

COSMOS: Making Robots and Making Robots Intelligent

Lecture 4: Introduction to discrete-time dynamics - the logistic map*

Jorge Cortés and William B. Dunbar

October 27, 2005

Abstract

In this lecture, we play with the logistic equation and see some very wild behavior. If you are interested in more related material, read the cool book S. H. Strogatz. *Sync: the emerging science of spontaneous order*. Hyperion, New York, 2003.

Contents

1	Introduction	1
2	Wild behavior of the logistic map	2
2.1	Period doubling	2
2.2	What the heck happens for $r > r_\infty$? Chaos	2
2.3	Analysing the logistic map	4
3	Homework	5

1 Introduction

Recall that the logistic map is defined by

$$x_{n+1} = rx_n(1 - x_n).$$

As we know, the logistic map is used to model the population growth of some species. Here, $x_n \geq 0$ is a measure of the population in the n th generation and $0 \leq r \leq 4$ is the growth rate.

*The exposition in this lecture builds upon Chapter 10 of S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview Press, Cambridge, 1994.

2 Wild behavior of the logistic map

2.1 Period doubling

Let us play a bit with the logistic map and MATLAB[®] to see some interesting behavior. Recall the handy MATLAB[®] files `orbit.m` and `cobweb.m` that we have been using so far. Once you have started MATLAB[®], do the following

- (i) check that for small growth rate $r < 1$, the population goes extinct: $x_n \rightarrow 0$. Try with $r = 1/2$ and $x_0 = 0.1$.
- (ii) check that for $1 < r < 3$, the population grows, and eventually reaches a nonzero steady state. Try with $r = 2.8$. Plot the first 50 points of the orbit starting from $x_0 = 0.1$.
- (iii) check that for bigger r , say $r = 3.3$, the population builds up again but now *oscillates* about the former steady state, alternating between a large population in one generation and a smaller population in the next. This type of oscillation, in which x_n repeats every two iterations, is called a *period-2 cycle*.
- (iv) things get even more interesting when we continue making r bigger. Try $r = 3.5$. The population approaches a cycle that now repeats every four generations, i.e., a *period-4 cycle*.

Actually, further period doublings occur as r keeps increasing. Computer experiments show the next cool result. Let r_n denote the value of r where a 2^n -cycle first appears. Then we have the following sequence of *bifurcations*,

$r_1 = 3$	period 2 is born
$r_2 = 3.449\dots$	period 4 is born
$r_3 = 3.54409\dots$	period 8 is born
$r_4 = 3.5644\dots$	period 16 is born
\vdots	
$r_\infty = 3.569946\dots$	∞

Successive bifurcations come faster and faster. Ultimately, the r_n converge to the limiting value r_∞ above.

2.2 What the heck happens for $r > r_\infty$? Chaos

According to J. Gleick. *Chaos: Making a New Science*. Viking, New York, 1987 and S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview Press, Cambridge, 1994, Robert May, who was the professor who studied the logistic map, posed a similar question to his students on a corridor blackboard. The answer is complicated. For many values of r , the orbits never settle down to a fixed point or a periodic orbit – instead, the long-term behavior is aperiodic.

Task 2.1 Try $r = 3.9$, and plot using `orbit.m` the first 51 points of the orbit starting at $x_0 = .12$. Try to plot also the corresponding cobweb with `cobweb.m`. You'll see what we mean! (you should get something similar to Figure 1. Try also changing the initial condition).

