connect $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ and $\mathcal{F}$ via bijective maps.

Proposition 7.9: (Isomorphism Between $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ and $\mathcal{F}$): Assume $\mathcal{F}$ and $\mathcal{F}^{\mathrm{aug}}$ satisfy ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Define the *extension operator* $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ as

$$\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} f = \mathcal{E}^{\mathcal{F}^{\mathrm{aug}}}_{\mathcal{F}} f.$$

Then,

(a) $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}}$ is well-defined;
(b) $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} \mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} = \mathrm{id}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}$;
(c) $\mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} \mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$;
(d) $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}}$ and $\mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}$ are bijective.

Proof: (a) This follows from Proposition 7.6(a) and (e).

(b) For $g \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$, we have

$$\begin{aligned}\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} \mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} g(x,u) &= g \circ E^{\mathcal{X}\times\mathcal{U}}_{\mathcal{X},u\equiv u^*} \circ R^{\mathcal{X}}_{\mathcal{X}\times\mathcal{U}}(x,u) \\ &= g(x,u^*) = g(x,u), \forall x \in \mathcal{X},\ \forall u,u^* \in \mathcal{U},\end{aligned}$$

where in the last equality we have used the fact that g is control-independent.

(c) This follows from Proposition 7.6(d).

(d) This is a consequence of (c) and (d). ■

Proposition 7.9 shows that there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$. One can think of $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ as a copy of $\mathcal{F}$, with the domain changed from $\mathcal{X}$ to $\mathcal{X}\times\mathcal{U}$. For example, if $f \in \mathcal{F}$, then there exist $g = \mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}} f \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ where

$$f(x) = g(x,u),\ \forall x \in \mathcal{X},\ \forall u \in \mathcal{U}. \tag{19}$$

The inverse of this statement is also true: for all $g \in \mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$ there is a unique $f = \mathcal{R}^{\mathcal{F}}_{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}} g \in \mathcal{F}$ for which (19) holds.

The next result is a consequence of Propositions 7.7 and 7.9.

Corollary 7.10: (Isomorphism Between $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$, $\mathcal{F}^{\infty}_{\mathrm{CI}}$): Assume $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}$, and $\mathcal{F}^{\infty}$ satisfy (Ci)-(Cii) in Proposition 7.3 and ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Then,

(a) $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}$, and $\mathcal{F}^{\infty}_{\mathrm{CI}}$ are isomorphic;
(b) $\mathrm{card}(\mathcal{F}) = \mathrm{card}(\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}) = \mathrm{card}(\mathcal{F}^{\infty}_{\mathrm{CI}})$;
(c) $\mathrm{card}(\mathcal{F}) \le \mathrm{card}(\mathcal{F}^{\mathrm{aug}}) \le \mathrm{card}(\mathcal{F}^{\infty})$.

Proof: (a)-(b) follow from Propositions 7.7 and 7.9. Regarding (c), the inequality $\mathrm{card}(\mathcal{F}) \le \mathrm{card}(\mathcal{F}^{\mathrm{aug}})$ follows from (b) and the fact that $\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}} \subseteq \mathcal{F}^{\mathrm{aug}}$. The other inequality follows from the fact that the operator $\mathcal{E}^{\mathcal{F}^{\infty}}_{\mathcal{F}^{\mathrm{aug}}} : \mathcal{F}^{\mathrm{aug}} \to \mathcal{F}^{\infty}$ is injective since it has a left inverse based, cf. Proposition 7.3(d). ■

Corollary 7.10 ensures that the spaces $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}$, $\mathcal{F}^{\infty}$ have the same level of richness in terms of encoding the information about the trajectories of control system (4). However, one should keep in mind that the spaces $\mathcal{F}^{\mathrm{aug}}$ and $\mathcal{F}^{\infty}$ generally have larger cardinality than $\mathcal{F}$, cf. Figure 1, since the frameworks $\mathcal{K}^{\mathrm{aug}}$ and $\mathcal{K}^{\infty}$ embed the effect of the input sequence in the function spaces while in the KCF framework the input information is embedded in the switching signal (and kept separate from $\mathcal{F}$).

