

COSMOS: Making Robots and Making Robots Intelligent

Lecture 3: Introduction to discrete-time dynamics*

Jorge Cortés and William B. Dunbar

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Abstract

In this and the coming lecture, we will introduce the notion of discrete-time dynamical systems and observe some of their cool behaviors. We will learn what fixed points are, what stability means and how to determine it.

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1 Introduction

Set your calculator to radian mode. Suppose you repeatedly press the cosine button starting from an arbitrary number (choose one right now!), say x_0 . The successive readouts are $x_1 = \cos x_0$, $x_2 = \cos x_1$, and so on. Can you explain the surprising result that emerges after many iterations?

The rule

$$x_{n+1} = \cos x_n \tag{1}$$

*Part of the exposition in this lecture builds upon Chapter 10 of S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview Press, Cambridge, 1994.

is an example of *discrete-time dynamics*. If you have taken physics, you may have been introduced to the word “dynamics”. Here is an online encyclopedia entry for dynamics, courtesy of Answers.com¹:

dynamics A branch of mechanics that deals with the motion of objects; it may be further divided into kinematics, the study of motion without regard to the forces producing it, and kinetics, the study of the forces that produce or change motion. Motion is caused by an unbalanced force acting on a body. Such a force will produce either a change in the body’s speed or a change in the direction of its motion (acceleration). The motion may be either translational (straight-line) or rotational, or both.

Dynamics is very broadly referred to in engineering as the general word that described the evolution of any system, be it mechanical (as referenced in the description above), electrical, chemical or even biological. Mathematical equations are used to describe the evolution of a system’s dynamics, e.g., using differential equations.

The rule above in equation (1) is an example of *discrete-time* dynamics. It is referred to as “discrete-time” because this description of the evolution of the system’s behavior is characterized by the value of the variable x as it is evaluated at discrete increments. Specifically, we are looking at the value of x as n increments, starting from 0 and increasing by steps of size 1. Discrete-time dynamics are also referred to as “recursion”, “iterated map”, or sometimes “difference equation”. We will use the reference “map” often in this lecture. A more general iterated map description is

$$x_{n+1} = f(x_n),$$

where f could be any function. For example, $f(x) = x^2$.

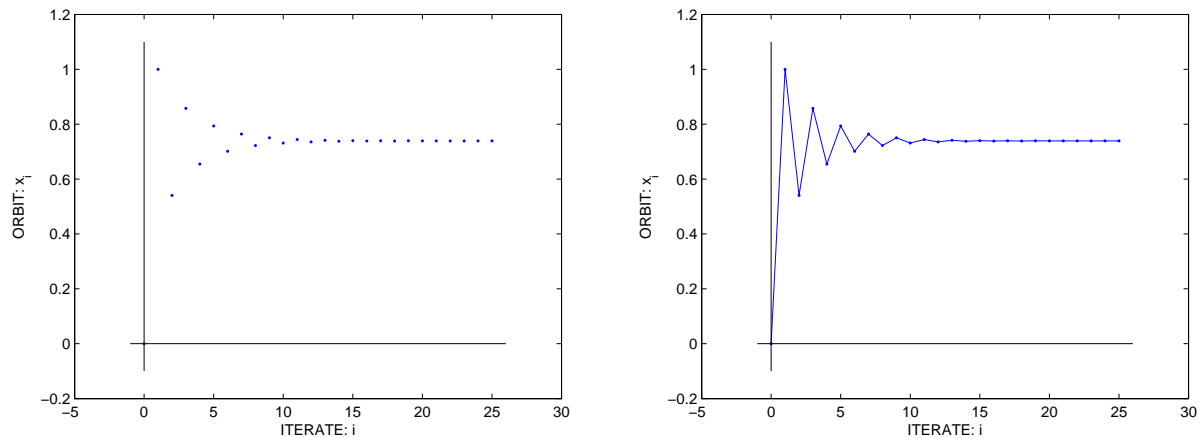
Anyone who has written an algorithm has in fact designed a discrete-time dynamical system (Computers are discrete-time dynamical systems). The nice thing about discrete-time dynamics is that the solution can be calculated by simple algebra; starting from a given x_0 , compute x_1 , and then x_2 , and so on.

