

Data-Driven Stabilization of Unknown Linear-Threshold Network Dynamics

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Abstract—This paper studies the data-driven control of unknown linear-threshold network dynamics to stabilize the state to a reference value. We consider two types of controllers: (i) a state feedback controller with feed-forward reference input and (ii) an augmented feedback controller with error integration. The first controller features a simpler structure and is easier to design, while the second offers improved performance in the presence of system parameter changes and disturbances. Our design strategy employs state-input datasets to construct data-based representations of the closed-loop dynamics. Since these representations involve linear threshold functions, we rewrite them as switched linear systems, and formulate the design problem as that of finding a common controller for all the resulting modes. This gives rise to a set of linear matrix inequalities (LMIs) whose solutions corresponds to the controller gain matrices. We analyze the computational complexity of solving the LMIs and propose a simplified, sufficient set of conditions that scales linearly with the system state. Simulations on two case studies involving regulation of firing rate dynamics in rodent brains and of arousal level dynamics in humans demonstrate the effectiveness of the controller designs.

Index Terms—Linear-threshold networks; data-driven control; switched systems; linear matrix inequalities.

I. INTRODUCTION

Linear-threshold dynamics are a class of nonlinear dynamics with wide scientific and practical applications. These dynamics model interactions within network systems, where the states of the nodes evolve based on inter-node connectivity and a linear-threshold activation function. Compared to linear systems, linear-threshold networks provide bounded system states and greater dynamical versatility, including mono- and multi-stability, limit cycles, and chaotic behavior. These properties have made them useful in diverse fields, including computational neuroscience [2]–[4], social networks [5], [6], and deep learning [7], [8]. Traditional approaches to regulate the dynamical behavior of linear-threshold networks are heavily model-based and rely on precise model parameters to design control schemes. Recent advances in data collection, processing, and computation have motivated a spur of activity on data-driven control methods for systems with unknown dynamics directly from its input-output data. This paper develops data-driven control methods for stabilization specifically tailored for linear-threshold network dynamics.

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Literature review: The existing literature for data-driven control can be classified into two major categories: linear and nonlinear systems. For linear systems, Willems’ Lemma [9] has been a foundational tool in constructing a data-based representation of input-output trajectories as a linear combination of measured data samples. Building on this result, various types of controllers can be reformulated as part of a closed-loop representation, thereby facilitating the design of controller parameters. Along this idea, [10] studies the stabilization controller and linear-quadratic regulator (LQR) of a system by associating the gains of linear feedback controllers with the solutions of a linear matrix inequality (LMI) and a semidefinite program (SDP), respectively. The idea has also been utilized in a receding horizon fashion for model predictive control (MPC) [11], [12] and distributed MPC [13] when data are only locally available to nodes of a network. Considering data richness and noise, [12], [14] study the online implementation of sample efficient data-driven control with noisy system measurements. Aligned with this body of work, the informativity approach to data-driven control [15] considers measurements that do not contain enough information to obtain a unique system. By making assumptions on the model class and noise model, this approach explicitly determines the set of all systems consistent with the measurements, thereby enabling the certification of desirable properties for the measured system. These have included stability [16], [17], controllability and observability [18], [19], and dissipativity [20], [21].

These ideas and techniques are currently been extended to nonlinear systems, see [22], [23] for recent comprehensive surveys on data-driven control of nonlinear systems. These include [24], [25], utilizing low-rank approximation technique to handle system nonliterary and treat reminders as disturbances; [26], studying data-driven optimal control of bilinear systems; and [27], studying the stabilization of bilinear systems and uses LMIs to create a guaranteed region of attraction. By generalizing Willems’ Lemma to particular types of nonlinear systems, [28] provides nonlinear data-driven control for Hammerstein-Wiener systems, and [29] for second-order discrete Volterra systems. Based on dual stability theory and Farkas’ lemma, [30] develops data-driven control methods for nonlinear continuous-time systems. Other methods for nonlinear data-driven control that are not based on Willems’ Lemma include unfalsified control [31], simultaneous perturbation stochastic approximation [32], model-free adaptive control [33], and iterative feedback tuning [34]. Different from the approaches described above, where the controller can be directly computed using the data, these approaches tune the parameters of the controller in an iterative manner and gradually improve system performance. To address stabilization to states

other than the origin, methods like [35] employ nonlinear basis functions and combine model-inverse techniques with virtual reference feedback tuning for reference tracking.

Here, we focus on developing data-driven control techniques for linear-threshold network dynamics. These systems have diverse applications across several domains. In computational neuroscience, linear-threshold networks have been used to model mesoscale brain activity [2]. Nodes of the system represent neuron populations, with states indicating their average firing rates. Edge weights capture excitatory or inhibitory interactions among physically adjacent neurons, and a linear-threshold activation function accounts for firing rate saturation due to hyperpolarization [36]. In social sciences, linear-threshold networks have been used to build influence propagation models [5], [6] to characterize the dynamics of public opinions. Nodes represent individuals and their opinions, while edges describe interactions influenced by others or social media (through external inputs to the network). A linear-threshold is introduced to each node to gauge the condition when individuals' opinions change. In deep learning applications with artificial neural networks, linear-threshold models are the same as modified rectified linear units (RELU with max-limits) [8]. RELUs with tunable parameters are generally used within the hidden nodes in the deep neural networks [37], which allows good robustness and versatility for function approximation [38]. Compared with other common activation functions such as sigmoid and tanh, RELU networks do not suffer vanishing gradient problems [39], because its gradient is either a constant or zero. For the same reason, ReLU networks have low computational complexity [40] when performing gradient propagation. One feasible approach to designing controllers for linear-threshold networks is to first identify the system parameters [41] and then design a model-based controller [4]. Instead, we pursue here a direct data-driven approach that bypasses the system identification step to avoid accumulating approximation errors.

Statement of Contributions: We study the data-driven control of linear-threshold network dynamics to stabilize the state of the system to a constant reference value. We do this by designing two types of controllers: (i) a state feedback controller with feed-forward reference input and (ii) an augmented feedback controller with error integration. By utilizing sampled input-output data of the system, we introduce a map that reconstructs the state-input pair as a transformation of data matrices. This allows us to obtain a purely data-based representation to describe the system dynamics, avoiding an explicit identification of the parametric model. We then use this result and the specific form of the state feedback controller with feed-forward reference input to obtain a data-driven representation of the closed-loop system. To design the feedback and feedforward gain matrices, we view the resulting system as a switched system and formulate conditions for the design of a common controller that stabilizes all modes. The stabilization conditions are characterized by a set of linear matrix inequalities (LMIs), whose solutions correspond to the gain matrices. We provide a theoretical guarantee for the controller's effectiveness in stabilizing the system state to the desired constant reference value. For non-zero reference

states, however, we observe that controllers with feed-forward inputs are often not robust to system parameter changes and external disturbances. To address this limitation, we design an augmented feedback controller with error integration adapting the approach developed for the state feedback controller to the new design. The process involves deriving a closed-loop data-based representation and formulating stabilization conditions in the form of LMIs. In both cases, we observe that the LMI formulations result in a number of equations that grow exponentially with the system state's dimensionality. To address this computational challenge, we introduce an alternative sufficient condition that reduces the complexity of solving LMIs without sacrificing performance. Finally, we validate the effectiveness of both algorithms in two case studies: regulating firing rate dynamics in rodent brains and regulating arousal level dynamics in humans. The results demonstrate that both controllers are effective and the controller with error integration has an improved performance compared with the one with feed-forward reference input for handling disturbances.

Notation: Let \mathbb{R} denote the set of real numbers. Let $\mathbf{1}_r \in \mathbb{R}^r$ and $\mathbf{1}_{r \times p} \in \mathbb{R}^{r \times p}$ denote the vector and matrix with all entries equal to 1, respectively. Let I_r denote the $r \times r$ identity matrix. We let $\text{col}\{A_1, A_2, \dots, A_r\} = [A_1^\top \ A_2^\top \ \dots \ A_r^\top]^\top$ be a vertical stack of matrices A_1, \dots, A_r possessing the same number of columns. We let $\text{diag}\{A_1, A_2, \dots, A_r\}$ denote the diagonal stack of matrices A_1, \dots, A_r . We use $\mathbf{x}[i] \in \mathbb{R}$ to denote the i th entry of vector \mathbf{x} ; correspondingly, $M[i, j] \in \mathbb{R}$ is the entry of matrix M on its i th row and j th column. We denote by M^\top the transpose of M . We let $\text{image}(M)$ denote the linear span of the columns of matrix M . Specially, $\text{image}(I_r) = \mathbb{R}^r$. For symmetric matrices M, N , $M > (\geq) N$ means $M - N$ is positive (semi-) definite. For $s > 0$ and $x \in \mathbb{R}$, the threshold function $[x]_0^s$ is defined as

$$[x]_0^s = \begin{cases} s & \text{for } x > s, \\ x & \text{for } 0 \leq x \leq s, \\ 0 & \text{for } x < 0, \end{cases}$$

For a vector $\mathbf{x} \in \mathbb{R}^r$, $[\mathbf{x}]_0^s$ denotes the component-wise application of this definition.

II. PROBLEM FORMULATION

Consider a linear-threshold network of n nodes governed by the following discrete-time dynamics:

$$\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + [W\mathbf{x}(t) + B\mathbf{u}(t)]_0^s, \quad t \in \mathbb{N}. \quad (1)$$

Here $\mathbf{x} \in \mathbb{R}_{\geq 0}^n$ is the network state, which is the compact stack of a scalar state in each node. $W \in \mathbb{R}^{n \times n}$ is the network connectivity matrix, characterizing the strength of interactions between different nodes. $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input vector, and $B \in \mathbb{R}^{n \times m}$ is the associated input matrix. For each node, the state of the node evolves according to an intrinsic decay rate $\alpha \in (0, 1)$, and the linear-threshold activation function denoted by $[\cdot]_0^s$, whose input is co-generated by the node's neighbors and external inputs. Given the decay rate α and the properties of linear-threshold functions, we assume the initial state of the dynamics satisfies $0 \leq \mathbf{x}(0) \leq \frac{s}{1-\alpha}$, so that $0 \leq \mathbf{x}(t) \leq \frac{s}{1-\alpha}$, for all $t \in \mathbb{N}$. In (1), we assume the

parameters α and s are known, but the matrices W and B are unknown. This type of network dynamics arise in the modeling of the dynamical behavior of the firing rates of neuronal populations [2], where α denotes the intrinsic decay rate of neurons' firing behaviors, W and B characterize the inhibition/excitation response of neurons from other connected neurons or external inputs, and the linear-threshold s characterizes the firing rate saturation of neurons due to hyperpolarization [36]. In such contexts, the assumption above about known parameters and unknown matrices are reasonable. The dynamics also model the opinion propagation of individuals in social networks [6], where α denotes decaying confidence [42], W and B characterize the change of individuals' agreement/disagreement on the matter impacted by other people or social media, and the linear-threshold characterizes the saturation threshold of public opinion [43]. More generally, model (1), with tunable matrices W and B , can be used as artificial neural networks to approximate nonlinear dynamics for learning and control tasks [8], [38].

