

# Systematic Circuit Analysis (T&R Chap 3)

## Node-voltage analysis

Using the voltages of the each node relative to a ground node, write down a set of consistent linear equations for these voltages

Solve this set of equations using, say, Cramer's Rule

## Mesh current analysis

Using the loop currents in the circuit, write down a set of consistent linear equations for these variables. Solve.

This introduces us to procedures for *systematically* describing circuit variables and solving for them

# Nodal Analysis

## Node voltages

Pick one node as the ground node 

Label all other nodes and assign voltages  $v_A, v_B, \dots, v_N$   
and currents with each branch  $i_1, \dots, i_M$

Recognize that the voltage across a branch  
is the difference between the end node  
voltages

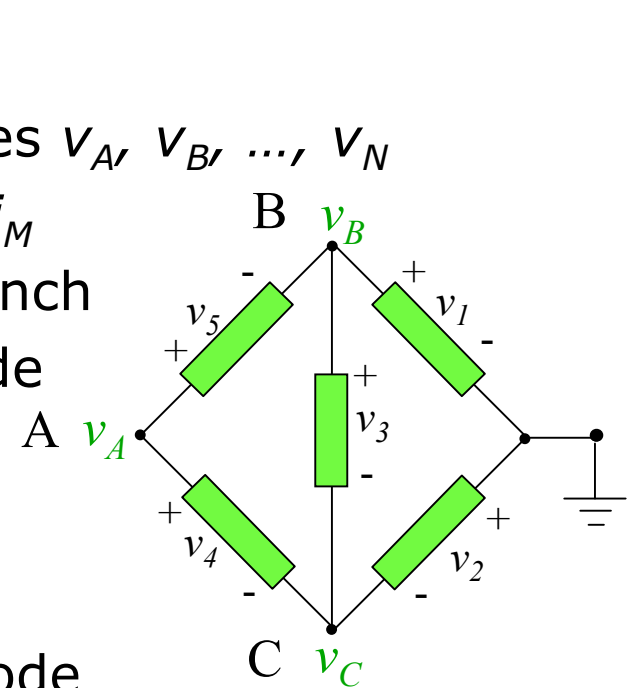
Thus  $v_3 = v_B - v_C$  with the direction  
as indicated

Write down the KCL relations at each node

Write down the branch  $i-v$  relations to express branch  
currents in terms of node voltages

Accommodate current sources

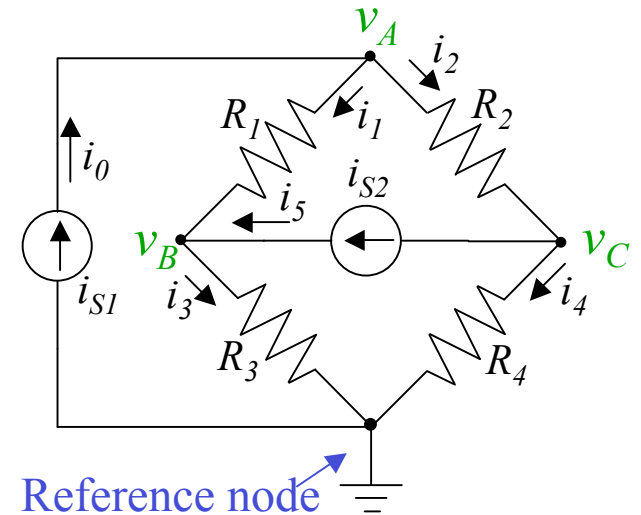
Obtain a set of linear equations for the node voltages



# Nodal Analysis – Ex 3-1 (T&R, 5th ed, p.72)

Apply KCL

Write the element/branch eqns



Substitute to get node voltage equations

Solve for  $v_A$ ,  $v_B$ ,  $v_C$  then  $i_0$ ,  $i_1$ ,  $i_2$ ,  $i_3$ ,  $i_4$ ,  $i_5$

# Nodal Analysis – Ex 3-1 (T&R, 5th ed, p.72)

## Apply KCL

$$\text{Node A: } i_0 - i_1 - i_2 = 0$$

$$\text{Node B: } i_1 - i_3 + i_5 = 0$$

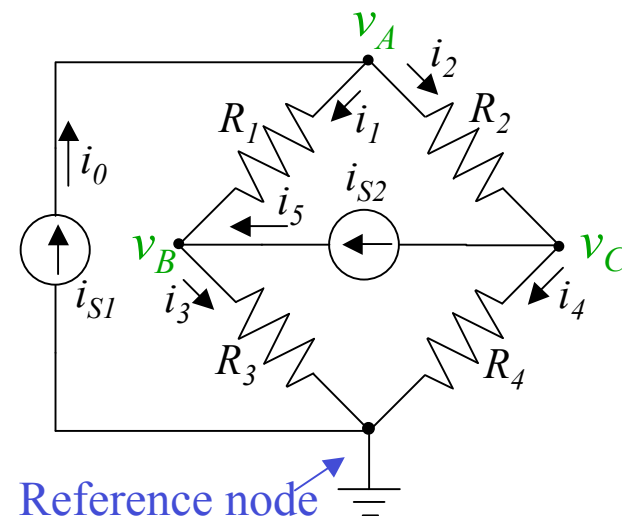
$$\text{Node C: } i_2 - i_4 - i_5 = 0$$

## Write the element/branch eqns

$$i_0 = i_{S1} \qquad i_3 = G_3 V_B$$

$$i_1 = G_1 (V_A - V_B) \qquad i_4 = G_4 V_C$$

$$i_2 = G_2 (V_A - V_C) \qquad i_5 = i_{S2}$$



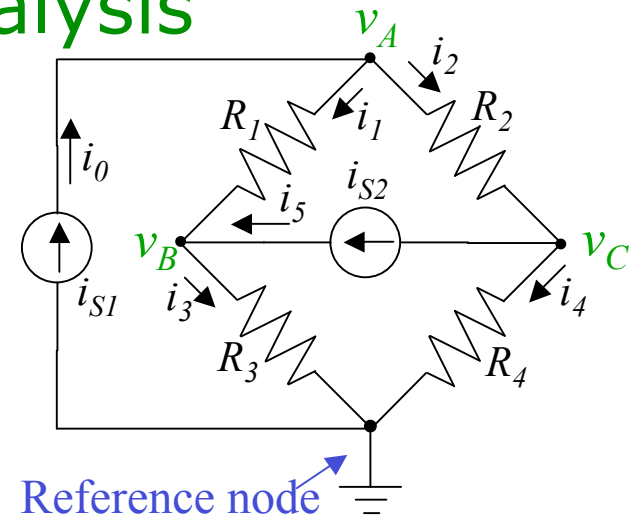
## Substitute to get node voltage equations

$$\begin{aligned} \text{Node A: } (G_1 + G_2)V_A - G_1V_B - G_2V_C &= i_{S1} \\ \text{Node B: } -G_1V_A + (G_1 + G_3)V_B &= i_{S2} \\ \text{Node C: } -G_2V_A + (G_2 + G_4)V_C &= -i_{S2} \end{aligned} \quad \begin{pmatrix} G_1 + G_2 & -G_1 & -G_2 \\ -G_1 & G_1 + G_3 & 0 \\ -G_2 & 0 & G_2 + G_4 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$