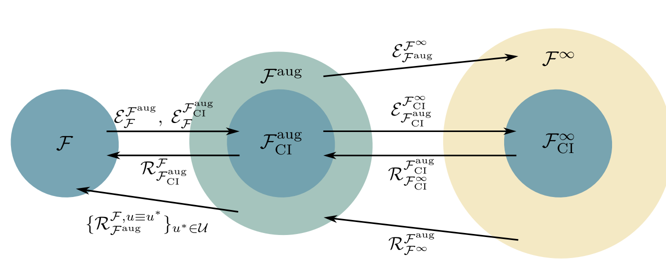

Fig. 1: Connections between the functions spaces. The action of $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}}_{\mathcal{F}}$ and $\mathcal{E}^{\mathcal{F}^{\mathrm{aug}}_{\mathrm{CI}}}_{\mathcal{F}}$ coincide even though the have different codomains.

VIII. Connections Between the Associated Koopman Operators

In this section, we use the augmented Koopman operator as an intermediary to connect the Koopman operators associated with the infinite input sequences framework and the Koopman Control Family. We perform this process at two levels: dynamics and operators acting on function spaces, cf. Table I.

A. Infinite Input Sequences Framework and Augmented Koopman Operator

We are ready to relate $\mathcal{T}^\infty$ and $\mathcal{T}^{\mathrm{aug}}$ using the notions introduced in Section VII-A.

Proposition 8.1: (Dynamics Connection Between Infinite Input Sequences and Augmented Koopman Operator Frameworks): Let $E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}$ and $R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}}$ be the maps defined in (16). Then,

(a) $\mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} = E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} \circ \mathcal{T}^{\mathrm{aug}}$;

(b) $\mathcal{T}^{\mathrm{aug}} = R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}$.

Proof: (a) For $(x,u) \in \mathcal{X}\times\mathcal{U}$, we have

$$\begin{aligned}\mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}(x,u) &= \mathcal{T}^\infty(x,(u,u,\ldots)) \\ &= (\mathcal{T}(x,u),(u,u,\ldots)).\end{aligned}$$

On the other hand,

$$\begin{aligned}E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} \circ \mathcal{T}^{\mathrm{aug}}(x,u) &= E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}(\mathcal{T}(x,u),u) \\ &= (\mathcal{T}(x,u),(u,u,\ldots)).\end{aligned}$$

Therefore, $\mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}(x,u) = E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} \circ \mathcal{T}^{\mathrm{aug}}(x,u)$.

(b) For $(x,u) \in \mathcal{X}\times\mathcal{U}$, we have

$$\begin{aligned}R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}(x,u) &= R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}^\infty(x,(u,u,\ldots)) \\ = R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}}(\mathcal{T}(x,u),(u,u,\ldots)) &= (\mathcal{T}(x,u),u) = \mathcal{T}^{\mathrm{aug}}(x,u).\end{aligned}$$

■

Proposition 8.1 provides a tool based on domain restriction and extension maps to connect the action of dynamical systems in the infinite input sequences and the augmented Koopman operator frameworks. The connections are via function compositions, allowing to state equivalent descriptions at the operator level.

Theorem 8.2: (Operator Connection Between Infinite Input Sequences and Augmented Koopman Operator Frameworks): Assume $\mathcal{F}^{\mathrm{aug}}$ and $\mathcal{F}^\infty$ satisfy (Ci)-(Cii) in Proposition 7.3. Then,

(a) $\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty = \mathcal{K}^{\mathrm{aug}} \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}}$;

(b) $\mathcal{K}^{\mathrm{aug}} = \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty \mathcal{E}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F}^\infty}$;

(c) $(\mathcal{K}^\infty f)\restriction_{\mathcal{X}\times\mathcal{U}} = \mathcal{K}^{\mathrm{aug}}(f\restriction_{\mathcal{X}\times\mathcal{U}})$, for all $f \in \mathcal{F}^\infty$;

(d) $\mathcal{K}^{\mathrm{aug}} g = (\mathcal{K}^\infty g^\infty)\restriction_{\mathcal{X}\times\mathcal{U}}$, for all $g \in \mathcal{F}^{\mathrm{aug}}$.

Proof: (a) For $h \in \mathcal{F}^\infty$, we have

$$\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty h = \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}}(h \circ \mathcal{T}^\infty) = h \circ \mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}.$$

This, together with Proposition 8.1(a), yields

$$\begin{aligned}h \circ \mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} &= h \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})} \circ \mathcal{T}^{\mathrm{aug}} \\ &= \mathcal{K}^{\mathrm{aug}}(h \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}) \\ &= \mathcal{K}^{\mathrm{aug}}(\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} h).\end{aligned}$$

Therefore, $\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty h = \mathcal{K}^{\mathrm{aug}} \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} h$.