To study dynamics (1), we employ a data-driven approach and assume that system inputs and states can be sampled from experiments. Let T_d be the total number of available data points, and let $\mathbf{x}_d^+(k)$, $\mathbf{x}_d(k)$, and $\mathbf{u}_d(k)$, $k \in \{1, \dots, T_d\}$ denote the data samples (corresponding to $\mathbf{x}(t+1)$, $\mathbf{x}(t)$, and $\mathbf{u}(t)$, respectively). We employ the index k as an indicator that distinguishes one data sample from another. It is possible that all the sampling instances of the data are chosen consecutively from a system trajectory, where all the data samples are head-tail connected, i.e., $\mathbf{x}_d^+(k)$ of the former data can be used as the $\mathbf{x}_d(k)$ of the latter one. In general, we allow data samples to be collected at independent time instances, and even from various trajectories of the same system.

Problem 1: Consider system (1) with known parameters α and s , and unknown matrices W and B . Let $\mathbf{r} \in \mathbb{R}^n$, with $0 \leq \mathbf{r} \leq \frac{s}{1-\alpha}$, be a constant reference value the desired state should converge to. Given data samples $\mathbf{x}_d(k)$, $\mathbf{x}_d^+(k)$ and $\mathbf{u}_d(k)$, $k \in \{1, \dots, T_d\}$, design a feedback controller which asymptotically stabilizes the system state $\mathbf{x}(t)$ to the reference value \mathbf{r} .

In the context of the applications mentioned above, the data-driven control Problem 1 is motivated by the following considerations. In neuroscience, for instance, our brain continuously regulates the firing rate of neuronal cells to specific patterns, i.e., depending on different brain functions, the activity of certain neurons should be excited or inhibited. In social science, the control problem arises when certain actors seek to steer public opinion in a particular direction. In more general control applications where a linear-threshold artificial neural network approximates certain nonlinear dynamics, the manipulation of the latter can be achieved by controlling the neural network model.

We assume Problem 1 is solvable. This means that: (i) the matrices W and B are such that there exist controllers that stabilize the system (1) to \mathbf{r} ; (ii) the data samples available are sufficiently rich to allow us to design the controller without knowing the matrices W and B . [4, Theorem IV.8] describes classes of linear-threshold systems for which (i) holds. We provide below conditions that ensure that (ii) holds. In the

following, we develop a data-driven approach (instead of model-based) that directly synthesizes a controller to solve Problem 1.

III. DATA-BASED REPRESENTATION OF LINEAR THRESHOLD MODELS

In this section, we provide a data-based reformulation of the system (1). This representation describes the system update using the available data samples and does not require knowledge of the unknown matrices W and B .

We start by defining $\mathbf{z}(t) = \mathbf{x}(t+1) - \alpha\mathbf{x}(t)$ and $\mathbf{p}(t) = \text{col}\{\mathbf{x}(t), \mathbf{u}(t)\}$. Then, the update (1) can be rewritten as

$$\mathbf{z}(t) = [H\mathbf{p}(t)]_0^s, \quad (2)$$

where

$$H = \begin{bmatrix} W & B \end{bmatrix} = \begin{bmatrix} - & h_1^\top & - \\ - & h_2^\top & - \\ & \vdots & \\ - & h_n^\top & - \end{bmatrix} \in \mathbb{R}^{n \times (n+m)}.$$

Let $\mathbf{h} = \text{col}\{h_1, h_2, \dots, h_n\} \in \mathbb{R}^{n(n+m)}$ be a vectorized system parameter encoding the matrices W and B . Then,

$$H\mathbf{p}(t) = \begin{bmatrix} h_1^\top \mathbf{p}(t) \\ h_2^\top \mathbf{p}(t) \\ \vdots \\ h_n^\top \mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(t)^\top h_1 \\ \mathbf{p}(t)^\top h_2 \\ \vdots \\ \mathbf{p}(t)^\top h_n \end{bmatrix} = (I_n \otimes \mathbf{p}(t)^\top) \mathbf{h}.$$

Thus, equation (2) reads

$$\mathbf{z}(t) = [(I_n \otimes \mathbf{p}(t)^\top) \mathbf{h}]_0^s. \quad (3)$$

In order to obtain a data-based representation for the system, a key step is to represent \mathbf{h} with the data samples $\{\mathbf{x}_d^+(k), \mathbf{x}_d(k), \mathbf{u}_d(k)\}_{k=1}^{T_d}$. Towards this end, let

$$\mathcal{Z}_d = \begin{bmatrix} \mathbf{x}_d^+(1) - \alpha\mathbf{x}_d(1) \\ \mathbf{x}_d^+(2) - \alpha\mathbf{x}_d(2) \\ \vdots \\ \mathbf{x}_d^+(T_d) - \alpha\mathbf{x}_d(T_d) \end{bmatrix}, \quad \mathcal{P}_d = \begin{bmatrix} I_n \otimes \mathbf{p}_d^\top(1) \\ I_n \otimes \mathbf{p}_d^\top(2) \\ \vdots \\ I_n \otimes \mathbf{p}_d^\top(T_d) \end{bmatrix},$$

with $\mathbf{p}_d(k) = \text{col}\{\mathbf{x}_d(k), \mathbf{u}_d(k)\}$, $\mathcal{Z}_d \in \mathbb{R}^{nT_d}$ and $\mathcal{P}_d \in \mathbb{R}^{nT_d \times n(n+m)}$. According to (3), we have

$$\mathcal{Z}_d = [\mathcal{P}_d \mathbf{h}]_0^s. \quad (4)$$

Define $f(\mathcal{Z}_d) = \mathcal{P}_d \mathbf{h} - [\mathcal{P}_d \mathbf{h}]_0^s \in \mathbb{R}^{nT_d}$ to represent the part of $\mathcal{P}_d \mathbf{h}$ that is truncated by the linear threshold $[\cdot]_0^s$. Clearly, $f(\mathcal{Z}_d)[i] \neq 0$ only if $\mathcal{Z}_d[i] = s$ or $\mathcal{Z}_d[i] = 0$. Thus, we define a diagonal matrix $E_d \in \mathbb{R}^{nT_d \times nT_d}$ such that for all $i \in \{1, \dots, nT_d\}$,

$$E_d[i, i] = \begin{cases} 1 & \text{if } \mathcal{Z}_d[i] = s \text{ or } \mathcal{Z}_d[i] = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Then, there exists a vector $\mathbf{v} \in \mathbb{R}^{nT_d}$ such that

$$f(\mathcal{Z}_d) = E_d \mathbf{v}, \quad (6)$$

which, together with (4), yields $\mathcal{P}_d \mathbf{h} = \mathcal{Z}_d + E_d v$. Using the property that $(I - E_d)E_d = E_d - E_d = 0$, one has

$$(I - E_d)\mathcal{P}_d \mathbf{h} = (I - E_d)\mathcal{Z}_d. \quad (7)$$

This equation describes the vectorized system parameter \mathbf{h} in the form of a data-based equality constraint. We make the following assumption on the data samples.

Assumption 1: (Data richness): Given data samples $\{\mathbf{x}_d^+(k), \mathbf{x}_d(k), \mathbf{u}_d(k)\}_{k=1}^{T_d}$, the matrix $(I - E_d)\mathcal{P}_d$, has full column rank.

Assumption 1 can be directly verified by computation. Note that Assumption 1 becomes easier to satisfy as the number of data samples grows. Based on this assumption, we can combine equations (3) and (7) to obtain a data-based representation of the system dynamics, as stated next.

Lemma 3.1: (Data-based representation): Under Assumption 1, let $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n \times nT_d}$ be such that

$$F(\mathbf{p}) \cdot (I - E_d)\mathcal{P}_d = I_n \otimes \mathbf{p}^\top, \quad (8)$$

for any state-input pair $\mathbf{p} = \text{col}\{\mathbf{x}, \mathbf{u}\}$. Then the dynamics (1) has the following data-based representation,

$$\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + [F(\mathbf{p}(t)) \cdot (I - E_d)\mathcal{Z}_d]_0^s. \quad (9)$$

Proof: Because $(I - E_d)\mathcal{P}_d$ has full column rank, by the Rouché-Capelli Theorem [44], a map $F : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{n \times nT_d}$ satisfying (8) must exist (but may not be unique). Then, from equations (3), (7) and (8), it follows that

$$\begin{aligned} \mathbf{z}(t) &= [(I_n \otimes \mathbf{p}(t)^\top) \mathbf{h}]_0^s \\ &= [F(\mathbf{p}(t)) \cdot (I - E_d)\mathcal{P}_d \mathbf{h}]_0^s \\ &= [F(\mathbf{p}(t)) \cdot (I - E_d)\mathcal{Z}_d]_0^s. \end{aligned}$$

Equation (9) follows since $\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + \mathbf{z}(t)$. ■

We refer to (9) as a data-based representation of system (1) because it does not involve the matrices W and B (or their vectorized version \mathbf{h}), and instead allows, based on the data, to determine the state at the next timestep based on the current state and the control input. The representation is valid for an arbitrary input.

IV. DATA-DRIVEN CONTROL WITH FEED-FORWARD REFERENCE INPUT

In this section, we introduce a data-driven approach to solve Problem 1 with a controller of the following form:

$$\mathbf{u}(t) = K_1 \mathbf{x}(t) + K_2 \mathbf{r}, \quad (10)$$

which is composed of a feed-back gain $K_1 \in \mathbb{R}^{m \times n}$ over the system's current state and a feed-forward gain $K_2 \in \mathbb{R}^{m \times n}$ over the reference input. Our strategy consists of first leveraging the data-based representation obtained Section III to describe the closed-loop system. We then view the resulting linear-threshold system as a switched system and design a common linear feedback controller that stabilizes all modes.

A. Closed-loop data-based representation

We start with a data-based representation of the closed-loop system dynamics under the controller (10).

Lemma 4.1: (Closed-loop data-based representation): Let Assumption 1 hold. The system (1) under the controller (10) admits the data-based representation,

$$\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + [G(\mathbf{x}(t), \mathbf{r}) \cdot (I - E_d)\mathcal{Z}_d]_0^s, \quad (11)$$

where $G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times nT_d}$ satisfies

$$\begin{aligned} G(\mathbf{x}, \mathbf{r}) \cdot (I - E_d)\mathcal{P}_d \\ = I_n \otimes (\mathbf{x}^\top [I_n \quad K_1^\top] + \mathbf{r}^\top [\mathbf{0}_n \quad K_2^\top]). \end{aligned} \quad (12)$$

Proof: By defining

$$G(\mathbf{x}, \mathbf{r}) = F \left(\begin{bmatrix} I_n \\ K_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0}_n \\ K_2 \end{bmatrix} \mathbf{r} \right),$$

the result follows from Lemma 3.1. ■

Equation (11) provides a closed-loop data-based description of system (1). Nevertheless, a closed-form expression for the map G is not readily available from equation (12). In what follows, we provide an explicit construction of this map.