## Solve for $v_A, v_B, v_C$ then $i_0, i_1, i_2, i_3, i_4, i_5$

# Systematic Nodal Analysis

$$\begin{pmatrix} G_1+G_2 & -G_1 & -G_2 \\ -G_1 & G_1+G_3 & 0 \\ -G_2 & 0 & G_2+G_4 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_{S1} \\ i_{S2} \\ -i_{S2} \end{pmatrix}$$



## Writing node equations by inspection

Note that the matrix equation looks

just like  $\underline{G}\underline{v}=\underline{i}$  for matrix  $\underline{G}$  and vector  $\underline{v}$  and  $\underline{i}$   
 $\underline{G}$  is symmetric (and non-negative definite)

Diagonal  $(i,i)$  elements: sum of all conductances connected to node  $i$

Off-diagonal  $(i,j)$  elements: -conductance between nodes  $i$  and  $j$

Right-hand side: current sources entering node  $i$

There is no equation for the ground node – the column sums give the conductance to ground

# Nodal Analysis Ex. 3-2 (T&R, 5th ed, p.74)

## Node A:

Conductances

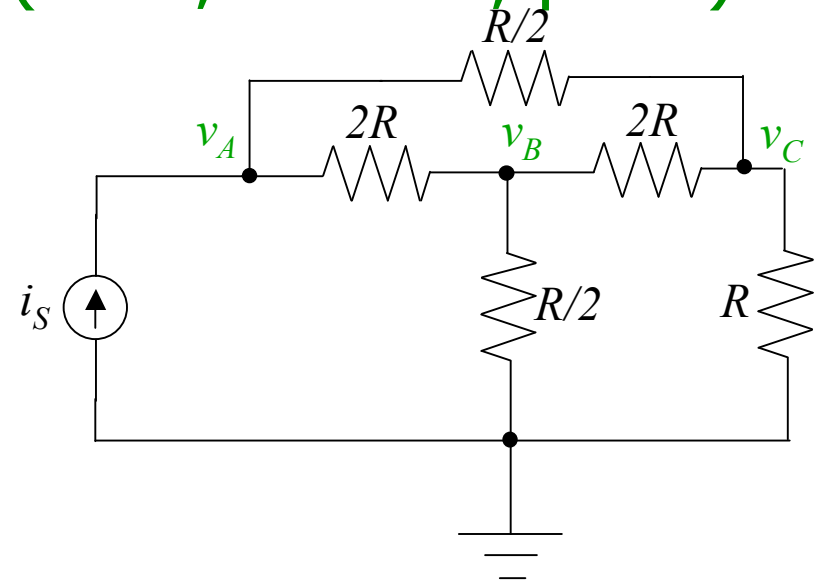
Source currents entering

## Node B:

Conductances

## Node C:

Conductances



Source currents entering

Source currents entering

# Nodal Analysis Ex. 3-2 (T&R, 5th ed, p.74)

## Node A:

Conductances

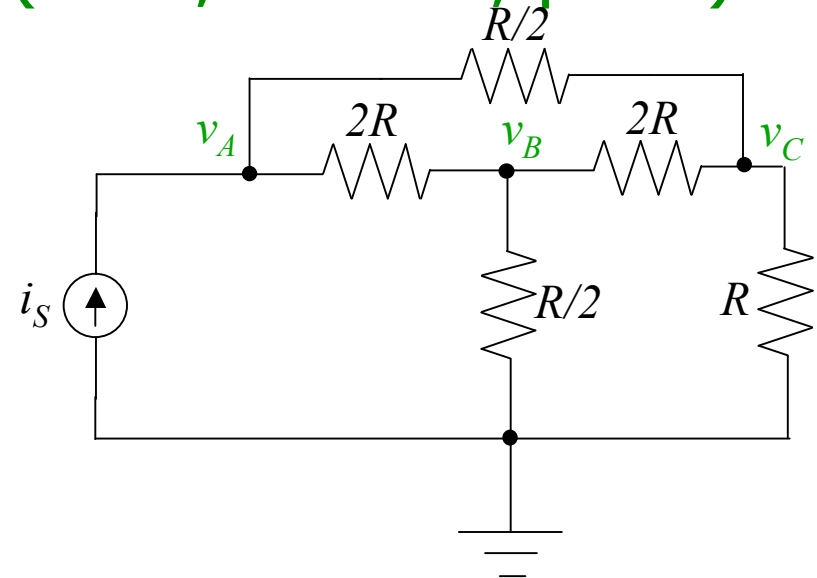
$$G/2_B + 2G_C = 2.5G$$

Source currents entering =  $i_S$

## Node B:

Conductances

$$G/2_A + G/2_C + 2G_{ground} = 3G$$



Source currents entering = 0

## Node C:

Conductances

$$2G_A + G/2_B + G_{ground} = 3.5G$$

Source currents entering = 0

$$\begin{pmatrix} 2.5G & -0.5G & -2G \\ -0.5G & 3G & -0.5G \\ -2G & -0.5G & 3.5G \end{pmatrix} \begin{pmatrix} v_A \\ v_B \\ v_C \end{pmatrix} = \begin{pmatrix} i_S \\ 0 \\ 0 \end{pmatrix}$$

## Nodal Analysis – some points to watch

1. The formulation given is based on KCL with the sum of currents **leaving** the node

$$0 = i_{total} = G_{AtoB}(v_A - v_B) + G_{AtoC}(v_A - v_C) + \dots + G_{AtoGround}v_A + i_{leavingA}$$

This yields

$$0 = (G_{AtoB} + \dots + G_{AtoGround})v_A - G_{AtoB}v_B - G_{AtoC}v_C \dots - i_{enteringA}$$

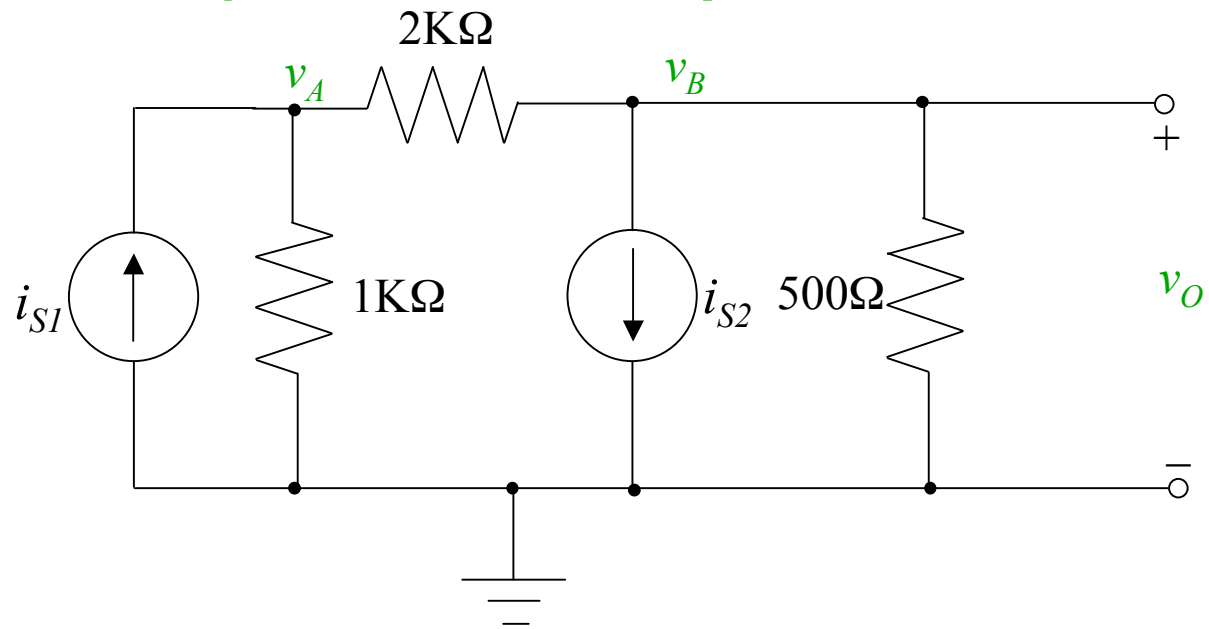
$$(G_{AtoB} + \dots + G_{AtoGround})v_A - G_{AtoB}v_B - G_{AtoC}v_C \dots = i_{enteringA}$$