(b) For $g \in \mathcal{F}^{\mathrm{aug}}$, we have

$$\begin{aligned}\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty \mathcal{E}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F}^\infty} g &= \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty (g \circ R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}}) \\ &= \mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}}(g \circ R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}^\infty) \\ &= g \circ R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}^\infty \circ E_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}\times\ell(\mathcal{U})}.\end{aligned}$$

This, together with Proposition 8.1(b), yields

$$\mathcal{R}_{\mathcal{F}^\infty}^{\mathcal{F}^{\mathrm{aug}}} \mathcal{K}^\infty \mathcal{E}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F}^\infty} g = g \circ \mathcal{T}^{\mathrm{aug}} = \mathcal{K}^{\mathrm{aug}} g.$$

(c)-(d) This follows from (a)-(b) in conjunction with Proposition 7.3. ■

Figure 2 shows the commutative diagram for the operators' actions described in Theorem 8.2.

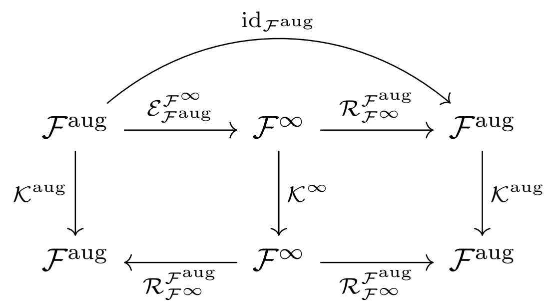

Fig. 2: Commutative diagram illustrating Theorem 8.2.

B. Koopman Control Family and Augmented Koopman Operator

We are ready to relate $\{\mathcal{T}_{u^*}\}_{u^*\in\mathcal{U}}$ and $\mathcal{T}^{\mathrm{aug}}$ using the notions introduced in Section VII-B.

Proposition 8.3: (Dynamics Connection Between Koopman Control Family and Augmented Koopman Operator Frameworks): Let $R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}}$ and $\{E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}\}_{u^*\in\mathcal{U}}$ be the maps defined in (17). Then, for all $u^* \in \mathcal{U}$,

(a) $\mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} = E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}_{u^*}$;

(b) $\mathcal{T}_{u^*} = R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}} \circ \mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}$.

Proof: Let $u^* \in \mathcal{U}$. (a) for $x \in \mathcal{X}$, we have

$$\mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}(x) = \mathcal{T}^{\mathrm{aug}}(x,u^*) = (\mathcal{T}(x,u^*),u^*).$$

On the other hand,

$$E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}_{u^*}(x) = E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}(\mathcal{T}(x,u^*)) = (\mathcal{T}(x,u^*),u^*).$$

Therefore, $\mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} = E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} \circ \mathcal{T}_{u^*}$.

(b) for $x \in \mathcal{X}$, we have

$$\begin{aligned}R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}} \circ \mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}(x) &= R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}} \circ \mathcal{T}^{\mathrm{aug}}(x,u^*) \\ = R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}}(\mathcal{T}(x,u^*),u^*) &= \mathcal{T}(x,u^*) = \mathcal{T}_{u^*}(x).\end{aligned}$$

Therefore, $R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}} \circ \mathcal{T}^{\mathrm{aug}} \circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}} = \mathcal{T}_{u^*}$. ■

This result allows us to state equivalent descriptions at the operator level between the augmented Koopman operator and the Koopman control family frameworks.

Theorem 8.4: (Operator Connection Between Koopman Control Family and Augmented Koopman Operator Frameworks): Assume $\mathcal{F}$ and $\mathcal{F}^{\mathrm{aug}}$ satisfy ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Then, for all $u^* \in \mathcal{U}$

(a) $\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}} = \mathcal{K}_{u^*}\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}$;
(b) $\mathcal{K}_{u^*} = \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}^{\mathrm{aug}}}$;
(c) $(\mathcal{K}^{\mathrm{aug}}f)\restriction_{\mathcal{X},u\equiv u^*} = \mathcal{K}_{u^*}(f\restriction_{\mathcal{X},u\equiv u^*})$, for all $f \in \mathcal{F}^{\mathrm{aug}}$;
(d) $\mathcal{K}_{u^*}g = (\mathcal{K}^{\mathrm{aug}}g_e)\restriction_{\mathcal{X},u\equiv u^*}$, for all $g \in \mathcal{F}$.