Looking at the right-hand side of (12), we observe a block-diagonal matrix resulting from the Kronecker product. We make use of such special pattern as follows. Define

$$\bar{\mathcal{P}}_d = I_n \otimes \begin{bmatrix} \mathbf{p}_d^\top(1) \\ \mathbf{p}_d^\top(2) \\ \vdots \\ \mathbf{p}_d^\top(T_d) \end{bmatrix} \in \mathbb{R}^{nT_d \times n(n+m)}.$$

Because $\bar{\mathcal{P}}_d$ and \mathcal{P}_d share the same rows but in different orders, there exist a permutation matrix $T_F \in \mathbb{R}^{nT_d \times nT_d}$ such that

$$\bar{\mathcal{P}}_d = T_F \mathcal{P}_d. \quad (13)$$

Since E_d is a diagonal matrix, $\bar{E}_d = T_F E_d T_F^{-1} \in \mathbb{R}^{nT_d \times nT_d}$ is also a diagonal matrix¹ which we write as

$$\bar{E}_d = \text{diag}\{\bar{\mathbf{E}}_1, \dots, \bar{\mathbf{E}}_n\},$$

with $\bar{\mathbf{E}}_i \in \mathbb{R}^{T_d \times T_d}$. Now, let

$$\begin{bmatrix} \bar{\mathbf{z}}_1 \\ \vdots \\ \bar{\mathbf{z}}_n \end{bmatrix} = T_F \mathcal{Z}_d,$$

with $\bar{\mathbf{z}}_i \in \mathbb{R}^{T_d}$, $i \in \{1, \dots, n\}$, and define $Z = \text{diag}\{Z_1, \dots, Z_n\}$ by

$$Z_i = \bar{\mathbf{z}}_i^\top (I_{T_d} - \bar{\mathbf{E}}_i) \in \mathbb{R}^{1 \times T_d}. \quad (14a)$$

Further define $Q = \text{diag}\{Q_1, \dots, Q_n\} \in \mathbb{R}^{nT_d \times n(n+m)}$ by

$$Q_i = (I_{T_d} - \bar{\mathbf{E}}_i) \begin{bmatrix} \mathbf{p}_d^\top(1) \\ \mathbf{p}_d^\top(2) \\ \vdots \\ \mathbf{p}_d^\top(T_d) \end{bmatrix} \in \mathbb{R}^{T_d \times (n+m)}. \quad (14b)$$

¹Let the permutation matrix T_F correspond to the permutation tuple π , which is a reordering of the set $\{1, 2, \dots, nT_d\}$. By definition of a permutation matrix, one has $T_F^{-1} = T_F^\top$, and therefore $\bar{E}_d[j, k] = E_d[\pi[j], \pi[k]]$ [45]. Consequently, $\bar{E}_d[j, k] = 0$ if $j \neq k$.

By definition, we have $Q = (I_{nT_d} - \bar{E}_d)\bar{\mathcal{P}}_d$.

Lemma 4.2: (*Modified closed-loop data-based representation*): Let Assumption 1 hold and consider the data matrices Z and Q defined in (14). The system (1) under the controller (10) can be represented by

$$\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + [Z(M\mathbf{x}(t) + N\mathbf{r})]_0^s, \quad (15)$$

where $M, N \in \mathbb{R}^{nT_d \times n}$ satisfy

$$Q^\top M = \mathbf{1}_n \otimes \begin{bmatrix} I_n \\ K_1 \end{bmatrix}, \quad Q^\top N = \mathbf{1}_n \otimes \begin{bmatrix} \mathbf{0}_{n \times n} \\ K_2 \end{bmatrix}. \quad (16)$$

Proof: We first establish the existence of M, N . Note that

$$\begin{aligned} Q &= (I - \bar{E}_d)\bar{\mathcal{P}}_d = (T_F T_F^{-1} - T_F E_d T_F^{-1}) T_F \mathcal{P}_d \\ &= T_F (I - E_d) \mathcal{P}_d. \end{aligned}$$

Since $(I - E_d)\mathcal{P}_d$ has full column rank, cf. Assumption 1, we deduce that Q^\top must have full row rank. From the Rouché-Capelli Theorem [44], for any K_1, K_2 , there always exist M, N (which may not be unique) satisfying equation (16).

Next, we show that M, N can be used to construct the map G satisfying (12). Let $M = \text{col}\{M_1, \dots, M_n\}$, $N = \text{col}\{N_1, \dots, N_n\}$, with $M_i, N_i \in \mathbb{R}^{T_d \times n}$. From the diagonal structure of Q , (16) can be equivalently rewritten as

$$Q_i^\top M_i = \begin{bmatrix} I_n \\ K_1 \end{bmatrix}, \quad Q_i^\top N_i = \begin{bmatrix} \mathbf{0}_{n \times n} \\ K_2 \end{bmatrix}, \quad (17)$$

for $i \in \{1, \dots, n\}$. Let $\bar{G}_i(\mathbf{x}, \mathbf{r}) = (M_i \mathbf{x} + N_i \mathbf{r})^\top \in \mathbb{R}^{T_d}$, $i \in \{1, \dots, n\}$. Note that

$$\begin{aligned} \bar{G}_i(\mathbf{x}, \mathbf{r}) Q_i &= \mathbf{x}^\top M_i^\top Q_i + \mathbf{r}^\top N_i^\top Q_i \\ &= \mathbf{x}^\top \begin{bmatrix} I_n & K_1^\top \end{bmatrix} + \mathbf{r}^\top \begin{bmatrix} \mathbf{0}_{n \times n} & K_2^\top \end{bmatrix}. \end{aligned}$$

Define $\bar{G}(\mathbf{x}, \mathbf{r}) = \text{diag}\{\bar{G}_1(\mathbf{x}, \mathbf{r}), \dots, \bar{G}_n(\mathbf{x}, \mathbf{r})\}$. One then has

$$\begin{aligned} \bar{G}(\mathbf{x}, \mathbf{r})(I - \bar{E}_d)\bar{\mathcal{P}}_d &= I_n \otimes (\mathbf{x}^\top \begin{bmatrix} I_n & K_1^\top \end{bmatrix} + \mathbf{r}^\top \begin{bmatrix} \mathbf{0}_{n \times n} & K_2^\top \end{bmatrix}). \end{aligned} \quad (18)$$

By letting $G(\mathbf{x}, \mathbf{r}) = \bar{G}(\mathbf{x}, \mathbf{r}) T_F$, it follows that

$$\begin{aligned} G(\mathbf{x}, \mathbf{r})(I - E_d)\mathcal{P}_d &= \bar{G}(\mathbf{x}, \mathbf{r}) T_F T_F^{-1} (I - \bar{E}_d)\bar{\mathcal{P}}_d \\ &= \bar{G}(\mathbf{x}, \mathbf{r})(I - \bar{E}_d)\bar{\mathcal{P}}_d. \end{aligned}$$

This, together with (18), implies that G satisfies (12).

Finally, we show that (15) follows from this fact and equation (11), as follows

$$\begin{aligned} [Z(M\mathbf{x} + N\mathbf{r})]_0^s &= \begin{bmatrix} \bar{z}_1^\top (I - \bar{E}_1)(M_1 \mathbf{x} + N_1 \mathbf{r}) \\ \vdots \\ \bar{z}_n^\top (I - \bar{E}_n)(M_n \mathbf{x} + N_n \mathbf{r}) \end{bmatrix}_0^s \\ &= \begin{bmatrix} \bar{z}_1^\top (I - \bar{E}_1)(\bar{G}_1(\mathbf{x}, \mathbf{r}))^\top \\ \vdots \\ \bar{z}_n^\top (I - \bar{E}_n)(\bar{G}_n(\mathbf{x}, \mathbf{r}))^\top \end{bmatrix}_0^s = \begin{bmatrix} \bar{G}_1(\mathbf{x}, \mathbf{r})(I - \bar{E}_1)\bar{z}_1 \\ \vdots \\ \bar{G}_n(\mathbf{x}, \mathbf{r})(I - \bar{E}_n)\bar{z}_n \end{bmatrix}_0^s \\ &= [\bar{G}(\mathbf{x}, \mathbf{r})(I - \bar{E}_d) T_F \mathcal{Z}_d]_0^s = [G(\mathbf{x}, \mathbf{r})(I - E_d) \mathcal{Z}_d]_0^s, \end{aligned}$$

which completes the proof. \blacksquare

Comparing the statements in Lemmas 4.1 and 4.2, we see that finding $G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times nT_d}$ satisfying (12) can be accomplished by finding the matrices $M, N \in \mathbb{R}^{nT_d \times n}$ satisfying (16). Regarding the solution of this latter equation, we note that when K_1, K_2 are given, M, N can be readily computed. In fact, equations (17) prescribe exactly how to find $\{M_i, N_i\}_{i=1}^n$. However, since our ultimate goal is to design controller gain matrices K_1, K_2 themselves, the solution in M, N and K_1, K_2 of equation (16) poses the challenge of jointly solving for n coupled systems of linear equations. The following result provides a reformulation of the equation showing that K_1, K_2 can be expressed as functions of M, N .

Lemma 4.3: (*Decoupling constraints for closed-loop data-based representation*): Define matrices $L \in \mathbb{R}^{n \times n}$, $C_1 \in \mathbb{R}^{n \times n(n+m)}$ and $C_2 \in \mathbb{R}^{m \times n(n+m)}$ as

$$\begin{aligned} L[i, j] &= \begin{cases} n-1 & \text{if } i = j \\ -1 & \text{otherwise,} \end{cases} \\ C_1 &= [I_n \quad \mathbf{0}_{n \times n(n+m)-n}], \\ C_2 &= [\mathbf{0}_{m \times n} \quad I_m \quad \mathbf{0}_{m \times (n-1)(n+m)}], \end{aligned}$$

and let $\mathcal{L} = L \otimes I_{n+m} \in \mathbb{R}^{n(n+m) \times n(n+m)}$. Then, equation (16) can be equivalently written as

$$\mathcal{L} Q^\top M = \mathbf{0}_{n(n+m) \times n}, \quad \mathcal{L} Q^\top N = \mathbf{0}_{n(n+m) \times n}, \quad (19a)$$

$$C_1 Q^\top M = I_n, \quad C_1 Q^\top N = \mathbf{0}_{n \times n}, \quad (19b)$$

$$C_2 Q^\top M = K_1, \quad C_2 Q^\top N = K_2. \quad (19c)$$

Proof: From the definition of L , one has $\ker(L) = \text{image}(\mathbf{1}_n)$. Thus, $\ker(\mathcal{L}) = \mathbf{1}_n \otimes \text{image}(I_{n+m})$. From (19a), for $i, j \in \{1, \dots, n\}$, it holds

$$Q_i^\top M_i = Q_j^\top M_j, \quad Q_i^\top N_i = Q_j^\top N_j. \quad (20)$$

Furthermore, from (19b) and (19c), one has

$$Q_1^\top M_1 = \begin{bmatrix} I_n \\ K_1 \end{bmatrix}, \quad Q_1^\top N_1 = \begin{bmatrix} \mathbf{0}_{n \times n} \\ K_2 \end{bmatrix}. \quad (21)$$

Thus, equations (20) and (21) are equivalent to (17) for $i \in \{1, \dots, n\}$, which corresponds to (16). Equation (19c) readily follows by left multiplying (16) by C_2 . \blacksquare

The advantage of the constraint formulation (19) over (16) is that, in the former, K_1, K_2 can be expressed as functions of M, N , respectively. In fact, as we vary M, N among all possible solutions of (19a)-(19b), K_1, K_2 in (19c) take every possible value in $\mathbb{R}^{m \times n}$. This means that we can use (15) and (19a)-(19b) to design the closed-loop behavior of the system, with M, N as the only variables. Then, to implement the desired closed-loop system, one can compute K_1, K_2 using (19c) and apply it to the controller (10) of the closed-loop system. In the equations (15) and (19a)-(19b), note that \mathcal{L}, C_1, C_2 are constant matrices; and Z and Q are matrices constructed from the data samples.