2. If in doubt about the sign of the current source, go back to this basic KCL formulation
3. This formulation works for independent current sources

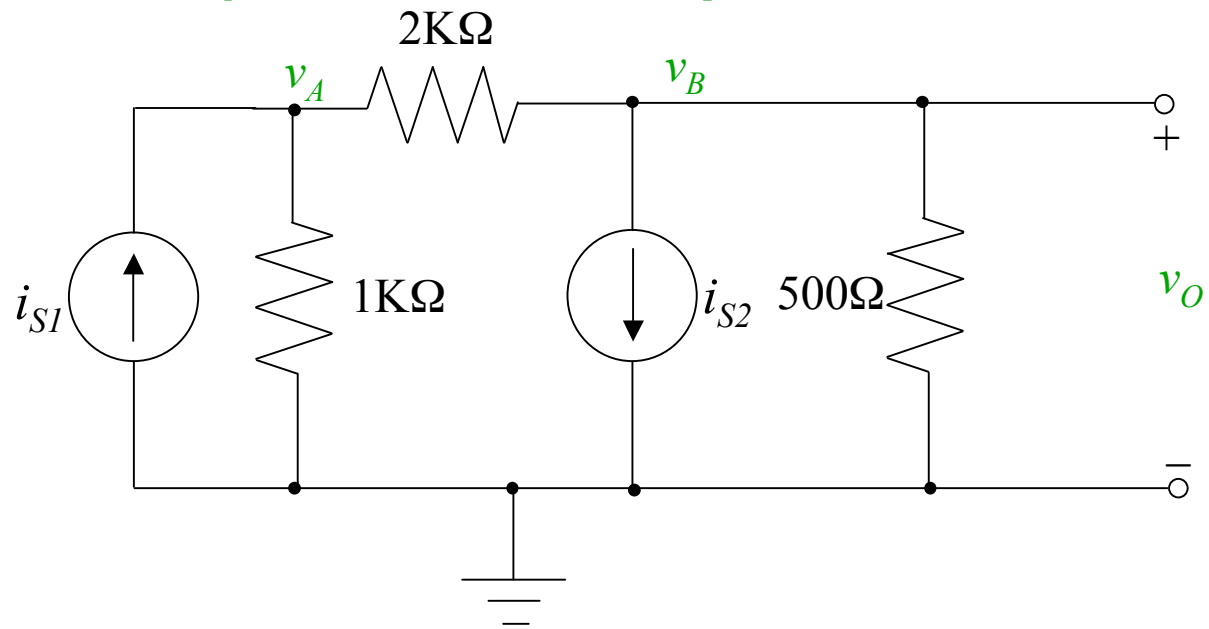
For dependent current sources (introduced later) use your wits



# Nodal Analysis Ex. 3-2 (T&R, 5th ed, p. 72)



## Nodal Analysis Ex. 3-2 (T&R, 5th ed, p. 72)



$$\begin{pmatrix} 1.5 \times 10^{-3} & -0.5 \times 10^{-3} \\ -0.5 \times 10^{-3} & 2.5 \times 10^{-3} \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix} = \begin{pmatrix} i_{S1} \\ -i_{S2} \end{pmatrix}$$

Solve this using standard linear equation solvers

Cramer's rule

Gaussian elimination

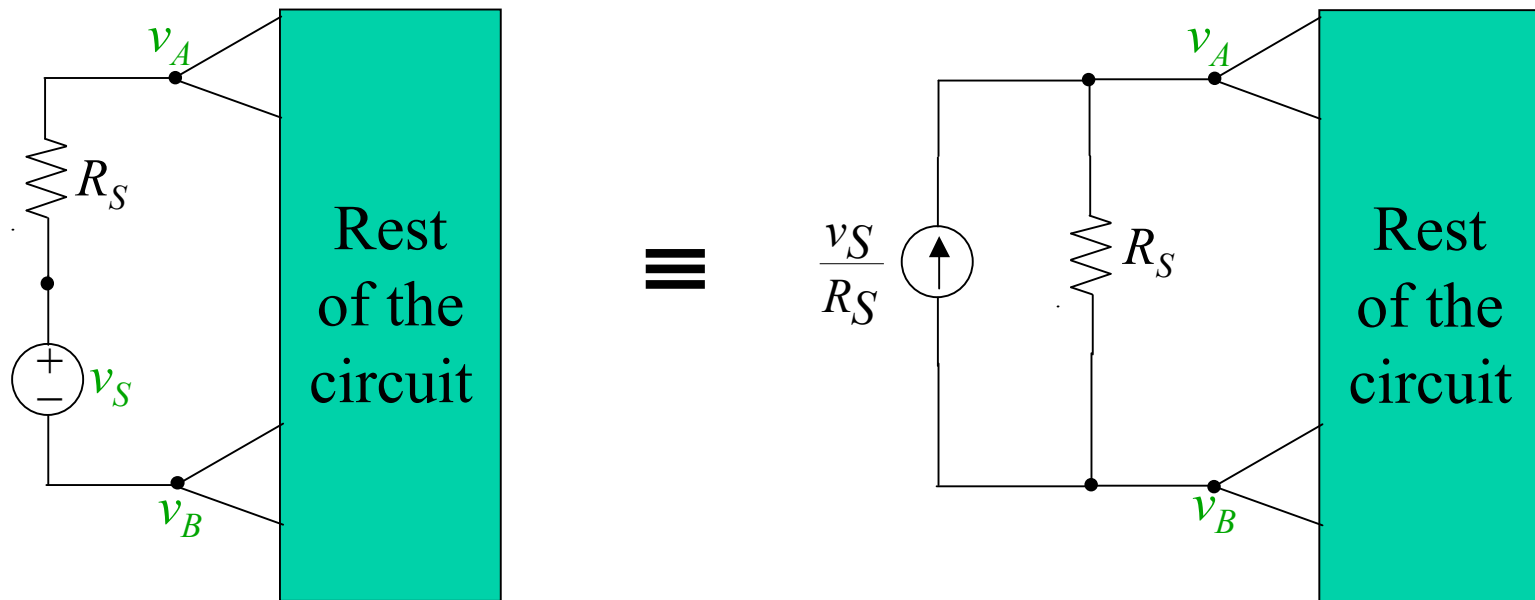
Matlab

# Nodal Analysis with Voltage Sources

Current through voltage source is not computable from voltage across it. We need some tricks!