Proof: Let $u^* \in \mathcal{U}$. (a) For $f \in \mathcal{F}^{\mathrm{aug}}$, we have

$$\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}f = \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}(f\circ\mathcal{T}^{\mathrm{aug}}) = f\circ\mathcal{T}^{\mathrm{aug}}\circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}.$$

On the other hand,

$$\mathcal{K}_{u^*}\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}f = \mathcal{K}_{u^*}(f\circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}) = f\circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}\circ\mathcal{T}_{u^*}.$$

Using Proposition 8.3(a), we can write $\mathcal{K}_{u^*}\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}f = f\circ\mathcal{T}^{\mathrm{aug}}\circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}$, and the result follows.

(b) For $g \in \mathcal{F}$, we have

$$\begin{aligned}\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}^{\mathrm{aug}}}g &= \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}(g\circ R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}})\\ &= \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}(g\circ R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}}\circ\mathcal{T}^{\mathrm{aug}})\\ &= g\circ R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}}\circ\mathcal{T}^{\mathrm{aug}}\circ E_{\mathcal{X},u\equiv u^*}^{\mathcal{X}\times\mathcal{U}}.\end{aligned}$$

Using Proposition 8.3(b), we deduce that $\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}^{\mathrm{aug}}}g = g\circ\mathcal{T}_{u^*} = \mathcal{K}_{u^*}g$.

(c)-(d) This follows from (a)-(b) in conjunction with Lemma 7.5. ∎

Figure 3 shows the commutative diagram for the operators' actions described in Theorem 8.4.

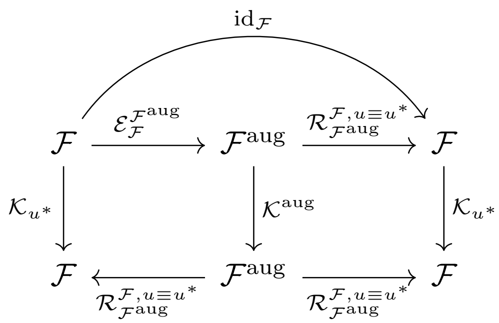

Fig. 3: Commutative diagram illustrating Theorem 8.4. Here, $u^* \in \mathcal{U}$ is arbitrary.

C. Equivalence Results on Control-Independent Functions

Following up on Section VII-C, here we turn our attention to the operators' action on control-independent functions. The next result reveals how the actions of $\{\mathcal{K}_{u^*}\}_{u^*\in\mathcal{U}}$, $\mathcal{K}^{\mathrm{aug}}$, and $\mathcal{K}^{\infty}$ are connected on the isomorphic spaces $\mathcal{F}$, $\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}$, and $\mathcal{F}_{\mathrm{CI}}^{\infty}$.

Theorem 8.5: (Operators' Actions on Control-Independent Functions): Assume $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}$, and $\mathcal{F}^{\infty}$ satisfy (Ci)-(Cii) in Proposition 7.3 and ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Then,

(a) $\mathcal{K}_{u^*}f = \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{K}^{\mathrm{aug}}\,\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}f$, for all $f \in \mathcal{F}$ and $u^* \in \mathcal{U}$;
(b) $\mathcal{K}^{\mathrm{aug}}g = \mathcal{R}_{\mathcal{F}^{\infty}}^{\mathcal{F}^{\mathrm{aug}}}\mathcal{K}^{\infty}\mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}g$, for all $g \in \mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}$;
(c) $\mathcal{K}_{u^*}f = \mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\mathcal{R}_{\mathcal{F}^{\infty}}^{\mathcal{F}^{\mathrm{aug}}}\mathcal{K}^{\infty}\mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}f$, for all $f \in \mathcal{F}$ and $u^* \in \mathcal{U}$.

Proof: (a) This follows from Theorem 8.4(b) and the definition of $\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}$.

(b) This follows from Theorem 8.2(b) and the definition of $\mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}$.