B. LMI-based design of feedback gain matrices

Based on the closed-loop data-based representations obtained in Section IV-A, we now introduce a data-driven approach to design the gain matrices for (10). Note that the system (1) is nonlinear due to the presence of the threshold

function. Our strategy is to view it as a switched system and design a common controller that applies to all modes.

We start by defining an error term $\epsilon(t) = \mathbf{x}(t) - \mathbf{r}$. By subtracting \mathbf{r} on both sides of (15), one has

$$\epsilon(t+1) = \alpha\epsilon(t) + [Z(M\mathbf{x}(t) + N\mathbf{r})]_0^s - (1-\alpha)\mathbf{r}. \quad (22)$$

To ensure $\mathbf{x}(t) = \mathbf{r}$, i.e., $\epsilon(t) = 0$, is the equilibrium of the system, one needs $Z(M\mathbf{r} + N\mathbf{r}) = (1-\alpha)\mathbf{r}$. Since the reference signal might be anything satisfying $0 \leq \mathbf{r} \leq \frac{s}{1-\alpha}$, this necessitates

$$Z(M+N) = (1-\alpha)I_n. \quad (23)$$

Based on (23), we rewrite (22) as

$$\begin{aligned} \epsilon(t+1) &= \alpha\epsilon(t) + [ZM\epsilon(t) + (1-\alpha)\mathbf{r}]_0^s - (1-\alpha)\mathbf{r} \\ &= \alpha\epsilon(t) + \bar{R}(\epsilon(t), \mathbf{r})ZM\epsilon(t), \end{aligned} \quad (24)$$

where $\bar{R}(\epsilon, \mathbf{r}) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with each entry $\bar{R}(\epsilon, \mathbf{r})[i, i]$ defined as

$$\begin{cases} \frac{([ZM\epsilon + (1-\alpha)\mathbf{r}]_0^s - (1-\alpha)\mathbf{r})[i]}{(ZM\epsilon)[i]} & \text{if } (ZM\epsilon + (1-\alpha)\mathbf{r})[i] > s \\ & \text{or } (ZM\epsilon + (1-\alpha)\mathbf{r})[i] < 0, \\ 1 & \text{otherwise.} \end{cases}$$

This represents equation (24) as a switched system, where the matrix $\bar{R}(\epsilon, \mathbf{r})$ depends on the state. This dependency makes challenging the stability analysis. To address this, we perform an overapproximation by constructing a convex hull that encompasses all possible values of $\bar{R}(\epsilon, \mathbf{r})$.

By definition, one can derive that, for $i \in \{1, \dots, n\}$,

$$0 < \bar{R}(\epsilon, \mathbf{r})[i, i] \leq 1. \quad (25)$$

Here $\bar{R}(\epsilon, \mathbf{r})[i, i] \neq 0$ because the equality holds only if $([ZM\epsilon + (1-\alpha)\mathbf{r}]_0^s - (1-\alpha)\mathbf{r})[i] = 0$, which means $(ZM\epsilon)[i] = 0$, contradicting the condition that $(ZM\epsilon + (1-\alpha)\mathbf{r})[i] > s$ or $(ZM\epsilon + (1-\alpha)\mathbf{r})[i] < 0$.

To continue, define diagonal matrices $R_j \in \mathbb{R}^{n \times n}$, $j \in \{1, \dots, 2^n\}$, covering all possibilities such that $R_j[i, i] \in \{0, 1\}$ (note that this means that one of these matrices is $\mathbf{0}_{n \times n}$). We let $R_1 = I_n$. Since $\bar{R}(\epsilon, \mathbf{r})$ is a diagonal matrix satisfying (25), regardless of the values of ϵ, \mathbf{r} , it can always be represented as a convex combination:

$$\bar{R}(\epsilon, \mathbf{r}) = \sum_{j=1}^{2^n} \mu_j R_j, \quad \text{with } \sum_{j=1}^{2^n} \mu_j = 1 \quad \text{and } \mu_1 > 0. \quad (26)$$

Here, $\mu_1 > 0$ because $R_1 = I_n$ and all diagonal entries of $\bar{R}(\epsilon, \mathbf{r})$ are strictly positive.

Based on equation (26), the following result provides a way to synthesize the controller.

Theorem 4.4: (Data-driven synthesis via LMIs): Let the matrices $\bar{P} \in \mathbb{R}^{n \times n}$ and $S_1, S_2 \in \mathbb{R}^{n \times n}$ satisfy

$$\begin{bmatrix} \bar{P} & (\alpha\bar{P} + ZS_1)^\top \\ \alpha\bar{P} + ZS_1 & \bar{P} \end{bmatrix} > 0, \quad (27a)$$

$$\begin{bmatrix} \bar{P} & (\alpha\bar{P} + R_j ZS_1)^\top \\ \alpha\bar{P} + R_j ZS_1 & \bar{P} \end{bmatrix} \geq 0, \quad (27b)$$

$$\mathcal{L}Q^\top S_1 = \mathbf{0}_{n(n+m) \times n}, \quad \mathcal{L}Q^\top S_2 = \mathbf{0}_{n(n+m) \times n}, \quad (27c)$$

$$C_1 Q^\top S_1 = \bar{P}, \quad C_1 Q^\top S_2 = \mathbf{0}_{n \times n}, \quad (27d)$$

$$Z(S_1 + S_2) = (1-\alpha)\bar{P}, \quad (27e)$$

for R_j , $j \in \{2, 3, \dots, 2^n\}$. Then, the controller (10), with

$$K_1 = C_2 Q^\top S_1 \bar{P}^{-1}, \quad K_2 = C_2 Q^\top S_2 \bar{P}^{-1}, \quad (28)$$

ensures that \mathbf{r} is asymptotically stable for the closed-loop system.

Proof: To prove the statement, we construct a quadratic Lyapunov function of the form $V(\epsilon) = \epsilon^\top P \epsilon$, with $P > 0$. The convergence result holds if for any $\epsilon(t) \neq 0$, the function satisfies

$$V(\epsilon(t+1)) < V(\epsilon(t)).$$

From (24), this requires

$$\epsilon^\top ((\alpha I + \bar{R}(\epsilon, \mathbf{r})ZM)^\top P (\alpha I + \bar{R}(\epsilon, \mathbf{r})ZM)) \epsilon < \epsilon^\top P \epsilon, \quad (29)$$

for all $\epsilon \neq 0$. To make sure the above equation holds, a typical and sufficient approach in switched systems [46] is to guarantee the following matrix inequality holds

$$(\alpha I + \bar{R}(\epsilon, \mathbf{r})ZM)^\top P (\alpha I + \bar{R}(\epsilon, \mathbf{r})ZM) - P < 0, \quad (30)$$

for all possible values of $\bar{R}(\epsilon, \mathbf{r})$. This effectively bypasses the dependency of $\bar{R}(\epsilon, \mathbf{r})$ on ϵ . Using the Schur complement [47], (30) is equivalent to

$$\begin{bmatrix} P & (\alpha I + \bar{R}(\epsilon, \mathbf{r})ZM)^\top \\ \alpha I + \bar{R}(\epsilon, \mathbf{r})ZM & P^{-1} \end{bmatrix} > 0. \quad (31)$$

Based on the convex combination in (26), it is sufficient to consider the following condition

$$\begin{bmatrix} P & (\alpha I + ZM)^\top \\ \alpha I + ZM & P^{-1} \end{bmatrix} > 0, \quad (32a)$$

$$\begin{bmatrix} P & (\alpha I + R_j ZM)^\top \\ \alpha I + R_j ZM & P^{-1} \end{bmatrix} \geq 0. \quad (32b)$$

for a common $P > 0$, for all $j \in \{2, 3, \dots, 2^n\}$. Here (32a) is obtained by letting $R_1 = I_n$. Since in (26), $\mu_1 > 0$, the convex combination of (32) guarantees (31).

Now, let $\bar{P} = P^{-1}$. Since $\bar{P} > 0$, without losing generality, we introduce two new matrices $S_1 \triangleq M\bar{P}$ and $S_2 \triangleq N\bar{P}$. Pre- and post-multiplying equation (32) by $\begin{bmatrix} \bar{P} & 0 \\ 0 & I \end{bmatrix}$ yields conditions (27a)-(27b), respectively.

To continue, recall that matrices M, N need to satisfy constraints (19a)-(19b) and (23). Since $\bar{P} > 0$, $S_1 = M\bar{P}$ and $S_2 = N\bar{P}$, these constraints can be equivalently written as

$$\mathcal{L}Q^\top M\bar{P} = \mathcal{L}Q^\top S_1 = \mathbf{0}_{n(n+m) \times n},$$

$$\mathcal{L}Q^\top N\bar{P} = \mathcal{L}Q^\top S_2 = \mathbf{0}_{n(n+m) \times n},$$

$$C_1 Q^\top M\bar{P} = C_1 Q^\top S_1 = \bar{P},$$

$$C_1 Q^\top N\bar{P} = C_1 Q^\top S_2 = \mathbf{0}_{n \times n},$$

$$Z(M+N)\bar{P} = Z(S_1 + S_2) = (1-\alpha)\bar{P},$$

which correspond to (27c)-(27e). ■

Note that Theorem 4.4 only provides a sufficient condition

for stabilizing the system (1) to \mathbf{r} . The conservativeness stems from the facts that we search for a *quadratic* Lyapunov function and that, in the derivation from (29) to (32), we ignore the dependency of $\bar{R}(\epsilon, \mathbf{r})$ on ϵ . This means that, even if (32) does not hold, (29) may still be true.

Remark 4.5: (Computational complexity of solving LMIs): LMIs with linear constraints can be efficiently solved using existing algorithms, cf. [48]. The computational complexity of solving an equation of the form (27b) is polynomial in T_d , which is the number of data samples and determines the sizes of matrices Z and Q . However, due to the combinatorial nature in the definition of the vertices R_j , the number of equations in (27b) that need to be solved is of order 2^n , which grows exponentially with the dimension of the system. We address this issue later in Section VI by providing a sufficient condition with reduced computational complexity. \square

It is known [49] that feed-forward loops are non-robust to system disturbances, meaning that slight changes in system parameters may lead to a large degradation in tracking performance. To address this issue, in the next section, we take advantage of the classical idea of integral feedback and design an augmented feedback controller with error integration for better robustness against disturbance.

