

COSMOS: Making Robots and Making Robots Intelligent

Lecture 6: Introduction to feedback control

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Abstract

In this lecture we give our first steps into the realm of feedback control. First, we will get a rough idea of what feedback is. Then, we will see how to turn unstable discrete-time dynamical systems into stable ones using feedback. What we have learned in previous lectures will be very helpful to do this. We will even be able to assign fixed points of the system wherever we want.

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1 Introduction: the magic of feedback

The following paragraph, taken from the introduction of the book “K. J. Åström and R. M. Murray. *Feedback Systems: An Introduction for Scientists and Engineers*¹. Preprint, 2005,” nicely describes what the nature of feedback is.

The term *feedback* is used to refer to a situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled. By dynamical system, we refer to a system whose behavior changes over time, often in response to external stimulation or forcing. Simple causal reasoning about a feedback system is difficult because the first system influences the second and the second system influences the first, leading to a circular argument. This makes reasoning based on

¹Available online at <http://www.cds.caltech.edu/~murray/amwiki>.

cause and effect tricky and it is necessary to analyze the system as a whole. A consequence of this is that the behavior of feedback systems is often counterintuitive and it is therefore necessary to resort to formal methods to understand them.

So, various important things to notice: (i) feedback is “interconnection” between systems; (ii) just when we think we understand how a feedback system works, the interconnection between different systems might have surprises in store for us. As said above, reasoning based on cause and effect gets tricky (it’s like asking: who came first, the chicken or the egg? It gets crazy!); and (iii) you’ll be exposed in the coming lectures to powerful formal methods that will get you through the tricky business and allow you to analyze the behavior of feedback systems.

Feedback allows us to make a system behave as desired. It keeps variables constant and stabilizes unstable systems. Feedback also reduces the effects of disturbances and component variations, allowing a new freedom to designers.

Feedback is ubiquitous in both natural and engineered systems. Everyday life applications of feedback include:

- (i) flight control
- (ii) feedback amplifier for long-distance telephone networks and television networks
- (iii) CD player
- (iv) And so many more...

2 Controlled dynamics

From the standpoint of control it is interesting to explore the possibilities of shaping the dynamic behavior of a system by using feedback control. Let us first introduce the class of systems we are talking about. We consider controlled systems of the form

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

In general, external inputs (forces, electrical currents, chemical concentrations) are bounded (i.e., we cannot produce a force as infinitely large!). Therefore, u is restricted to belong to some bounded set, let’s call it U . We only know about 1-dimensional systems, so for us $U \subset \mathbb{R}$, which in English, means that the set U is a subset of the real numbers. For example, $U = [0, 1]$, which is all numbers x satisfying $0 \leq x \leq 1$. Later in the course we will also consider higher-dimensional systems.

If we do not exert any control over the system (i.e., the system is **unforced**), then $u_k = 0$, and we simply have

$$x_{k+1} = f(x_k, 0)$$

This is a discrete-time dynamical system like the ones we saw in Lecture 3.

Example 2.1 (Cruise control example) Recall the cruise control model

$$v_{k+1} = v_k + \frac{\Delta}{m}[-bv_k + u_{eng,k} + u_{hill}]. \quad (2)$$

Recall also that the speed of the car v_k is the *state* (so $x = v$ here) and the gas pedal $u_{eng,k}$ is the control input. We've included the subscript k now to emphasize that the control signal can change with time. The map f is $f(v, u) = v + \frac{\Delta}{m}[-bv + u_{eng,k} + u_{hill}]$. The unforced system (i.e., when the engine is off and the road is flat) is

$$v_{k+1} = v_k + \frac{\Delta}{m}[-bv_k] \quad (3)$$

Task 2.2 What are the equilibrium points of (3)?

3 Shaping the dynamics: forcing desired equilibrium points

The first thing we may be interested in is to find controls that give desired equilibrium points. For that purpose we will consider a controlled system of the form (1). The equilibrium of this system are given by $f(x_{eq}, u_{eq}) = x_{eq}$. Therefore, the equilibria that can be achieved the points x such that there exist $u \in U$ with $f(x, u) = x$. (Note: The notation $u \in U$ simply means that u is in the set U).

Task 3.1 (Cruise control example) Let v_{ss} a desired velocity for our car. Assume for simplicity that we are driving on a flat road, so that $u_{hill} = 0$ in (2). Find the control $u_{eng,ss}$ that makes v_{ss} an equilibrium.

Based on this discussion, here it is a simple algorithm to make your desired point an equilibrium of the system:

- (i) Select you desired “wanna-be” equilibrium point x_{ss}
- (ii) Find the corresponding control u_{ss} such that $f(x_{ss}, u_{ss}) = x_{ss}$
- (iii) Choose the control input to be $u_k = u_{ss}$ for the system (1).