(c) This follows from (a) and (b). ∎

Figure 4 illustrates the action of operators in Theorem 8.5 on different function spaces starting from control-independent functions.

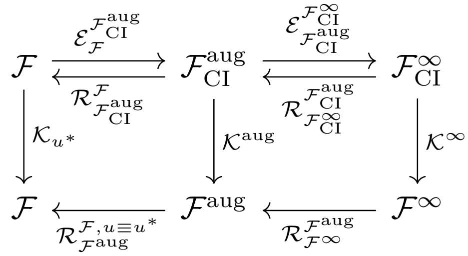

Fig. 4: Commutative diagram illustrating Theorem 8.5 and Propositions 7.7 and 7.9. Here, $u^* \in \mathcal{U}$ is arbitrary.

Figure 4 reveals that one can start from any of the spaces $\mathcal{F}$, $\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}$, or $\mathcal{F}_{\mathrm{CI}}^{\infty}$ through equivalent functions linked by isomorphisms $\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}$, $\mathcal{R}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}}$, $\mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}$, and $\mathcal{R}_{\mathcal{F}_{\mathrm{CI}}^{\infty}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}$ and capture dynamical information from system (4) by applying $\{\mathcal{K}_{u^*}\}_{u^*\in\mathcal{U}}$, $\mathcal{K}^{\mathrm{aug}}$, or $\mathcal{K}^{\infty}$. Note that, after applying $\mathcal{K}^{\mathrm{aug}}$ and $\mathcal{K}^{\infty}$, the output functions will not be necessarily control-independent anymore, since the system behavior depends on the input. It also worth mentioning that the output functions in $\mathcal{F}^{\mathrm{aug}}$ and $\mathcal{F}^{\infty}$ capture both the information of the system's trajectories as well as the information of the input sequence (since the domain of functions in $\mathcal{F}^{\mathrm{aug}}$ and $\mathcal{F}^{\infty}$ are $\mathcal{X}\times\mathcal{U}$ and $\mathcal{X}\times\ell(\mathcal{U})$, resp.). Therefore, the operators $\mathcal{R}_{\mathcal{F}^{\infty}}^{\mathcal{F}^{\mathrm{aug}}}$ and $\{\mathcal{R}_{\mathcal{F}^{\mathrm{aug}}}^{\mathcal{F},u\equiv u^*}\}_{u^*\in\mathcal{U}}$ act as filters, detaching the information about the input sequence except the first element influencing the systems' trajectory for the next time step.

Next, we focus our attention on a different kind of equivalence result pertaining to the evolution of function values on multi-step trajectories under the Koopman operator via infinite input sequences and the Koopman Control Family framework.

Theorem 8.6: (Equivalence of KCF and Koopman Operator via Infinite Input Sequences on Trajectories): Assume $\mathcal{F}$, $\mathcal{F}^{\mathrm{aug}}$, and $\mathcal{F}^{\infty}$ satisfy (Ci)-(Cii) in Proposition 7.3 and ($\mathfrak{C}$i)-($\mathfrak{C}$ii) in Proposition 7.6. Let $\{x_k\}_{k\in\mathbb{N}_0}$ be the trajectory of (4) from initial condition x_0 with input sequence $\mathbf{u} = (u_0, u_1, \ldots)$. Then, for all $k \in \mathbb{N}_0$,

(a) for all $f \in \mathcal{F}$, we have
$$\begin{aligned}f(x_k) &= [\mathcal{K}_{u_0}\mathcal{K}_{u_1}\ldots\mathcal{K}_{u_{k-1}}f](x_0)\\ &= [(\mathcal{K}^{\infty})^k\mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}f](x_0,\mathbf{u});\end{aligned}$$
(b) for all $h \in \mathcal{F}_{\mathrm{CI}}^{\infty}$ with decomposition $h = h_{\mathcal{X}}\,1_{\ell(\mathcal{U})}$ (cf. Definition 4.1), we have
$$\begin{aligned}h_{\mathcal{X}}(x_k) &= [(\mathcal{K}^{\infty})^k h](x_0,\mathbf{u})\\ &= [\mathcal{K}_{u_0}\mathcal{K}_{u_1}\ldots\mathcal{K}_{u_{k-1}}\mathcal{R}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}}\mathcal{R}_{\mathcal{F}_{\mathrm{CI}}^{\infty}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}h](x_0).\end{aligned}$$