From now on, the ordered sequence x_0, x_1, x_2, \dots is called the *orbit* starting from x_0 . The sequence defining the orbit could be finite, that is, if you stop after some number N of iterations, or it could be infinite. As we will soon see, discrete-time dynamics are capable of wild behavior... even *chaos!*

Task 1.1 As usual, begin by going to your working directory. Let’s use MATLAB[®] to plot some orbits of the map $x_{n+1} = \cos x_n$. To do so, you can use a little program - taken from the web² - called `orbit.m`. Make sure you have a copy of this program and `iterates.m` in your working directory. Type `help orbit` in your command window to learn how to use it. For instance, `orbit('cos',0,25)` should yield what you see in Figure 1 (left). Can you figure out how to ‘join the points’ to make the plot look like Figure 1 (right)?

¹<http://www.answers.com/topic/dynamics>

²The routines `orbit.m`, `iterates.m` and `cobweb.m` are taken from the webpage <http://pages.pomona.edu/~aer04747/MatlabFiles.htm>

Figure 1: Orbit of the equation $x_{n+1} = \cos x_n$.

1.1 Cobwebs

A *cobweb* is a graphical approach to iterating a map. It requires that one first have a figure with a plot of $f(x)$ versus x and a plot of x versus x (a diagonal 45 degree line). Figure 2 is an example of such plots for equation (1) with a plot for $\cos(x)$ versus x indicated by the curved line. Given such a figure, we want to find the orbit from some initial point.

In general, the approach works as follows: given the map $x_{n+1} = f(x_n)$ and an initial point x_0 , draw a vertical line at $x = x_0$. The vertical value where the line intersects the graph of $f(x)$ is assigned to x_1 . Now, trace a horizontal line through this intersection point until it intersects the diagonal line x versus x . Draw a vertical line through the diagonal intersection point, which corresponds to a vertical line at $x = x_1$. As before, the vertical location where this line intersects the graph of $f(x)$ is assigned to x_2 . Repeating the process N times yields the first N points in the orbit. Cobwebs can be very helpful, especially for nonlinear systems.

Task 1.2 We can use MATLAB[®] to plot some cobwebs. Consider the map $x_{n+1} = \cos x_n$. You can use a little program called `cobweb.m`. Make sure you have a copy of this program and `iterates.m` in your working directory. Type `help cobweb` in your command window to learn how to use it. For instance, `cobweb('cos', 0, 50, -.25, 1.25)` should yield what you see in Figure 2.

2 Fixed points

Let f be a function. For instance, you can think of $f(x) = \cos x$, or $f(x) = e^{-x}$. Let us start analyzing maps of the form

$$x_{n+1} = f(x_n) \quad (2)$$

A point x_* is a *fixed point* of (2) if $f(x_*) = x_*$. A fixed point is also referred to as an *equilibrium point*, or *equilibrium state*.

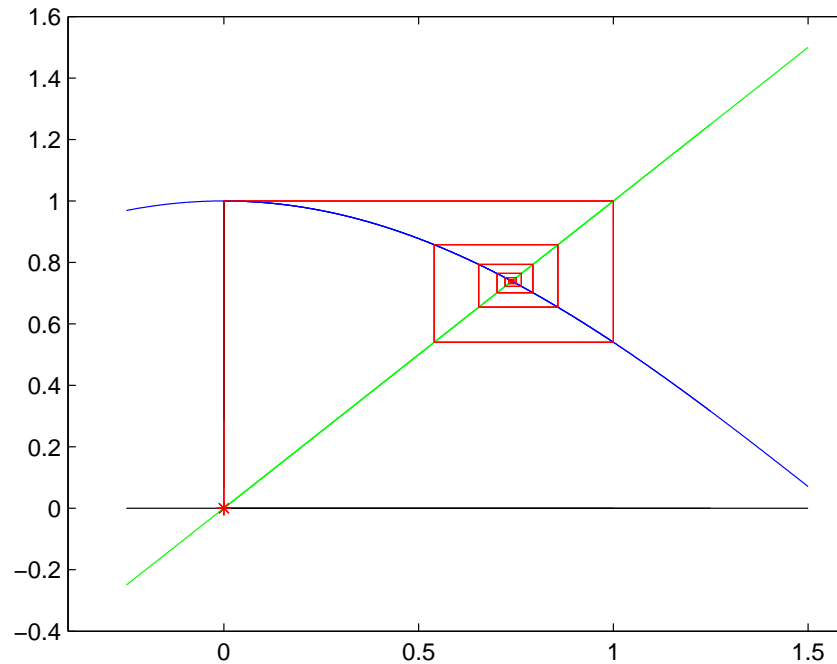


Figure 2: Cobweb of the equation $x_{n+1} = \cos x_n$.