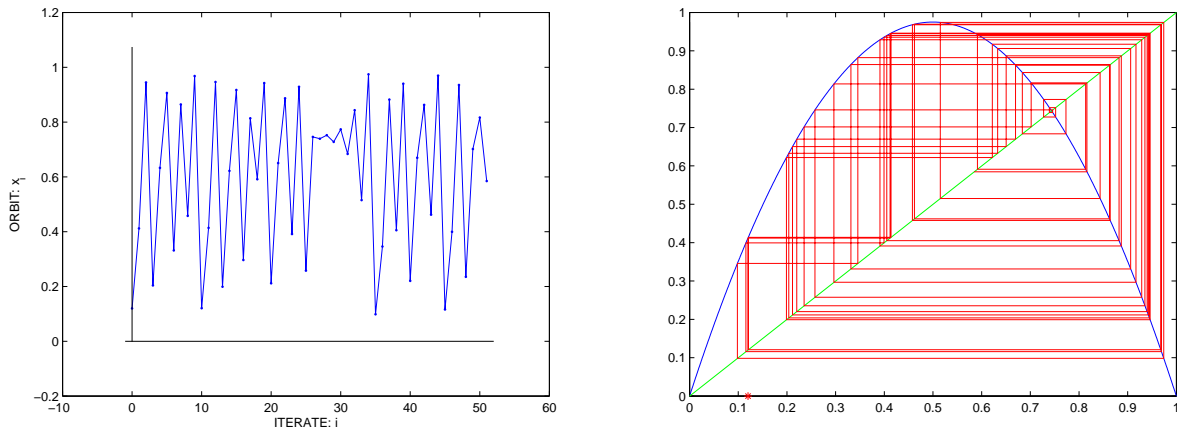


Figure 1: Orbit and corresponding cobweb of the logistic equation.

One good guess is that the dynamics becomes more and more complicated as r increases. Actually, the situation is even more subtle than that. To see the long-term behavior of the system for all values of r at once, we are going to plot the *orbit diagram*. To do that, we need to create a computer program. The idea is described next.

Task 2.2 The orbit diagram is a plot of the orbits of the logistic equation against the values of r . We have to do a computer program with two loops, something like the following.

Name: Orbit diagram algorithm
Goal: Plot orbit diagram of logistic equation

The idea is to have a figure where many orbits of the logistic map are plotted (one for each value of r between 3.4 and 4). So the x -axis of the figure will correspond to values of r , and the y -axis will correspond to values of the orbits.

- 1: Set $i = 0$.
- 2: Set $r = 3.4 + i$.
- 3: Iterate the logistic map for 200 cycles (after 200 cycles, the system should settle down to its eventual behavior – the settling portion of the response is called the *transient*) starting from $x_0 = .6$.
- 4: Once the transients have decayed, plot many points, say x_{201}, \dots, x_{400} , versus the current value of r in the figure.
- 5: Set $i = i + 0.005$. If $r = 4.0$, exit the algorithm. Otherwise, return to step 2.

Once you have the algorithm setup, you should get something similar to Figure 2.

Looking at Figure 2, you can see that at $r = 3.4$, the *attractor* (the set of points to which the orbits eventually tend) is a period-2-cycle, as indicated by the two branches. As r increases, both branches split simultaneously, yielding a period-4 cycle. This splitting process continues as r increases, until

at $r = r_\infty \simeq 3.57$ the map becomes chaotic, and the attractor changes from a finite to an infinite set of points.

For $r > r_\infty$, the orbit diagram reveals an unexpected mixture of order and chaos, with *periodic windows* interspersed between chaotic clouds of dots. The large window beginning near $r \simeq 3.83$ contains a stable period-3 cycle. A blow-up of part of the period-3 window is shown in the right plot of Figure 2. A copy of the orbit diagram reappears in miniature!

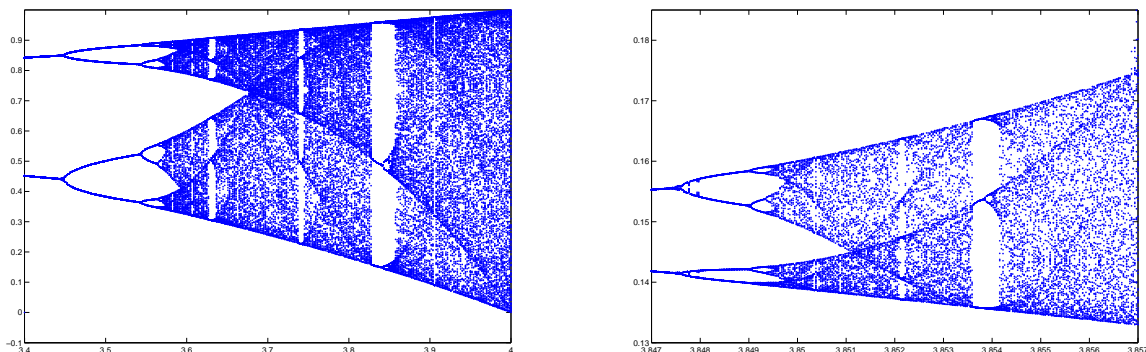


Figure 2: Orbit diagram (left, zoom on the right) of the logistic equation.