V. DATA-DRIVEN CONTROL WITH ERROR INTEGRATION

In this section, we design an augmented feedback controller with error integration in the following form

$$\mathbf{u}(t) = K_1(\mathbf{x}(t) - \mathbf{r}) + K_2\boldsymbol{\xi}(t) \quad (33a)$$

$$\boldsymbol{\xi}(t+1) = \boldsymbol{\xi}(t) + (\mathbf{x}(t) - \mathbf{r}) \quad (33b)$$

to solve Problem 1, where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times n}$ are controller gains, and $\boldsymbol{\xi}(t) \in \mathbb{R}^n$ is an integrator that accumulates the system error. Compared with the linear feedback controller (10), the integral controller offers better robustness against state disturbances.

Remark 5.1: (Limitations of the controller with error integration): The augmented feedback controller with error integration in the form of (33) is only applicable to a reference value which excludes the lower and upper bounds of system states, $0 < \mathbf{r} < \frac{\mathbf{s}}{1-\alpha}$. This limitation, which happens to all integral controls when applied to dynamics with threshold saturation, is caused by the following reason. The equilibrium of dynamics (1) under the controller (33) requires the convergence of $\boldsymbol{\xi}(t)$ to a specific value $\boldsymbol{\xi}^*$, which depends on the given \mathbf{r} and the choice of K_2 . If any entry of \mathbf{r} equals 0 or $\frac{s}{1-\alpha}$, then, due to the constraints $0 \leq \mathbf{x}(t) \leq \frac{\mathbf{s}}{1-\alpha}$, the corresponding entry in $(\mathbf{x}(t) - \mathbf{r})$ is either always non-negative or always non-positive. Consequently, the monotonic dynamics of $\boldsymbol{\xi}(t)$ cannot guarantee convergence to the desired $\boldsymbol{\xi}^*$. This issue does not occur when $0 < \mathbf{r} < \frac{\mathbf{s}}{1-\alpha}$. It is worth noting that, in engineering practice, it is rare to require the system states to reach their exact saturation bounds. \square

A. Closed-loop data-based representation

Based on the results in Section III, we first derive a closed-loop data-based representation for system (1) under the controller (33).

Lemma 5.2: (Closed-loop data-based representation for augmented feedback controller with error integration): Consider a controller in the form of (33). Given data matrices Z and Q in (14), let Assumption 1 hold. Then, the system (1) has the data-based representation,

$$\mathbf{x}(t+1) = \alpha\mathbf{x}(t) + [Z(M\mathbf{x}(t) + N\mathbf{r} + U\boldsymbol{\xi}(t))]_0^s, \quad (34)$$

where $M, N, U \in \mathbb{R}^{nT_d \times n}$ satisfy

$$Q^\top M = \mathbf{1}_n \otimes \begin{bmatrix} I_n \\ K_1 \end{bmatrix}, \quad Q^\top U = \mathbf{1}_n \otimes \begin{bmatrix} \mathbf{0}_{n \times n} \\ K_2 \end{bmatrix}, \quad (35a)$$

$$Q^\top N = \mathbf{1}_n \otimes \begin{bmatrix} \mathbf{0}_{n \times n} \\ -K_1 \end{bmatrix}. \quad (35b)$$

Proof: Based on Lemma 3.1, define a new function $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times nT_d}$ satisfying:

$$\Gamma(\mathbf{x}, \mathbf{r}, \boldsymbol{\xi}) = F \left(\begin{bmatrix} I_n \\ K_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0}_n \\ -K_1 \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{0}_n \\ K_2 \end{bmatrix} \boldsymbol{\xi} \right).$$

Then equation (8) yields:

$$\begin{aligned} & \Gamma(\mathbf{x}, \mathbf{r}, \boldsymbol{\xi}) \cdot (I - E_d) \mathcal{P}_d \\ &= I_n \otimes (\mathbf{x}^\top [I_n \ K_1^\top] + \mathbf{r}^\top [\mathbf{0}_{n \times n} \ -K_1^\top] + \boldsymbol{\xi}^\top [\mathbf{0}_{n \times n} \ K_2^\top]), \end{aligned} \quad (36)$$

and the closed-loop data-based representation (9) can be written as

$$\mathbf{x}(t+1) = \alpha\mathbf{x}(t) + [\Gamma(\mathbf{x}(t), \mathbf{r}, \boldsymbol{\xi}) \cdot (I - E_d) \mathcal{Z}_d]_0^s. \quad (37)$$

Based on (36)-(37), and similarly to the proof of Lemma 4.2, one can associate matrices M, N, U with terms $\mathbf{x}, \mathbf{r}, \boldsymbol{\xi}$, respectively, and obtain equations (34)-(35) as counterparts of equations (15)-(16). We omit the details for brevity. \blacksquare

In equation (35), the coupled constraints introduced by the Kronecker product can be equivalently reformulated as follows.

Lemma 5.3: (Decoupling constraints for closed-loop data-based representation): Given $L \in \mathbb{R}^{n \times n}$, $C_1 \in \mathbb{R}^{n \times n(n+m)}$ and $C_2 \in \mathbb{R}^{m \times n(n+m)}$ as in Lemma 4.3, the constraint (35a) can be equivalently written as

$$\mathcal{L}Q^\top M = \mathbf{0}_{n(n+m) \times n}, \quad \mathcal{L}Q^\top U = \mathbf{0}_{n(n+m) \times n} \quad (38a)$$

$$C_1 Q^\top M = I_n, \quad C_1 Q^\top U = \mathbf{0}_{n \times n} \quad (38b)$$

$$C_2 Q^\top M = K_1, \quad C_2 Q^\top U = K_2 \quad (38c)$$

Proof: Given the shared structure of (35a) and (16), the derivation of the equations (38) follows directly from Lemma 4.3. \blacksquare

Note that Lemma 5.3 only characterizes the constraint (35a) and ignores (35b). This is because Q^\top has full row rank. Then, based on the Rouché-Capelli Theorem [44], for any K_1 in (35b), there always exists N such that the equation holds. Furthermore, since K_1 can be determined from M , the constraints (35) can be simplified to only considering (35a). The control matrices K_1 and K_2 for (33) can therefore be designed by determining matrices M, U satisfying (38).

B. LMI-based design of integral feedback gain matrices

We follow an approach similar to that of Section IV-B to propose a data-driven approach to design the feedback gain

matrices in (33) for solving Problem 1. We start by letting $\epsilon(t) = \mathbf{x}(t) - \mathbf{r}$. Based on (33) and (34), one has

$$\begin{aligned}\epsilon(t+1) &= \alpha\epsilon(t) + [Z(M\mathbf{x}(t) + N\mathbf{r} + U\xi(t))]_0^s - (1-\alpha)\mathbf{r}, \\ \xi(t+1) &= \xi(t) + \epsilon(t).\end{aligned}$$

For now, we make the assumption that ZU is non-singular (we show later in Proposition 5.5 that this property actually holds). Based on this assumption, for any \mathbf{r} , there must exist ξ^* such that

$$Z((M+N)\mathbf{r} + U\xi^*) = (1-\alpha)\mathbf{r}. \quad (39)$$

Using this ξ^* , let $\mathbf{e}(t) = \xi(t) - \xi^*$. Then we have

$$\begin{aligned}\epsilon(t+1) &= \alpha\epsilon(t) + [ZM\epsilon(t) + ZU\mathbf{e}(t) + (1-\alpha)\mathbf{r}]_0^s \\ &\quad - (1-\alpha)\mathbf{r} \\ &= \alpha\epsilon(t) + \bar{R}(\epsilon, \mathbf{e}, \mathbf{r})(ZM\epsilon(t) + ZU\mathbf{e}(t)), \\ \mathbf{e}(t+1) &= \mathbf{e}(t) + \epsilon(t),\end{aligned} \quad (40)$$

where $\bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with each entry $\bar{R}(\epsilon, \mathbf{e}, \mathbf{r})[i, i]$ defined as

$$\begin{cases} \frac{([ZM\epsilon + ZU\mathbf{e} + (1-\alpha)\mathbf{r}]_0^s - (1-\alpha)\mathbf{r})[i]}{(ZM\epsilon + ZU\mathbf{e})[i]} & \text{if } (ZM\epsilon + ZU\mathbf{e} + (1-\alpha)\mathbf{r})[i] > s \\ & \text{or } (ZM\epsilon + ZU\mathbf{e} + (1-\alpha)\mathbf{r})[i] < 0 \\ 1 & \text{otherwise} \end{cases}$$

Note that $\bar{R}(\epsilon, \mathbf{e}, \mathbf{r})[i, i]$ is always well defined for any ϵ, \mathbf{e} , and $0 < \mathbf{r} < \frac{s}{1-\alpha}$, because the first two conditions hold only if $(ZM\epsilon + ZU\mathbf{e})[i] \neq 0$. It is also worth noting that, in (40), the matrix N does not appear. This aligns with the fact that the equations (38) do not involve N , and allow us to design K_1, K_2 by only considering M, U .

To continue, similar to the analysis in Section IV-B, we have $0 < \bar{R}(\epsilon, \mathbf{e}, \mathbf{r})[i, i] \leq 1$, for all $i \in \{1, \dots, n\}$. We employ the same matrices $R_j \in \mathbb{R}^{n \times n}$, $j \in \{1, \dots, 2^n\}$, to create a convex combination:

$$\bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) = \sum_{j=1}^{2^n} \mu_j R_j, \quad \text{with } \sum_{j=1}^{2^n} \mu_j = 1 \quad \text{and} \quad \mu_1 > 0. \quad (41)$$

The following result provides a way to synthesize the controller leveraging (41).

Theorem 5.4: (*Data-driven synthesis for augmented feed-back controller with error integration via LMIs*): Given $\bar{P} \in \mathbb{R}^{2n \times 2n}$, consider the block decomposition

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^\top & \bar{P}_{22} \end{bmatrix}$$

with $\bar{P}_{11}, \bar{P}_{12}, \bar{P}_{22} \in \mathbb{R}^{n \times n}$. Let the matrices \bar{P} and $S_1, S_2 \in \mathbb{R}^{n \times n}$ satisfy

$$\begin{bmatrix} \bar{P} & * \\ \begin{pmatrix} \alpha\bar{P}_{11} + ZS_1 & \alpha\bar{P}_{12} + ZS_2 \\ \bar{P}_{11} + \bar{P}_{12}^\top & \bar{P}_{12} + \bar{P}_{22} \end{pmatrix} & \bar{P} \end{bmatrix} > 0, \quad (42a)$$

$$\begin{bmatrix} \bar{P} & * \\ \begin{pmatrix} \alpha\bar{P}_{11} + R_j ZS_1 & \alpha\bar{P}_{12} + R_j ZS_2 \\ \bar{P}_{11} + \bar{P}_{12}^\top & \bar{P}_{12} + \bar{P}_{22} \end{pmatrix} & \bar{P} \end{bmatrix} \geq 0, \quad (42b)$$

$$\mathcal{L}Q^\top S_1 = \mathbf{0}_{n(n+m) \times n}, \quad \mathcal{L}Q^\top S_2 = \mathbf{0}_{n(n+m) \times n} \quad (42c)$$

$$C_1 Q^\top S_1 = \bar{P}_{11}, \quad C_1 Q^\top S_2 = \bar{P}_{12}, \quad (42d)$$

for R_j , $j \in \{2, 3, \dots, 2^n\}$, where $*$ represents the symmetric part of the matrix. Then the controller (33), with

$$[K_1 \quad K_2] = C_2 Q^\top [S_1 \quad S_2] \bar{P}^{-1}, \quad (43)$$

ensures that (\mathbf{r}, ξ^*) is asymptotically stable for the closed-loop system.