Task 2.1 What is the orbit starting from x_* ?

Task 2.2 What are the fixed points of $f(x) = x + x^2 - 1$?

We are interested in understanding the stability properties of fixed points. The idea is the following: what happens if instead of starting exactly at x_* , we start at a point close to x_* , for example, at $x_* + 0.00001$? Will the orbit approach x_* or will it get repelled from x_* ? Understanding these questions is very important in a wide range of biological and engineering examples. If the orbit tends to stay close to or approach x_* when starting nearby, then x_* is referred to as a **stable** equilibrium state. If, on the other hand, the orbit diverges away from x_* , then x_* is referred to as an **unstable** equilibrium state.

Example 2.3 (Common examples of stable and unstable equilibrium states from S. H. Strogatz. *Sync: the emerging science of spontaneous order*. Hyperion, New York, 2003) Imagine placing a glass of water on the kitchen table. For a second or two, the water splashes around in the glass, then comes to rest. Now the water surface is flat and horizontal. This is an equilibrium state. Additionally, it is stable, because if we shake the glass a little and then stop, the water surface will return to the level. By this example, another way of interpreting stability is that slight disturbances die out. Now, let us consider another example. Take a pencil and sharpen it, the stand it upright and carefully balance it on its point. Let go. If the pencil is poised perfectly, it will continue standing upright, so this is an equilibrium state. However, it is obviously unstable.

The slightest breeze will tip the pencil over and it won't re-right itself. Thus, the upright position of a pencil is an unstable equilibrium state.

Task 2.4 Given an example of a system that has a stable equilibrium state. Given an example of a system that has an unstable equilibrium state.

2.1 Linear maps

Let us start by studying linear maps (which are easy!). Consider a linear map $f(x) = ax$, with a a constant.

Task 2.5 What are the fixed points of $x_{n+1} = f(x_n) = ax_n$?

You should have discovered that the only fixed point is 0, unless $a = 1$, in which case all points are fixed points. Now, what happens if $a \neq 1$ and instead of starting 0, we start close to it? Let's examine this question for some specific values of a .

Task 2.6 Consider the map $f(x) = x/2$, start at $x_0 = 1/2$ and show what happens after many iterations with a cobweb. Do the same with $f(x) = 2x$.

Now, let's generalize our result for any a . After one iteration, we have $x_1 = ax_0$, After two, $x_2 = ax_1 = a^2x_0$. After n iterations, we get $x_n = a^n x_0$. The conclusion is that if $|a| < 1$, then $x_n \rightarrow 0$, and instead if $|a| > 1$, then $x_n \rightarrow \infty$. So the fixed point 0 is **stable** if $|a| \leq 1$ and **unstable** if $|a| > 1$.

2.2 Nonlinear maps

To determine the stability of x_* , let's define the variables $\eta_n = x_n - x_*$, which represent the deviation of the updated variable x_n from the fixed point x_* . Given a starting $\eta_0 \neq 0$, we have the initial point $x_0 = x_* + \eta_0$. Let us see whether the orbit is attracted or repelled from x_* . In other words, let's see whether the deviation η_n grows or decays as n increases. Substitution for the first iteration yields

$$x_1 = f(x_0) = f(x_* + \eta_0) = f(x_*) + f'(x_*)\eta_0 + \text{error}$$

In this equation, we have used what is called a "Taylor approximation". Don't worry if you don't know about it. It simply says that, up to small error, one can approximate the value of the function f at $x_* + \eta$ by the value of f at x_* and a small correction involving the derivative of f at x_* . (Don't worry also if you don't know what a derivative is - we'll provide it for you when necessary).