2.3 Analysing the logistic map

Let us try to understand the dynamics of the logistic map. Recall that we already know that the equilibrium points are 0 and $1 - 1/r$. The origin is stable for $r < 1$ and unstable for $r > 1$. The other equilibrium point is stable for $1 < r < 3$, and unstable for $r > 3$. Figure 3 gives a graphical clarification of these results. To find the fixed points, you have to look for the intersection between

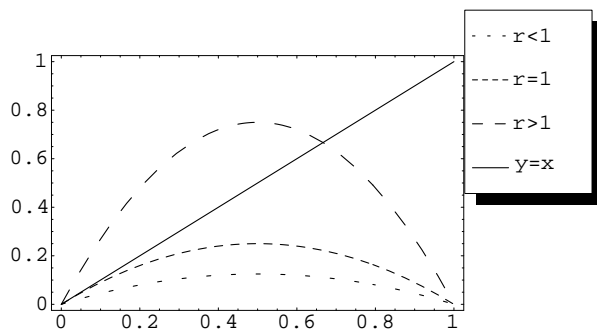


Figure 3: Plot of the function $rx(1-x)$ for different values of r .

the plot of f and the line $y = x$. For $r < 1$, there is only one intersection at $x = 0$ in the interval $[0, 1]$. Right at $r = 1$, a new intersection point forms at $x = 0$. For $r > 1$, there are two intersection points.

Now, we are going to prove the following neat result: “the logistic map has a 2-cycle for any $r > 3$ ”.

How do we do that? Let us first look at what a 2-cycle really means. It simply means that there exists p and q such that

$$f(p) = q, \quad f(q) = p.$$

Now, the cute thing is to realize that this equation is equivalent to

$$f^2(p) = f(f(p)) = p$$

i.e., all we are looking for is for fixed points of f^2 !

Task 2.3 Find an explicit expression for f^2 , where $f(x) = rx(1-x)$. Because f is a quadratic polynomial, then f^2 is a quartic polynomial. Its graph is given in Figure 4.

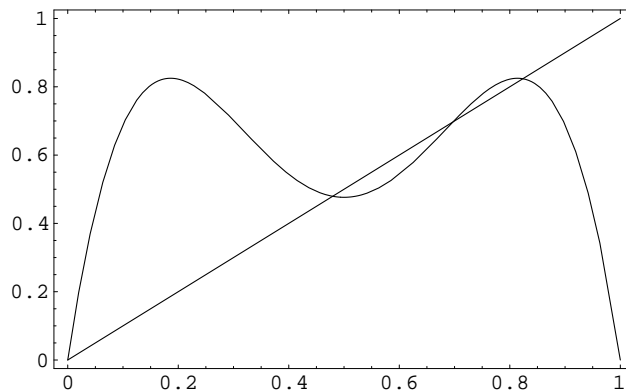


Figure 4: Plot of the function $f^2(x)$

Task 2.4 Find the 4 fixed points of f^2 . As a hint, realize that we already know two – the fixed points of f are necessarily fixed points of f^2 .

How do we determine the stability of a 2-cycle? That's a very good question. Actually, the solution is simple: we reduce the question to study the stability of a fixed point, which we know how to do. Both p and q are fixed points of f^2 , right? Therefore, the original 2-cycle is stable precisely if p and q are stable fixed points of f^2 .

Task 2.5 Building on the previous discussion, determine the stability of the 2-cycle of the logistic map for $r > 3$.

3 Homework

If you have not already done so, attempt the following Tasks introduced in this lecture:

- (i) Task 2.3
- (ii) Task 2.4
- (iii) Task 2.5