Proof: To prove the statement, we construct a quadratic function of the form

$$V(\epsilon, \mathbf{e}) = \begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix}^\top P \begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix} \quad (44)$$

with $P > 0$. The convergence result holds if, for any $\text{col}\{\epsilon(t), \mathbf{e}(t)\} \neq 0$, the function satisfies

$$V(\epsilon(t+1), \mathbf{e}(t+1)) < V(\epsilon(t), \mathbf{e}(t)).$$

From (40), this requires

$$\begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix}^\top \Phi(\epsilon, \mathbf{e}, \mathbf{r})^\top P \Phi(\epsilon, \mathbf{e}, \mathbf{r}) \begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix} < \begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix}^\top P \begin{bmatrix} \epsilon \\ \mathbf{e} \end{bmatrix} \quad (45)$$

when either $\epsilon \neq 0$ or $\mathbf{e} \neq 0$, where

$$\Phi(\epsilon, \mathbf{e}, \mathbf{r}) = \begin{bmatrix} \alpha I + \bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) ZM & \bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) ZU \\ I & I \end{bmatrix}. \quad (46)$$

To make sure (45) holds for the switched system, it is sufficient to guarantee the following matrix inequality holds

$$\Phi(\epsilon, \mathbf{e}, \mathbf{r})^\top P \Phi(\epsilon, \mathbf{e}, \mathbf{r}) < P \quad (47)$$

for all possible values of $\bar{R}(\epsilon, \mathbf{e}, \mathbf{r})$. Using the Schur complement [47], (47) is equivalent to

$$\begin{bmatrix} P & * \\ \begin{pmatrix} \alpha I + \bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) ZM & \bar{R}(\epsilon, \mathbf{e}, \mathbf{r}) ZU \\ I & I \end{pmatrix} & P^{-1} \end{bmatrix} > 0. \quad (48)$$

Leveraging (41), it is sufficient to consider instead

$$\begin{bmatrix} P & * \\ \begin{pmatrix} \alpha I + ZM & ZU \\ I & I \end{pmatrix} & P^{-1} \end{bmatrix} > 0, \quad (49a)$$

$$\begin{bmatrix} P & * \\ \begin{pmatrix} \alpha I + R_j ZM & R_j ZU \\ I & I \end{pmatrix} & P^{-1} \end{bmatrix} \geq 0. \quad (49b)$$

for a common $P > 0$, for all $j \in \{2, 3, \dots, 2^n\}$. Here (49a) is obtained by letting $R_1 = I_n$. Since in (41), $\mu_1 > 0$, the convex combination of (49) guarantees (48).

Now, let

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^\top & \bar{P}_{22} \end{bmatrix} = P^{-1}.$$

Since $\bar{P} > 0$, without losing generality, we introduce

a pair of matrices $\begin{bmatrix} S_1 & S_2 \end{bmatrix} \triangleq \begin{bmatrix} M & U \end{bmatrix} \bar{P} = \begin{bmatrix} M\bar{P}_{11} + U\bar{P}_{12}^\top & M\bar{P}_{12} + U\bar{P}_{22} \end{bmatrix}$. Pre- and post-multiplying (49) by $\begin{bmatrix} \bar{P} & 0 \\ 0 & I \end{bmatrix}$ yields (42a)-(42b), respectively.

To continue, recall that matrices M, U need to satisfy constraints (38a) and (38b). Since $\bar{P} > 0$, and $\begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} M & U \end{bmatrix} \bar{P}$, these constraints can be equivalently written as

$$\begin{aligned} \mathcal{L}Q^\top \begin{bmatrix} M & U \end{bmatrix} \bar{P} &= \mathcal{L}Q^\top \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \mathbf{0}_{n(n+m) \times 2n} \\ C_1 Q^\top \begin{bmatrix} M & U \end{bmatrix} \bar{P} &= C_1 Q^\top \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \end{bmatrix} \end{aligned}$$

which correspond to (42c-d). ■

Theorem 5.4 relies on the fact that ZU is a non-singular matrix, so that ξ^* is well defined. We show that this holds next.

Proposition 5.5: (Non-singular matrix): The matrix ZU , with U obtained from Theorem 5.4, is non-singular.

Proof: We reason by contradiction. Assume ZU is singular. Note that (49a) corresponds to (47) when $\bar{R}(\epsilon, e, r) = I_n$. Considering the block structure of $\Phi(\epsilon, e, r)$ in (46), since ZU is singular, then at least one eigenvalue of $\Phi(\epsilon, e, r)$ is 1. This contradicts (47), which requires all all eigenvalues of $\Phi(\epsilon, e, r)$ to have magnitude strictly less than 1. ■

Remark 5.6: (Computational complexity of solving LMIs, cont'd): The LMIs in (42) have twice the dimensions compared with those in (27). Apart from this, similar to the complexity of solving (27b), the computational complexity of solving a single (42b) is polynomial in T_d . However, the number of equations in (42b) that we need to solve is of order 2^n , which grows exponentially with the dimension of the system. This observation motivates Section VI, which provides a sufficient condition to reduce the computational complexity for both Theorems 4.4 and 5.4. □

VI. SUFFICIENT STABILITY CONDITIONS WITH REDUCED COMPUTATIONAL COMPLEXITY

The high computational cost of Theorems 4.4 and 5.4 arises from the use of matrices R_j in (26) and (41) to construct convex combinations representing $\bar{R}(\epsilon, r)$ and $\bar{R}(\epsilon, e, r)$, respectively. To reduce this complexity, we propose using a smaller set of matrices whose convex combination can still represent $\bar{R}(\epsilon, r)$ and $\bar{R}(\epsilon, e, r)$.

Towards this end, define $\tilde{R}_0 = \mathbf{0}_{n \times n}$. For $k \in \{1, \dots, n\}$, let $\tilde{R}_k \in \mathbb{R}^{n \times n}$ be a diagonal matrix such that

$$\tilde{R}_k[j, j] \triangleq \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

The next result provides an alternative set of LMIs whose number scales linearly with n , that can be used to replace conditions (27b) and (42b), respectively.

Proposition 6.1: (A sufficient condition with reduced computational complexity for solving LMIs):

(i) Let $\bar{P} \in \mathbb{R}^{n \times n}$ and $S_1, S_2 \in \mathbb{R}^{nT_d \times n}$ satisfy

$$\begin{bmatrix} \bar{P} & * \\ \alpha \bar{P} + n \tilde{R}_k Z S_1 & \bar{P} \end{bmatrix} \geq 0 \quad (50)$$

for all \tilde{R}_k , $k \in \{0, 1, \dots, n\}$. Then, these matrices satisfy (27b) for all $i \in \{2, 3, \dots, 2^n\}$;

(ii) Let

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^\top & \bar{P}_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

with $\bar{P}_{11}, \bar{P}_{12}, \bar{P}_{22} \in \mathbb{R}^{n \times n}$ and $S_1, S_2 \in \mathbb{R}^{nT_d \times n}$ satisfy

$$\begin{bmatrix} \bar{P} & * \\ \left(\begin{array}{cc} \alpha \bar{P}_{11} + n \tilde{R}_k Z S_1 & \alpha \bar{P}_{12} + n \tilde{R}_k Z S_2 \end{array} \right) & \bar{P} \end{bmatrix} \geq 0 \quad (51)$$

for all \tilde{R}_k , $k \in \{0, 1, \dots, n\}$. Then these matrices satisfy (42b) for all $i \in \{2, 3, \dots, 2^n\}$.

Proof: For brevity, we only provide the proof for case (ii). The proof for (i) is analogous. Given $j \in \{1, \dots, 2^n\}$, let $a_{jk} = R_j[k, k]$, $k \in \{1, \dots, n\}$. It follows that

$$R_j = \sum_{k=1}^n a_{jk} \tilde{R}_k \quad \text{and} \quad \sum_{k=1}^n a_{jk} \leq n,$$

where the inequality holds because $a_{jk} = R_j[k, k] \in \{0, 1\}$. For convenience, let $\sigma_j = \sum_{k=1}^n a_{jk}$.

For $k \in \{1, \dots, n\}$, we multiply (51) by $\frac{a_{jk}}{n}$ and sum the inequalities over all k to yield

$$\begin{bmatrix} \frac{\sigma_j}{n} \bar{P} & * \\ \left(\begin{array}{cc} \alpha \frac{\sigma_j}{n} \bar{P}_{11} + R_j Z S_1 & \alpha \frac{\sigma_j}{n} \bar{P}_{12} + R_j Z S_2 \end{array} \right) & \frac{\sigma_j}{n} \bar{P} \end{bmatrix} \geq 0. \quad (52)$$

Consider (51) for $\tilde{R}_{k=0} = \mathbf{0}_{n \times n}$. Since $\sigma_j \leq n$, we have

$$\left(1 - \frac{\sigma_j}{n}\right) \begin{bmatrix} \bar{P} & * \\ \left(\begin{array}{cc} \alpha \bar{P}_{11} & \alpha \bar{P}_{12} \end{array} \right) & \bar{P} \end{bmatrix} \geq 0. \quad (53)$$

Thus, for $j \in \{2, 3, \dots, 2^n\}$, adding (52) and (53) results in (42b). ■

The conditions in Proposition 6.1 are more computationally tractable: we have only order n LMIs to consider, instead of order 2^n from (27b) or (42b). However, these conditions are stricter due to the rescaling factor on the off-diagonal elements of the matrix. As a consequence, there might exist matrices satisfying (27b) or (42b), but not the corresponding (50) or (51). Finally, note that when solving the LMIs, conditions (27a) and (42a) still need to be considered because they impose positive definite conditions, whereas from the proof of Proposition 6.1 we can only guarantee positive semi-definiteness for $j = 1$.

VII. CASE STUDIES: VALIDATING DATA-DRIVEN CONTROL IN BIOLOGICAL SYSTEMS

In this section, we present two biological examples [50], [51] to validate the effectiveness of the proposed data-driven control. The first example tackles the regulation of neuro-excitation levels in rodents' brains during selective listening tasks. This example assumes that the excitation levels of specific neuronal populations can be directly manipulated as system inputs. The second example simulates the regulation of human arousal levels using a brain-computer interface

(BCI) for a sensory-motor task. This example builds on a more realistic application with the potential for real-world implementation.