Proof: (a) Given $f \in \mathcal{F}$, the first equality follows from Lemma 4.4. Let $h := \mathcal{E}_{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}^{\mathcal{F}_{\mathrm{CI}}^{\infty}}\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\mathrm{CI}}^{\mathrm{aug}}}f \in \mathcal{F}_{\mathrm{CI}}^{\infty}$, cf. Propositions 7.7 and 7.9. Since h is control-independent, it can be decomposed as $h = h_{\mathcal{X}} 1_{\ell(\mathcal{U})}$. Based on Lemma 4.2, we have $h_{\mathcal{X}}(x_k) = [(\mathcal{K}^\infty)^k h](x_0, \mathbf{u})$. Hence, to prove the result, we seek to establish that $h_{\mathcal{X}} = f$. Based on the definition of the extension operators $\mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\rm CI}^{\rm aug}}$ and $\mathcal{E}_{\mathcal{F}_{\rm CI}^{\rm aug}}^{\mathcal{F}_{\rm CI}^\infty}$, one can write

$$\begin{aligned} h(x, \bar{\mathbf{u}}) &= [\mathcal{E}_{\mathcal{F}_{\rm CI}^{\rm aug}}^{\mathcal{F}_{\rm CI}^\infty} \mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\rm CI}^{\rm aug}} f](x, \bar{\mathbf{u}}) \\ &= [f \circ R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}} \circ R_{\mathcal{X}\times\ell(\mathcal{U})}^{\mathcal{X}\times\mathcal{U}}](x, \bar{\mathbf{u}}) = [f \circ R_{\mathcal{X}\times\mathcal{U}}^{\mathcal{X}}](x, \bar{\mathbf{u}}(0)) \\ &= f(x), \quad \forall x \in \mathcal{X}, \quad \forall \bar{\mathbf{u}} \in \ell(\mathcal{U}), \end{aligned}$$

and therefore $h_{\mathcal{X}} = f$.

(b) Given $h \in \mathcal{F}_{\rm CI}^\infty$, let $f := \mathcal{R}_{\mathcal{F}_{\rm CI}^{\rm aug}}^{\mathcal{F}} \mathcal{R}_{\mathcal{F}_{\rm CI}^\infty}^{\mathcal{F}_{\rm CI}^{\rm aug}} h \in \mathcal{F}$. Note that $h = \mathcal{E}_{\mathcal{F}_{\rm CI}^{\rm aug}}^{\mathcal{F}_{\rm CI}^\infty} \mathcal{E}_{\mathcal{F}}^{\mathcal{F}_{\rm CI}^{\rm aug}} f$ based on Propositions 7.7 and 7.9. The result now follows from (a). ■

Theorem 8.6 provides a direct connection between the KCF and the Koopman operator via infinite-sequences on the system trajectories. It is important to point out that this result does not include the augmented Koopman operator, $\mathcal{K}^{\rm aug}$, since it is not a Koopman operator associated with the control system (4) and does not directly capture multi-step trajectories.

We finish this section by explaining a fundamental difference between Theorems 8.5 and 8.6. Theorem 8.5 pertains to the operators' action on function spaces, revealing how the functions encode the system's information in general settings, and how one can separate the information of system trajectories from the information of input sequences and change the domain of functions to the state space of control system (4). On the other hand, Theorem 8.6 reveals information about the function *values* on trajectories as opposed to the functions themselves. This result is more useful in modeling and control applications while Theorem 8.5 pertains to deeper fundamental connections between the different frameworks.

IX. Conclusions

We have studied the connections between two extensions of Koopman operator theory to control systems, the Koopman operator via infinite input sequences and the Koopman control family. Since each extension relies on a different mechanism to encode system information into operators acting on vector spaces, we first examined how the information of trajectories and input sequences is captured in each. This understanding then enabled us to provide ways to connect the function spaces via linear composition operators. We relied on these operators as bridges to connect the actions of Koopman-based formulations in each framework. As a result, we provided a comprehensive analysis of their structure, along with constructive algebraic recipes to convert the actions of the operators. Finally, we showed that, under mild conditions on the function spaces, the frameworks are equivalent, both in terms of encoding system information in function spaces and the evolution of function values along system trajectories. Future work will explore the implications of these connections for studying nonlinear control systems as well as for control design procedures.

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