They actually help us simplify things

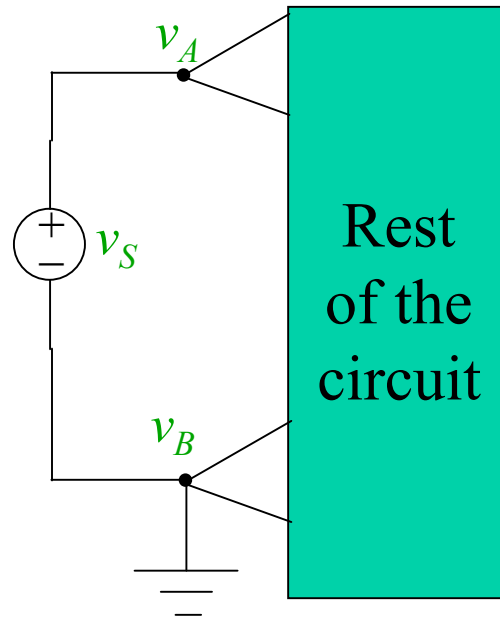
Method 1 – source transformation



Then use standard nodal analysis – one less node!

# Nodal Analysis with Voltage Sources 2

## Method 2 – grounding one node



This removes the  $v_B$  variable – simpler analysis  
But can be done once per circuit