A. Data-driven Regulation of Neuro-Excitation Levels in Rodents' Brain

We consider an experiment that studies the regulation of neural activation levels in rodents' brains for selective listening tasks [50], [52], [53]. The rodents are exposed to a (left/right) white noise burst and a (high/low pitch) narrow-band warble. Depending on the task, they need to focus on one of them and ignore the other. During the experiments, the firing rates of the neuron cells are recorded from two different regions of their brains: the prefrontal cortex (PFC) and the primary auditory cortex (A1).

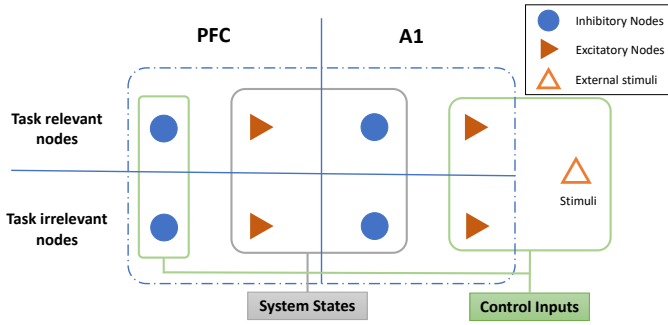


Fig. 1. The model includes 8 groups of neuron cells and one external stimulus. The nodes in the gray box are considered as system states; the nodes in green boxes are considered as control inputs.

We classify the neurons in the rodent brains into $8 = 2^3$ groups based on a combination of: region (PFC, A1); type (excitatory, inhibitory); and encoding (task relevant, irrelevant), as shown in Fig. 1. We choose 4 nodes as system states ($n = 4$), corresponding to the excitatory nodes in the PFC area and the inhibitory nodes in the A1 area. The other 4 nodes along with the external stimuli are considered as system inputs. Building on this configuration, we have identified the following system parameters in our previous work [41]:

$$\alpha = 0.9728, \quad s = 0.3984,$$

$$W = \begin{bmatrix} 0 & 0.0427 & -0.0122 & 0 \\ 0.0084 & 0 & -0.0003 & -0.0009 \\ 0.0421 & 0.0334 & 0 & 0 \\ 0.1031 & 0.0114 & -0.0036 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0114 & -0.0005 & -0.0749 & -0.0017 & 0 \\ -0.0270 & 0.0015 & 0.2107 & 0 & 0 \\ -0.6332 & 0.0044 & -0.2840 & 0 & 0.0358 \\ -0.7236 & 0.0162 & 0.5482 & 0 & 0.0207 \end{bmatrix}.$$

The upper bound of the system state is $\frac{s}{1-\alpha} = 14.647$. Note that the system matrices are assumed to be unknown during the data-driven controller design – we only use them to generate data samples. Based on W , B , α and s , we create data samples with the following discrete-time system model

$$\tilde{x}_d^+(k) = \alpha(\tilde{x}_d(k)) + [W(\tilde{x}_d(k)) + B(\tilde{u}_d(k))]_0^s$$

with $k \in \{1, \dots, T_d\}$ and $T_d = 250$. For different k , system states $\tilde{x}_d(k)$ and inputs $\tilde{u}_d(k)$ are chosen independently, i.e., the entries of $\tilde{x}_d(k)$ are randomly chosen from $[0 \ 14.647]$; the entries of $\tilde{u}_d(k)$ are randomly chosen from $[0 \ 10]$, with uniform distributions. By validation, the data samples satisfy the rank condition in Assumption 1.

To validate Theorem 4.4, we consider the following semi-definite programming (SDP) problem with a random reference input $r = [8.26 \ 4.42 \ 10.99 \ 6.95]^\top$,

$$\text{maximize } \gamma \quad (54a)$$

$$\text{subject to } \bar{P} \leq I_n, \quad (54b)$$

$$\begin{bmatrix} \bar{P} & * \\ \alpha \bar{P} + ZS_1 & \bar{P} \end{bmatrix} \geq \gamma I_{2n}, \quad (54c)$$

$$\begin{bmatrix} \bar{P} & * \\ \alpha \bar{P} + n\tilde{R}_k ZS_1 & \bar{P} \end{bmatrix} \geq 0, \quad (54d)$$

$$\mathcal{L}Q^\top S_1 = 0, \quad \mathcal{L}Q^\top S_2 = 0, \quad (54e)$$

$$C_1 Q^\top S_1 = \bar{P}, \quad C_1 Q^\top S_2 = 0, \quad (54f)$$

$$Z(S_1 + S_2) = (1 - \alpha)\bar{P}, \quad (54g)$$

for all $\tilde{R}_k, k \in \{1, \dots, n\}$. With the solution, we define $K_1 = C_2 Q^\top S_1 \bar{P}^{-1}$ and $K_2 = C_2 Q^\top S_2 \bar{P}^{-1}$ to design a controller

$$u(t) = K_1 x(t) + K_2 r.$$

With γ positive, conditions (54c-g) correspond to Theorem 4.4 and Proposition 6.1(i) (for the sake of reducing computation complexity). Building on this, we add (54a-b) to formulate an SDP problem. The motivation for these two terms is as follows. From the inequality (30), once P is given, the decrease of the Lyapunov function depends on the eigenvalues of the negative matrix in (30). Furthermore, since $0 < r < \frac{s}{1-\alpha}$, based on the definition of $\bar{R}(\epsilon, r)$ under (24), when ϵ is sufficiently small, we have $\bar{R}(\epsilon, r) = I_n$. Substituting this into (30) and applying the Schur complement, the decrease of the Lyapunov function is reflected by the magnitude of positive γ in (54c). This motivates the maximization of γ in (54a). Furthermore, since in (54c), γ can grow linearly (to infinity) with \bar{P} , we need to normalize the magnitude of \bar{P} , leading to the condition (54b). We validate that problem (54) is solvable, and compute the controller gains K_1, K_2 . Given a random initial state $x(0)$, we show in Figure 2 the system trajectory of the closed-loop system. One can observe that all the system states are stabilized at the reference input.

To also validate Theorem 5.4, we consider the following SDP problem

$$\text{maximize } \gamma \quad (55a)$$

$$\text{subject to } \bar{P} \leq I_n \quad (55b)$$

$$\begin{bmatrix} \bar{P} & * \\ \begin{pmatrix} \alpha \bar{I}_n + ZS_1 & \alpha \bar{P}_{12} + ZS_2 \\ \bar{P}_{11} + \bar{P}_{12}^\top & \bar{P}_{12} + \bar{P}_{22} \end{pmatrix} & \bar{P} \end{bmatrix} \geq \gamma I_{4n} \quad (55c)$$

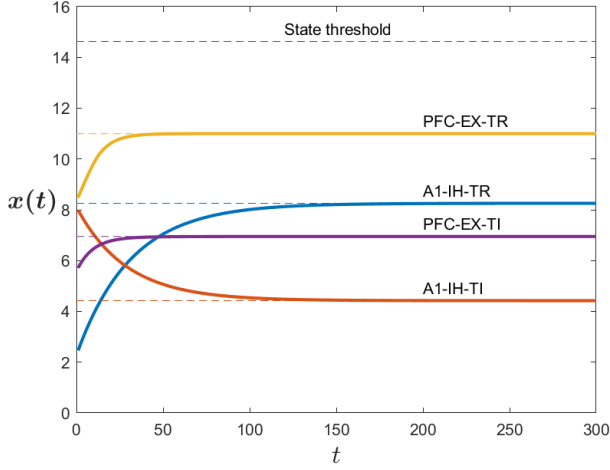


Fig. 2. Data-driven stabilization of a 4-node network. The synthesis of the feedback gain matrix is based on solving the SDP specified in (54).

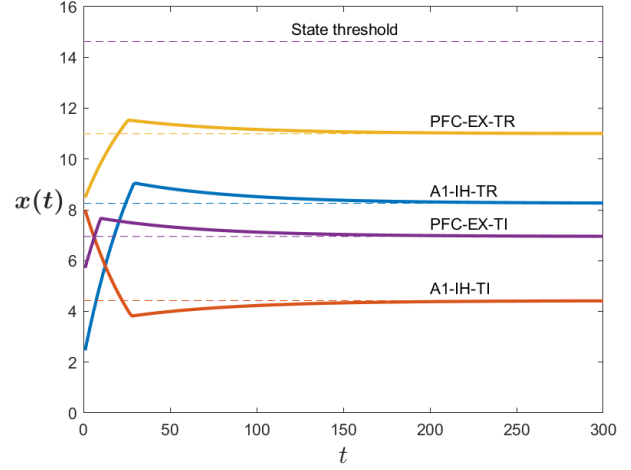


Fig. 3. Data-driven stabilization of a 4-node network. The synthesis of the feedback gain matrix is based on solving the SDP specified in (55).

$$\begin{bmatrix} \bar{P} & * \\ \left(\begin{array}{cc} \alpha \bar{P}_{11} + \tilde{R}_k Z S_1 & \alpha \bar{P}_{12} + \tilde{R}_k Z S_2 \\ \bar{P}_{11} + \bar{P}_{12}^\top & \bar{P}_{12} + \bar{P}_{22} \end{array} \right) & \bar{P} \end{bmatrix} \geq 0 \quad (55d)$$

$$\mathcal{L}Q^\top S_1 = \mathbf{0}, \quad \mathcal{L}Q^\top S_2 = \mathbf{0} \quad (55e)$$

$$C_1 Q^\top S_1 = \bar{P}_{11}, \quad C_1 Q^\top S_2 = \bar{P}_{12} \quad (55f)$$

for all \tilde{R}_k , $k \in \{1, \dots, n\}$.

This allows us to design a controller with integral feedback by defining $[K_1 \ K_2] = C_2 Q^\top [S_1 \ S_2] \bar{P}^{-1}$ and setting

$$\begin{aligned} \mathbf{u}(t) &= K_1(\mathbf{x}(t) - \mathbf{r}) + K_2 \boldsymbol{\xi}(t) \\ \boldsymbol{\xi}(t+1) &= \boldsymbol{\xi}(t) + (\mathbf{x}(t) - \mathbf{r}). \end{aligned}$$

The motivation for this SDP is the same as (54). We validate that this problem is also solvable, and compute the controller gains K_1 , K_2 . Given a random initial state $\mathbf{x}(0)$ and a random reference value $\mathbf{r} = [8.26 \ 4.42 \ 10.99 \ 6.95]^\top$, we show in Figure 3 the trajectory of the closed-loop system with integral feedback control. One can observe that all the states are stabilized at the reference value.

Compared with Figure 2, this result shows a faster response time in general. Furthermore, we observe larger overshoots and non-smoothness on these trajectories. This is caused by the joint effect of the integral term and the linear-threshold function, i.e., when the linear-thresholds are activated, the changing rates of the trajectories are bounded and the integral quickly accumulates; when the linear-thresholds changes from active to deactive, a sharp, non-smooth transition happens to the trajectory.