After performing these two steps, we have the new system

$$x_{k+1} = f(x_k, u_{ss})$$

We have managed to make the “wanna-be” equilibrium point x_{ss} into an actual fixed point of the system! But in general, this equilibrium can be stable or unstable. Let's see it in the cruise-controller example.

Task 3.2 (Cruise control example) Let's plot the orbits of cruise controller example on a flat road once we have chosen the input $u_{eng,ss}$ that makes our desired speed v_{ss} an equilibrium. Therefore, consider the equation

$$v_{k+1} = v_k + \frac{\Delta}{m}[-bv_k + u_{eng,ss}], \quad (4)$$

where you should substitute $u_{eng,ss}$ by the one you obtained in Task 3.1. Write a program in MATLAB[®] that asks for the values of the parameters Δ , b and m , a desired velocity v_{ss} , an initial velocity v_0 and the number of iterations N . The program should plot the first N points of the orbit of (4) starting from v_0 . Use the following set of values: (i) $\Delta = 1$, $m = 100$ and $b = 10$, and (ii) $\Delta = 30$, $m = 100$ and $b = 10$. Is the equilibrium stable in both cases? Why?

In Task 3.2, we have seen that our strategy has been only partially successful so far. We got v_{ss} to be an equilibrium of the system, but depending on the values of the parameters, this equilibrium might be stable or unstable. We certainly don't want this for our car! We want it to be stable, independently of the values that the parameters take (within certain limits). We can do this by means of control. That's what we explain in the next section.

Task 3.3 Consider the following one-dimensional controlled system

$$x_{k+1} = x_k^2 + u_k \tag{5}$$

Let $x_{ss} = 2$. Find the control u_{ss} that makes x_{eq} an equilibrium. Is x_{ss} a stable equilibrium point of the resulting system $x_{k+1} = f(x_k, u_{ss})$? Plot the first 7 points of the orbit starting from $x_0 = 2.001$. You should get something like Figure 1

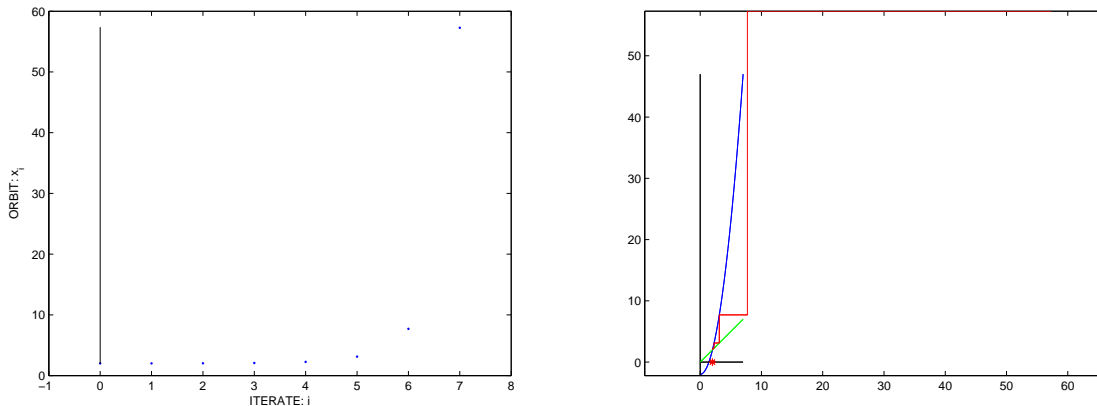


Figure 1: Orbit starting from $x_0 = 2.001$ of the system $x_{k+1} = x_k^2 + u_{ss}$.

4 Shaping the dynamics: making the equilibrium stable

Let's go over our former steps. Remember the simple algorithm we used to make our desired point an equilibrium. Let's slightly modify the last one to keep our options open. We rewrite it as follows:

- (i) Select your desired "wanna-be" equilibrium point x_{ss}
- (ii) Find the corresponding control u_{ss} such that $f(x_{ss}, u_{ss}) = x_{ss}$

- (iii) Choose the control input to be $u_k = u_{ss} + \tilde{u}_k$ for the system (1). You can think of \tilde{u}_k as a new control that we have not chosen yet. This is the way in which we keep our options open.

After performing these three steps, we have the new system

$$x_{k+1} = f(x_k, u_{ss} + \tilde{u}_k) = \tilde{f}(x_k, \tilde{u}_k) \quad (6)$$

with state x and input \tilde{u} . Now, the unforced new system looks like

$$x_{k+1} = \tilde{f}(x_k, 0)$$

which has x_{ss} as an equilibrium point.