# Nodal Analysis with Voltage Sources 3

## Method 3

Create a *supernode*

Act as if A and B were one node

KCL still works for this node

Sum of currents entering  
supernode box is 0

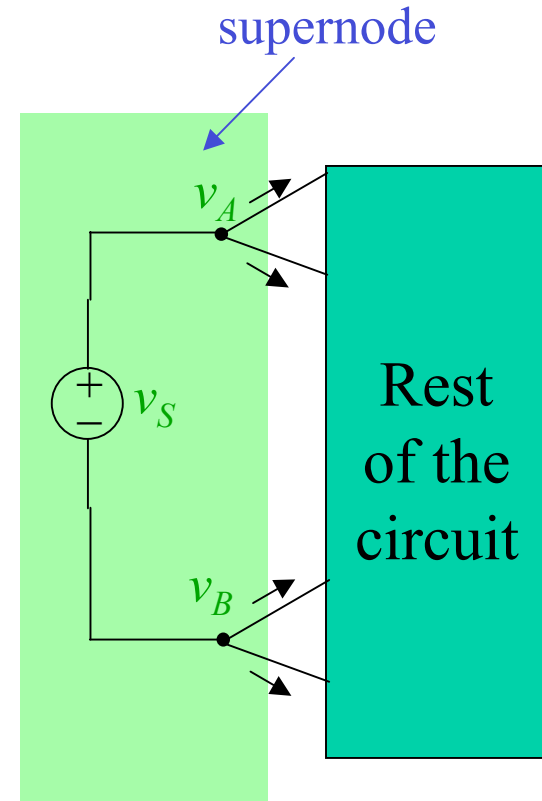
Write KCL at all N-3 other nodes

N-2 nodes less Ground node

using  $v_A$  and  $v_B$  as usual

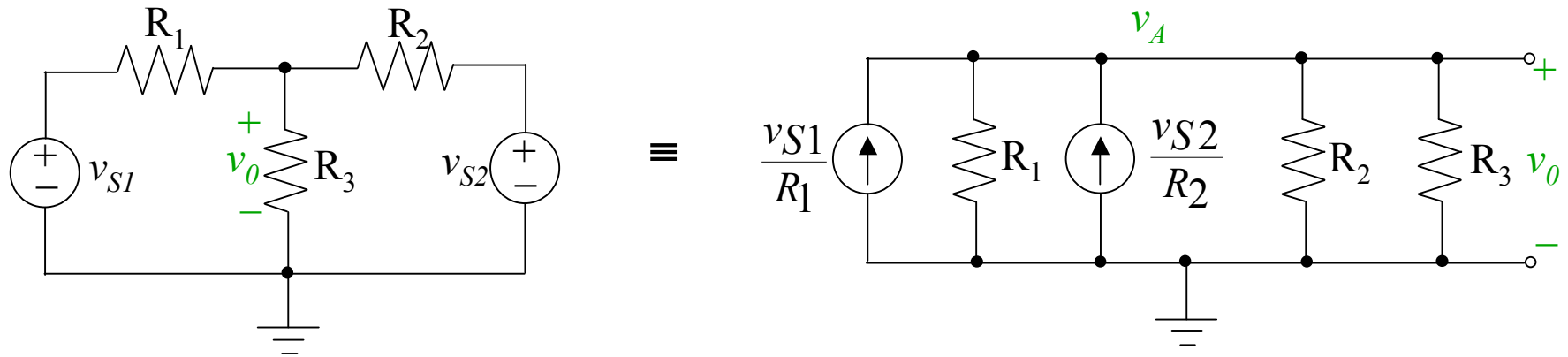
Write one supernode KCL

Add the constraint  $v_A - v_B = v_S$



These three methods allow us to deal with all cases

## Nodal Analysis Ex. 3-4 (T&R, 5th ed, p. 76)

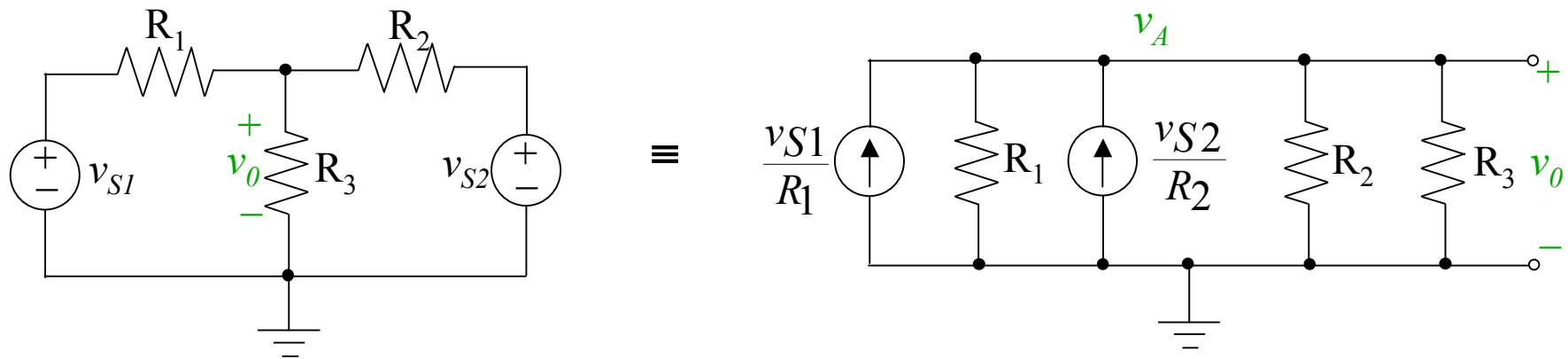


This is method 1 – transform the voltage sources

Applicable since voltage sources appear in series with Resist

Now use nodal analysis with one node, A

## Nodal Analysis Ex. 3-4 (T&R, 5th ed, p. 76)



This is method 1 – transform the voltage sources

Applicable since voltage sources appear in series with Resist

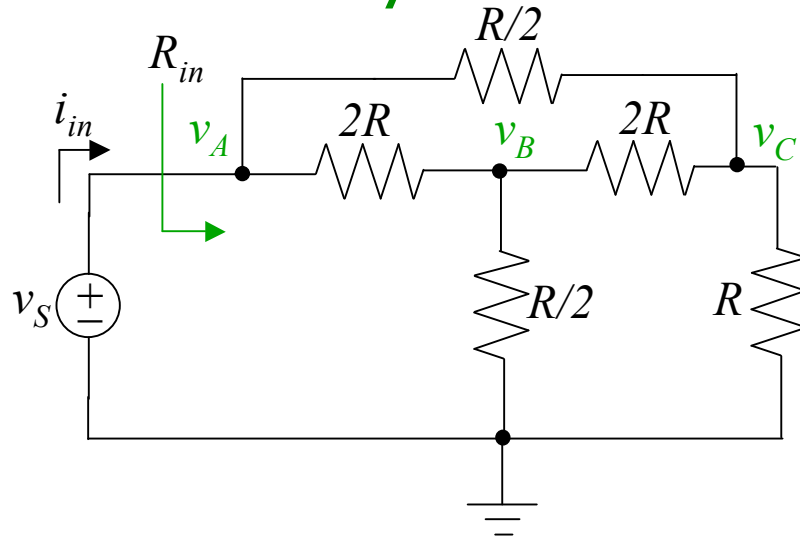
Now use nodal analysis with one node, A

$$(G_1 + G_2 + G_3)v_A = G_1v_{S1} + G_2v_{S2}$$

$$v_A = \frac{G_1v_{S1} + G_2v_{S2}}{G_1 + G_2 + G_3}$$

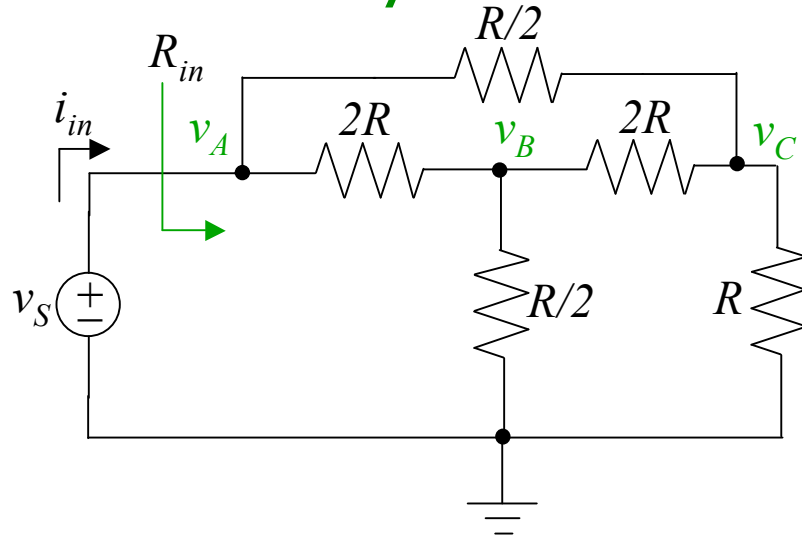
## Nodal Analysis Ex. 3-5 (T&R, 5th ed, p. 77)

What is the circuit input resistance viewed through  $v_S$ ?





# Nodal Analysis Ex. 3-5 (T&R, 5th ed, p. 77)



What is the circuit input resistance viewed through  $v_S$ ?

$$v_A = v_S$$

$$-0.5Gv_A + 3Gv_B - 0.5Gv_C = 0$$

$$-2Gv_A - 0.5Gv_B + 3.5Gv_C = 0$$

Rewrite in terms of  $v_S, v_B, v_C$

This is method 2

$$3Gv_B - 0.5Gv_C = 0.5Gv_S$$

$$-0.5Gv_B + 3.5Gv_C = 2Gv_S$$

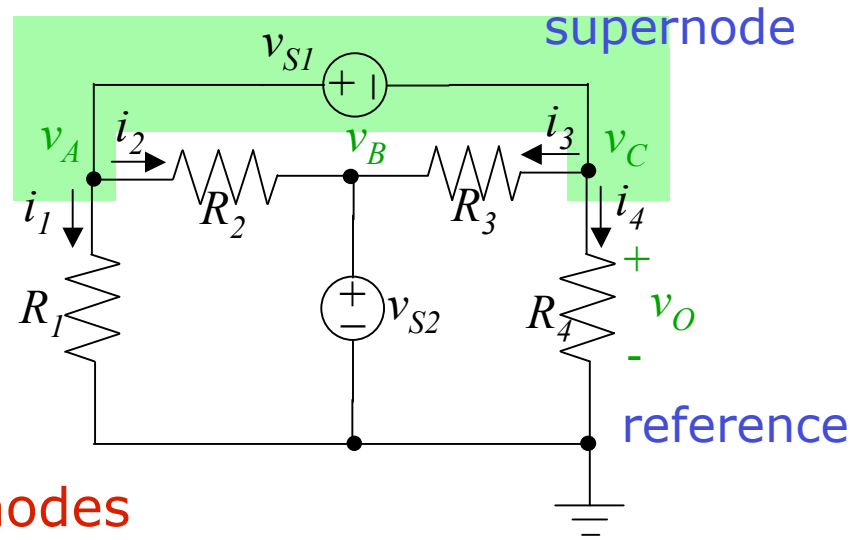
$$v_B = \frac{2.75v_S}{10.25}, v_C = \frac{6.25v_S}{10.25}$$

Solve

$$i_{in} = \frac{v_S - v_B}{2R} + \frac{v_S - v_C}{R/2} = \frac{11.75v_S}{10.25R}$$

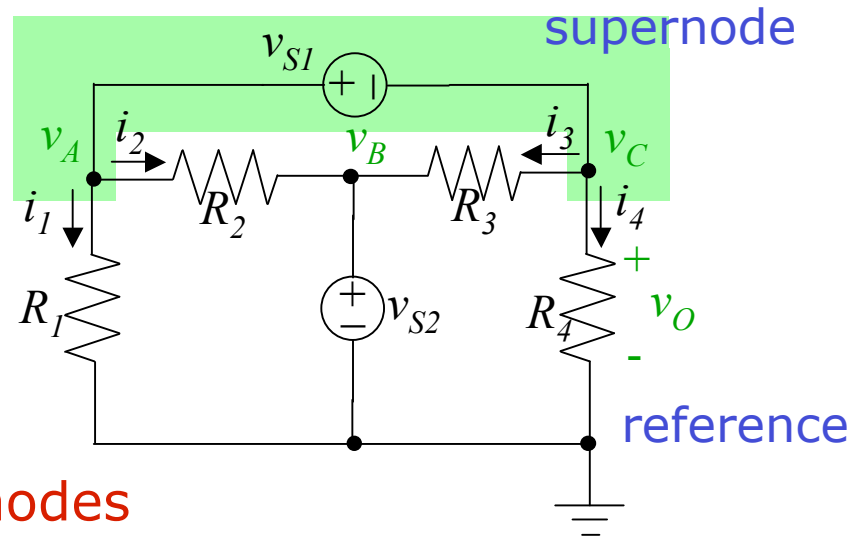
$$R_{in} = \frac{10.25R}{11.75} = 0.872R$$