Effect of system disturbances: We verify the robustness of the two controllers (10) and (33) under random disturbances. Consider the application of disturbance $\omega(t) \in \mathbb{R}^{4 \times 1}$ to system (1) to yield

$$\mathbf{x}(t+1) = \alpha \mathbf{x}(t) + [W\mathbf{x}(t) + B\mathbf{u}(t) + \omega(t)]_0^s. \quad (56)$$

We assume $\omega(t)$ is i.i.d. and at each time step, its entries are randomly chosen from $[0 \ 0.2]$ following a uniform distribu-

tion. Using the data corrupted by noise, we design controllers based on SDPs (54) and (55) to obtain controllers that can stabilize the system states to the desired value \mathbf{r} . Figure 4 shows that the controller (10) is not able to stabilize the system states to the desired values. This is because the controller uses a feed-forward mechanism to handle the reference value, which is not able to compensate the system disturbances.

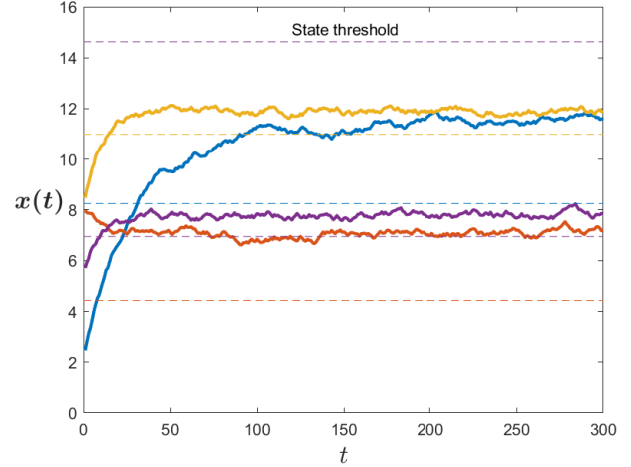


Fig. 4. Data-driven stabilization of the system with controller (10) in the presence of system disturbances. Dashed lines correspond to the values of \mathbf{r} .

In contrast, Figure 5 shows that the augmented feedback controller (55) with error integration is able to stabilize the system states to the desired values and rejects the tracking error caused by the system disturbances.

B. Data-driven Regulation of Arousal in Sensory-Motor Task

We validate the proposed data-driven control approach by stimulating a similar arousal regulation experiment studied in [51]. Arousal refers to the state of being physiologically alert, awake, and attentive, which significantly affects a human's ability to make decisions and take actions in dynamic

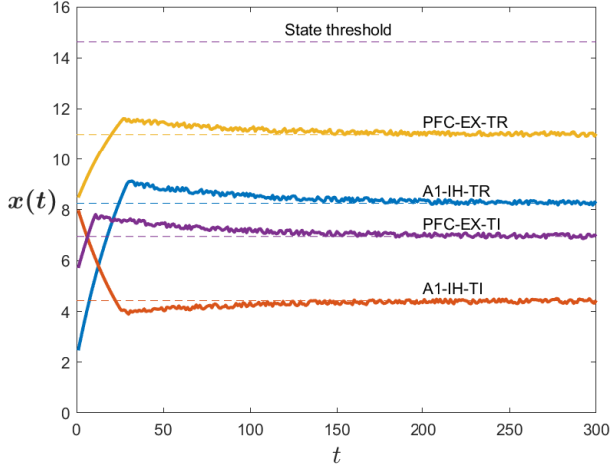


Fig. 5. Data-driven stabilization of the system with controller (55) in the presence of system disturbance. Dashed lines correspond to the values of r .

environments. According to the Yerkes–Dodson Law [54], there is an optimal mid-range level of arousal for peak performance. Deviations from this optimal level can hinder effectiveness, as shown in Figure 6 (right).

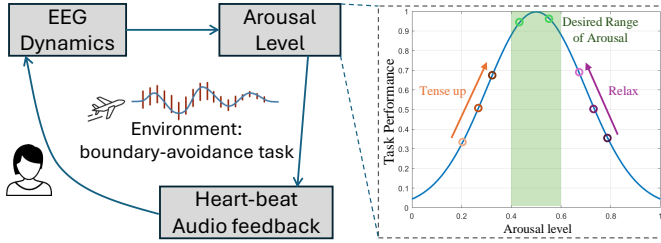


Fig. 6. Left: Regulation of arousal using audio feedback in sensory-motor tasks. Right: The relation between arousal level and task performance.

The experiment performed in [51] considers a boundary-avoidance task (BAT) paradigm, which demands the sensory-motor responsiveness of humans. The experiment uses a virtual reality (VR) environment, where participants navigate a plane through courses of rectangular red waypoints (‘rings’) and the game fails if the plane misses a ring. A neurofeedback mechanism, cf. Figure 6, is employed to regulate humans’ arousal levels. Using EEG signal features (specifically, spectral information), the system decodes a participant’s arousal level and generates auditory feedback in the form of a heartbeat sound. The volume of the sound increases linearly with the decoded arousal level, which ‘warns’ participants when their arousal level becomes excessively high. This feedback helps participants ‘relax’ and maintain their arousal within an optimal range to improve task performance. The experiment demonstrated the effectiveness of this feedback mechanism in regulating arousal levels. It is worth mentioning that in [51], both the arousal decoder and the auditory feedback coefficient derived from arousal levels are qualitative, since true arousal is difficult to quantify and accurately modeling human behavior for optimal control parameters is challenging. Consequently, the parameter choices employed there are based

on experience and experimental results. In contrast, the method proposed in this paper is data-driven, which can circumvent these limitations by enabling a systematic and quantitative design of the controller.

To simulate human EEG dynamics, we construct a linear threshold model in which the system state represents the spectral properties of the human’s EEG signal. The spectral properties of EEG signals can be approximated using linear-threshold networks, as justified by several works [55], [56]. We choose the dimension of the state as $x \in \mathbb{R}^{15}$, which can be justified as follows: we consider 5 frequency-bands (delta, theta, alpha, beta, and gamma: $[0.5, 4]$, $[4, 8]$, $[8, 15]$, $[15, 24]$, and $[24, 50]$ Hz, respectively) of the EEG signal and, for each band, we select three dominant frequencies and normalize them, resulting in a 15-dimensional representation. This approach is similar to the surrogate subspace introduced in [51]. The system input $u \in \mathbb{R}$ represents the volume of the auditory feedback. To map the EEG states to the arousal level a_{rou} , we introduce a linear mapping

$$a_{\text{rou}} = \phi^\top x \in [0\%, 100\%].$$

The model parameters are chosen as follows: the decay rate $\alpha = 0.7$ drives the EEG activities to resting states; we set $s = 0.3$, so that the normalized states x are bounded within $\frac{s}{1-\alpha} = 1$. The state update matrix W characterizes the interdependence of EEG channels and how they change due to factors such as the difficulty of the BAT game; the input matrix B represents how humans’ EEG states are impacted by auditory feedback. Since W and B can vary significantly among different subjects and are difficult to model explicitly, in this simulation, they are randomly generated with each entry chosen uniformly from $[-0.5, 0.5]$. However, to make sure these matrix are meaningful for the experiment, we ensure that an increase in u (auditory feedback volume) leads to a decrease in a_{rou} . This aligns with the experiment setup in [51], where subjects are instructed to relax in response to louder audio feedback. During the experiment, the mapping ϕ^\top that translates EEG states into arousal levels is unknown, so the arousal level cannot be directly decoded, which is different from [51]. Instead, we demonstrate the data-driven method proposed here, that uses EEG state feedback to regulate the arousal level.

Our control objective is to drive the subject’s EEG states to a target frequency pattern r_T . Such r_T is subject-specific, associated with the desired arousal level of the human, and can be determined during the testing trials. In addition, since the EEG states are subject to additive noise, we use the controller with error integration. We formulate the SDP problem (55) and solve for the controller gains K_1 and K_2 , which maps the EEG spectral properties x to a scaler volume of the auditory feedback input u . Figure 7 (left) demonstrates the effectiveness of the obtained data-driven controller in regulating the arousal into a desired range from an overly tensed-up state and maintaining it in this region. The overshoot of the trajectory is caused by the integral term.

Furthermore, Figure 7 (right), demonstrates the relationship between the arousal level and the audio feedback volume. This is done by taking random EEG states x and observing the

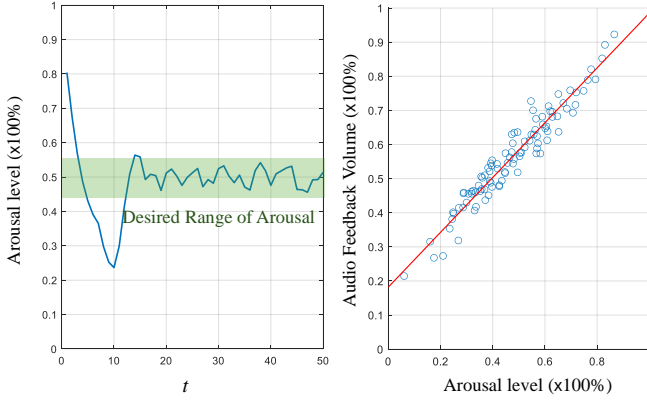


Fig. 7. Left: regulation of arousal using the proposed data-driven controller. Right: relationship between arousal level and audio feedback volume based on the state feedback matrix K_1 .

relations between $a_{\text{rou}} = \phi^T x$ and $u = K_1 x$. Here, since the integrator ξ is dynamically changing, except at equilibrium, its impact on the control input u is difficult to characterize. Therefore, we set $\xi = 0$ and ignore the $K_2 \xi$ term in u . The linear regression of the relation between a_{rou} and u (considering only the proportional term K_1) suggests that the volume should increase with the arousal level of the human, i.e., a higher volume reminds the human that their arousal level is high, and the human will try to calm down to reduce their arousal level. Our result hence aligns with the experiment design in [51] and provides a justification for this feedback mechanism.

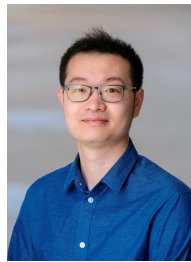
VIII. CONCLUSIONS AND FUTURE WORK

We have designed data-driven controllers to stabilize unknown linear-threshold network models to a given reference value. Exploiting the special structure of the linear-threshold model, we have established a data-based representation of the dynamics relying on a map that reconstructs the system's state-input datasets. Building on this, we have obtained closed-loop data-based representations for two types of data-driven controllers: state feedback with feed-forward reference input and augmented feedback controller with error integration. In both cases, we have shown how to combine these representations with techniques from switched systems theory to identify stabilization conditions in the form of a set of linear matrix inequalities (LMIs), whose solutions correspond to the controller gain matrices. We have formally established the correctness of the proposed designs. Given that the complexity of the LMI formulations grows exponentially with the system state, we have proposed alternative sufficient conditions to solve the LMIs that scale linearly without sacrificing performance. We have validated the effectiveness of the two controllers in two different case studies. Future work will investigate controller designs beyond time-invariant ones, extend our results to scenarios where access to full-state information is not available, and explore the applicability of the results to other case studies.

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