Our idea now is to find the feedback \tilde{u}_k that makes the equilibrium stable. Let us illustrate the basic idea of making this equilibrium stable. The controller

$$\tilde{u}_k = K(x_{ss} - x_k) \quad (7)$$

is called a P (P for proportional) controller, because the control action is proportional (with constant K) to the deviation of the state x_k from the desired equilibrium x_{ss} .

The idea is that we can tweak K to make the desired equilibrium stable! Let's see how this works.

Task 4.1 Substitute (7) into equation (6). What is the expression that you get?

After having done Task 4.1, you will have a system that looks like

$$x_{k+1} = \tilde{f}(x_k, K(x_{ss} - x_k))$$

For simplicity, let us denote $g(x_k) = \tilde{f}(x_k, K(x_{ss} - x_k))$. Then, our system simply looks like $x_{k+1} = g(x_k)$. We know that $g(x_{ss}) = x_{ss}$. Now, what do we know about the stability of this equilibrium point? Well, we can use what we learned in Lecture 3 to figure it out. Basically, we have to resort to computing the derivative of g at x_{ss} and see if it is larger than 1 (unstable) or smaller than 1 (stable). Let's do it. To compute the derivative of g , we have to use the chain rule. Don't worry if you do not know about these things. You can just assume that the result obtained in the next equation is correct (but do not forget to come back to these notes in the future once you learn how to compute partial derivatives!). We have that the derivative of g is

$$g'(x_{ss}) = \frac{\partial f}{\partial x}(x_{ss}, 0) - \frac{\partial f}{\partial u}(x_{ss}, 0) \cdot K \quad (8)$$

Let's use this in an example.

Example 4.2 Consider the system of Task 3.3. We have obtained the control $u_{ss} = -2$ that makes $x_{ss} = 2$ an equilibrium of the system $x_{k+1} = x_k^2 + u_k$. Choosing $u_k = u_{ss} + \tilde{u}_k = -2 + \tilde{u}_k$, we obtain the system

$$x_{k+1} = \tilde{f}(x_k, \tilde{u}_k) = x_k^2 - 2 + \tilde{u}_k$$

Note that the unforced system, $x_{k+1} = x_k^2 - 2$ has an unstable equilibrium at $x_{ss} = 2$. Following (7), we choose $\tilde{u}_k = K(x_{ss} - x_k)$ to get

$$x_{k+1} = g(x_k) = x_k^2 - 2 + K(2 - x_k)$$

From equation (8), we compute $g'(x_{ss}) = 2x_{ss} - K = 4 - K$. Therefore, if we choose K such that $3 < K < 5$, we get $|g'(x_{ss})| < 1$ and x_{ss} is stable.

Task 4.3 Pick K between 3 and 5, and plot various orbits starting close to the equilibrium point 2 of the system

$$x_{k+1} = x_k^2 - 2 + K(2 - x_k)$$

You should get orbits that look like the ones in Figure 2. What difference do you observe with respect to the plot in Task 3.3?

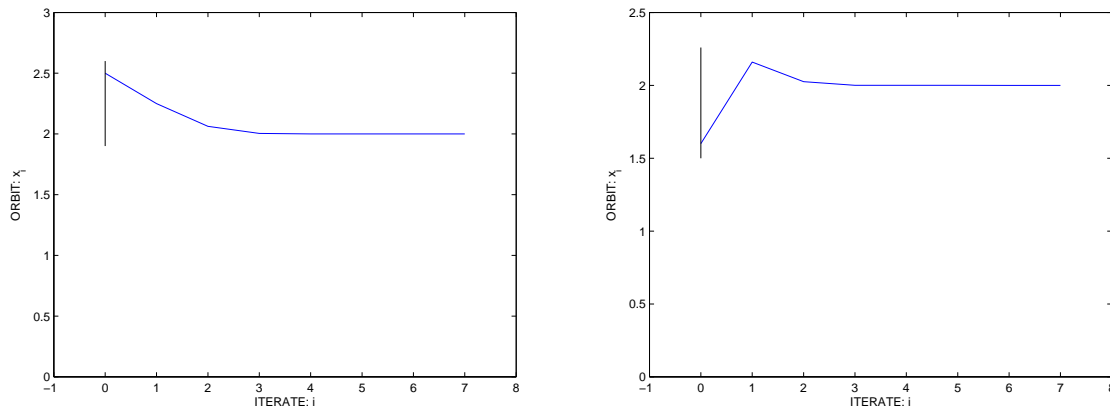


Figure 2: Orbits starting from $x_0 = 2.5$ (left) and $x_0 = 1.6$ (right) of the system $x_{k+1} = x_k^2 - 2 + 4(2 - x_k)$.

Example 4.2 reveals the magic of feedback. By appropriately choosing the constant K , we can make stable an otherwise unstable equilibrium! How to choose K is determined by equation (8): you should pick K such that $|g'(x_k)| < 1$.