# Nodal Analysis Ex. 3-6 (T&R, 5th ed, p. 78)



Method 3 – supernodes

# Nodal Analysis Ex. 3-6 (T&R, 5th ed, p. 78)



## Method 3 – supernodes

KCL for supernode:  $i_1 + i_2 + i_3 + i_4 = 0$

Or, using element equations

$$G_1 v_A + G_2 (v_A - v_B) + G_3 (v_C - v_B) + G_4 v_C = 0$$

Now use  $v_B = v_{S2}$

$$(G_1 + G_3) v_A + (G_3 + G_4) v_C = (G_2 + G_3) v_{S2}$$

Other constituent relation

$$v_A - v_C = v_{S1}$$

Two equations in two unknowns

# Summary of Nodal Analysis

1. Simplify the cct by combining elements in series or parallel
2. Select as reference node the one with most voltage sources connected
3. Label node voltages and supernode voltages – do not label the nodes directly connected to the reference
4. Use KCL to write node equations. Express element currents in terms of node voltages and ICSs
5. Write expressions relating node voltages and IVSs
6. Substitute from Step 5 into equations from Step 4  
Write the equations in standard form
7. Solve using Cramer, Gaussian elimination or matlab

# Solving sets of linear equations

$$\begin{pmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 6 \end{pmatrix}$$

Cramer's Rule Thomas & Rosa Appendix B pp. A-2 to A-11

$$\begin{aligned} \Delta &= \begin{vmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{vmatrix} = 5 \begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-5) \begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + (-3) \begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix} \\ &= 5(7 \times 8 - (-3) \times (-2)) + 5((-2) \times 8 - (-3) \times (-3)) - 3((-2) \times (-2) - 7 \times (-3)) \\ &= 250 - 125 - 75 = 50 \end{aligned}$$

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 4 & -2 & -3 \\ -10 & 7 & -2 \\ 6 & -3 & 8 \end{vmatrix} = 4 \begin{vmatrix} 7 & -2 \\ -3 & 8 \end{vmatrix} - (-10) \begin{vmatrix} -2 & -3 \\ -3 & 8 \end{vmatrix} + 6 \begin{vmatrix} -2 & -3 \\ 7 & -2 \end{vmatrix} \\ &= 4(7 \times 8 - (-3) \times (-2)) + 10((-2) \times 8 - (-3) \times (-3)) + 6((-2) \times (-2) - 7 \times (-3)) \\ &= 200 - 250 + 150 = 100 \end{aligned}$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{100}{50} = 2$$

## Solving sets of linear equations (contd)

$$\Delta_2 = \begin{vmatrix} 5 & 4 & -3 \\ -5 & -10 & -2 \\ -3 & 6 & 8 \end{vmatrix} = 5 \begin{vmatrix} -10 & -2 \\ 6 & 8 \end{vmatrix} - (-5) \begin{vmatrix} 4 & -3 \\ 6 & 8 \end{vmatrix} + (-3) \begin{vmatrix} 4 & -3 \\ -10 & -2 \end{vmatrix}$$
$$= -340 + 250 + 114 = 24$$

$$x_2 = \frac{\Delta_2}{\Delta} = \frac{24}{50} = 0.48$$

$$\Delta_3 = \begin{vmatrix} 5 & -2 & 4 \\ -5 & 7 & -10 \\ -3 & -3 & 6 \end{vmatrix} = 5 \begin{vmatrix} 7 & -10 \\ -3 & 6 \end{vmatrix} - (-5) \begin{vmatrix} -2 & 4 \\ -3 & 6 \end{vmatrix} + (-3) \begin{vmatrix} -2 & 4 \\ 7 & -10 \end{vmatrix}$$
$$= 60 - 0 + 24 = 84$$

$$x_3 = \frac{\Delta_3}{\Delta} = \frac{84}{50} = 1.68$$

### Notes:

This Cramer is not as much fun as Cosmo Kramer in *Seinfeld*  
I do not know of any tricks for symmetric matrices

# Solving Linear Equations: Gaussian elimination

$$\begin{pmatrix} 5 & -2 & -3 \\ -5 & 7 & -2 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \\ 6 \end{pmatrix} \xrightarrow{\text{Augment matrix}} \begin{pmatrix} 5 & -2 & -3 & 4 \\ -5 & 7 & -2 & -10 \\ -3 & -3 & 8 & 6 \end{pmatrix} \text{ Row operations only}$$

row<sub>2</sub>+row<sub>1</sub>  
row<sub>3</sub>+row<sub>1</sub>×3/5

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & \frac{-21}{5} & \frac{31}{5} & \frac{42}{5} \end{pmatrix}$$

row<sub>3</sub>×5

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & -21 & 31 & 42 \end{pmatrix}$$

row<sub>3</sub>+row<sub>2</sub>×21/5

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & 0 & 10 & \frac{84}{5} \end{pmatrix}$$

row<sub>3</sub>÷10

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 5 & -5 & -6 \\ 0 & 0 & 1 & \frac{84}{50} \end{pmatrix}$$

(row<sub>2</sub>+row<sub>3</sub>×5) ÷5

$$\begin{pmatrix} 5 & -2 & -3 & 4 \\ 0 & 1 & 0 & \frac{24}{50} \\ 0 & 0 & 1 & \frac{84}{50} \end{pmatrix}$$

(row<sub>1</sub>+row<sub>2</sub>×2  
+ row<sub>3</sub>×3) ÷5

$$\begin{pmatrix} 1 & 0 & 0 & \frac{100}{50} \\ 0 & 1 & 0 & \frac{24}{50} \\ 0 & 0 & 1 & \frac{84}{50} \end{pmatrix}$$

# Solving Linear Equations - matlab

```
A=[5 -2 -3; -5 7 -2; -3 -3 8]
```

```
A =  5 -2 -3  
    -5  7 -2  
    -3 -3  8
```

```
B=[4;-10;6]
```

```
B =  4  
    -10  
     6
```

```
inv(A)
```

```
ans =  1.0000  0.5000  0.5000  
       0.9200  0.6200  0.5000  
       0.7200  0.4200  0.5000
```

```
inv(A)*B
```

```
ans =  2.0000  
       0.4800  
       1.6800
```

```
A\B
```

```
ans =  2.0000  
       0.4800  
       1.6800
```