Now, using $x_1 = x_* + \eta_1$, the above equation can be rewritten as

$$x_* + \eta_1 = x_* + f'(x_*)\eta_0 + \text{error}$$

or equivalently,

$$\eta_1 = f'(x_*)\eta_0 + \text{error}$$

Let us forget about the term error for now. Then, we obtain the linear map

$$\eta_{n+1} = f'(x_*)\eta_n, \quad n = 0, 1, 2, \dots$$

Task 2.7 Based on our knowledge of linear maps, can you find conditions on $f'(x_*)$ that guarantee that the deviation η_n will asymptotically go to 0? Hint: the derivative of $f(x) = ax$ is $f'(x) = a$ for any x .

As for linear maps, we deduce that for $|f'(x_*)| < 1$, the equilibrium point is *linearly stable*, and for $|f'(x_*)| > 1$, the equilibrium point is *linearly unstable*.

Although these conclusions about stability are based on the linearization of the nonlinear map, it can be proved that they also hold for the original nonlinear map.

Task 2.8 Consider the map $x_{n+1} = x_n^2$. Find the fixed points and determine its stability. Draw a cobweb showing your results. Hint: the derivative of $f(x) = x^2$ is $f'(x) = 2x$, for any x .

It turns out that nothing can be said about the stability of a fixed point x_* of any nonlinear system when the linearization gives $|f'(x_*)| = 1$.

Task 2.9 Consider the map $x_{n+1} = \sin x_n$. Find the fixed points. Can you say something about the stability of the linearization? Use a cobweb to show that the fixed points are stable.

VERY IMPORTANT! We could develop similar stability tests for maps f which are not 1-dimensional. Think for instance of the map $f(x_1, x_2) = (\sin x_1, x_1 \cos x_2)$. In order to do this, we would need to learn about matrices and linear algebra (you will learn about this in the future, don't worry!).

Instead, in our course, for higher-dimensional maps, we will test stability by *simulation*. The idea is that, using MATLAB[®], you should check what happens to the orbits of the system if we start from an initial condition close to the equilibrium point. If the system is stable, then no matter what initial condition you choose, the orbit should stay close to the equilibrium point. Check for a lot of different initial conditions to see that this is the case. If the system is unstable, then you should be able to find an initial condition such that the corresponding orbit moves away from the equilibrium.

2.3 The logistic map

The logistic map is defined by

$$x_{n+1} = rx_n(1 - x_n)$$

Here, $x_n \geq 0$ is a measure of the population in the n th generation and $0 \leq r \leq 4$ is the growth rate.

Task 2.10 Plot the function $f(x) = rx(1-x)$ in MATLAB[®] for different values of r , say $r = 1/2, 1, 2$, with values of x between 0 and 1. Where does this function achieve its maximum, that is the value of x where $f(x)$ is the highest? Does this value of x where f is at a maximum depend on the value of r that you chose?

Task 2.11 Find the equilibrium points of the logistic map and determine its stability properties. Hint: the derivative of $f(x) = rx(1 - x)$ is $f'(x) = r(1 - 2x)$ for any x .

The results of the previous question are clarified by the graphical analysis of Figure 3.

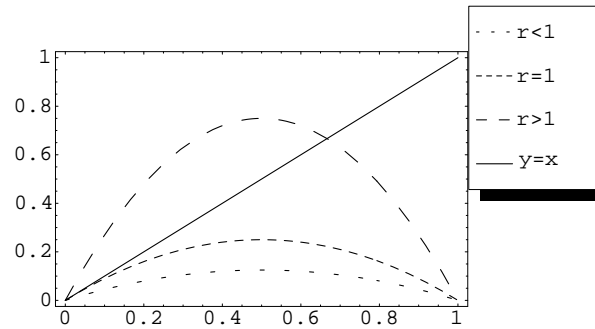


Figure 3: Plot of the function $rx(1 - x)$ for different values of r .

3 Homework

If you have not already done so, complete at least six of the following Tasks introduced in this lecture:

- (i) Task 2.1
- (ii) Task 2.2
- (iii) Task 2.4
- (iv) Task 2.5
- (v) Task 2.6
- (vi) Task 2.7
- (vii) Task 2.8
- (viii) Task 2.11