# Mesh Current Analysis

## Dual of Nodal Voltage Analysis with KCL

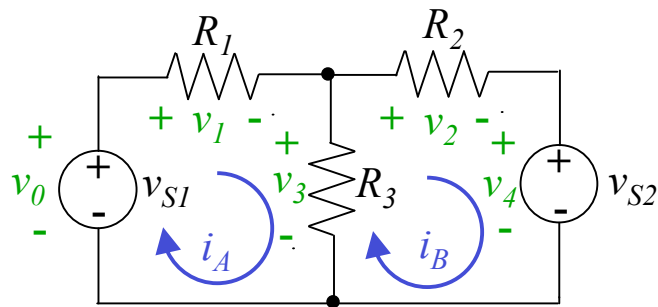
Mesh Current Analysis with KVL

Mesh = loop enclosing no elements

Restricted to Planar Ccts – no crossovers (unless you are really clever)

**Key Idea:** *If element  $K$  is contained in both mesh  $i$  and mesh  $j$  then its current is  $i_k = i_i - i_j$  where we have taken the reference directions as appropriate*

Same old tricks you already know



$$\begin{aligned} \text{Mesh A: } -v_0 + v_1 + v_3 &= 0 & v_1 &= R_1 i_A & v_0 &= v_{S1} \\ \text{Mesh B: } -v_3 + v_2 + v_4 &= 0 & v_2 &= R_2 i_B & v_4 &= v_{S2} \\ & & v_3 &= R_3 (i_A - i_B) & & \end{aligned}$$

$$\begin{aligned} (R_1 + R_3) i_A - R_3 i_B &= v_{S1} \\ -R_3 i_A + (R_2 + R_3) i_B &= -v_{S2} \end{aligned}$$

$$\begin{pmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{pmatrix} \begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} v_{S1} \\ -v_{S2} \end{pmatrix}$$

# Mesh Analysis by inspection $Ri = v_S$

## Matrix of Resistances $R$

Diagonal  $ii$  elements: sum of resistances around loop

Off-diagonal  $ij$  elements: - resistance shared by loops  $i$  and  $j$

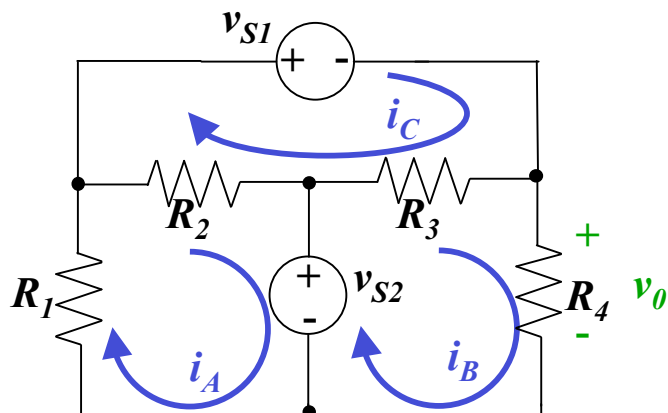
## Vector of currents $i$

As defined by you on your mesh diagram

## Voltage source vector $v_S$

Sum of voltage sources **assisting** the current in your mesh

If this is hard to fathom, go back to the basic KVL to sort these directions out



# Mesh Analysis by inspection $Ri = v_S$

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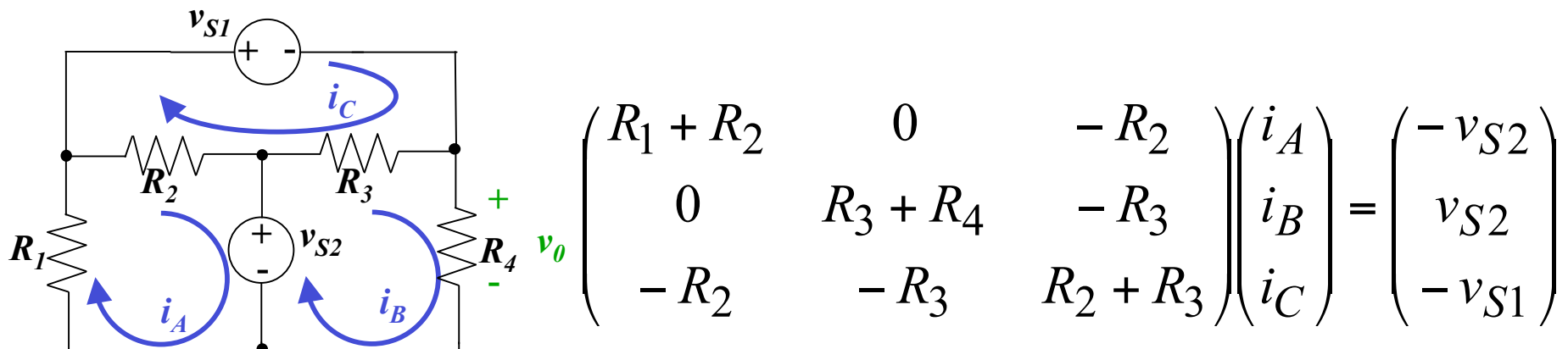
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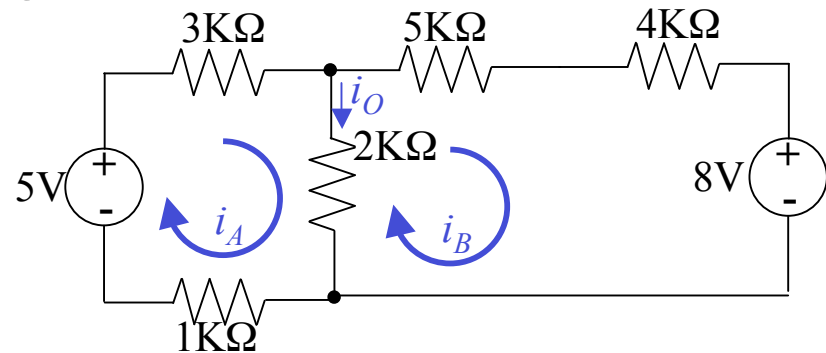
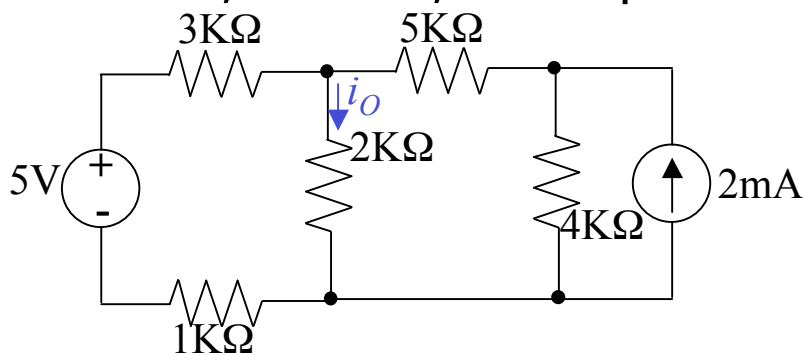


# Mesh Equations with Current Sources

## Duals of tricks for nodal analysis with voltage sources

1. Source transformation to equivalent

T&R, 5th ed, Example 3-8 p. 91



$$\begin{pmatrix} 6000 & -2000 \\ -2000 & 11000 \end{pmatrix} \begin{pmatrix} i_A \\ i_B \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$$

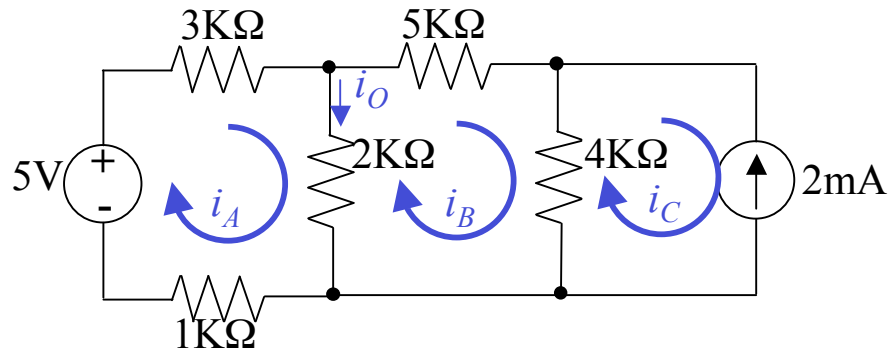
$$i_A = 0.6290 \text{ mA}$$

$$i_B = -0.6129 \text{ mA}$$

$$i_O = i_A - i_B = 1.2419 \text{ mA}$$

## Mesh Analysis with ICSs – method 2

Current source belongs to a single mesh



Same example

$$6000i_A - 2000i_B = 5$$

$$-2000i_A + 11000i_B - 4000i_C = 0$$

$$i_C = -2\text{mA}$$

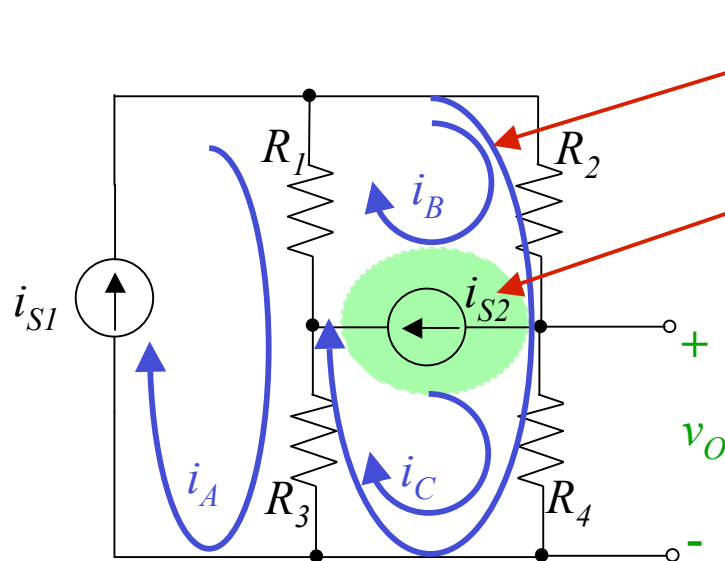
Same equations!  
Same solution

# Mesh Analysis with ICSs – Method 3

## Supermeshes – easier than supernodes

Current source in more than one mesh and/or not in parallel with a resistance

1. Create a supermesh by eliminating the whole branch involved
2. Resolve the individual currents last



Supermesh

Excluded branch

$$R_1(i_B - i_A) + R_2 i_B + R_4 i_C + R_3(i_C - i_A) = 0$$

$$i_A = i_{S1}$$

$$i_B - i_C = i_{S2}$$

# Summary of Mesh Analysis

1. Check if cct is planar or transformable to planar
2. Identify meshes, mesh currents & supermeshes
3. Simplify the cct where possible by combining elements in series or parallel
4. Write KVL for each mesh
5. Include expressions for ICSs
6. Solve for the mesh currents

# Linearity & Superposition

Linear cct – modeled by linear elements and independent sources

Linear functions

Homogeneity:  $f(Kx) = Kf(x)$

Additivity:  $f(x+y) = f(x) + f(y)$

Superposition – follows from linearity/additivity

Linear cct response to multiple sources is the sum of the responses to each source

1. “Turn off” all independent sources except one and compute cct variables
2. Repeat for each independent source in turn
3. Total value of all cct variables is the sum of the values from all the individual sources



# Superposition

## Turning off sources

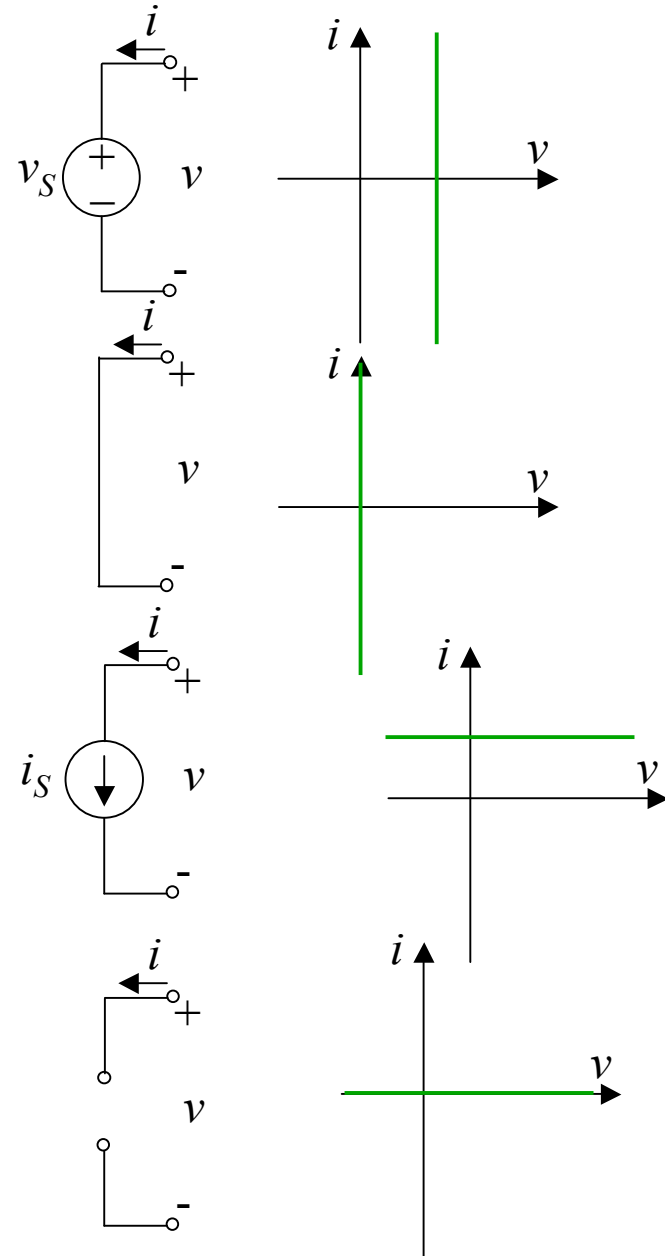
Voltage source

Turned off when  $v=0$  for all  $i$   
a short circuit

Current source

Turned off when  $i=0$  for all  $v$   
an open circuit

We have already used this in  
Thévenin and Norton equiv



# Where are we now?

## Finished resistive ccts with ICS and IVS

Two analysis techniques – nodal voltage and mesh current

Preference depends on simplicity of the case at hand

The aim has been to develop general techniques for access to analytical tools like matlab

## Where to now?

Active ccts with resistive elements – transistors, op-amps

Life starts to get interesting – design introduced

Capacitance and inductance – dynamic ccts

Frequency response –  $s$ -domain analysis

Filters