Laplace Transforms – recap for ccts

What's the big idea?

1. Look at initial condition responses of ccts due to capacitor voltages and inductor currents at time $t=0$

   Mesh or nodal analysis with s-domain impedances (resistances) or admittances (conductances)

   Solution of ODEs driven by their initial conditions

   Done in the s-domain using Laplace Transforms

2. Look at forced response of ccts due to input ICSs and IVSs as functions of time

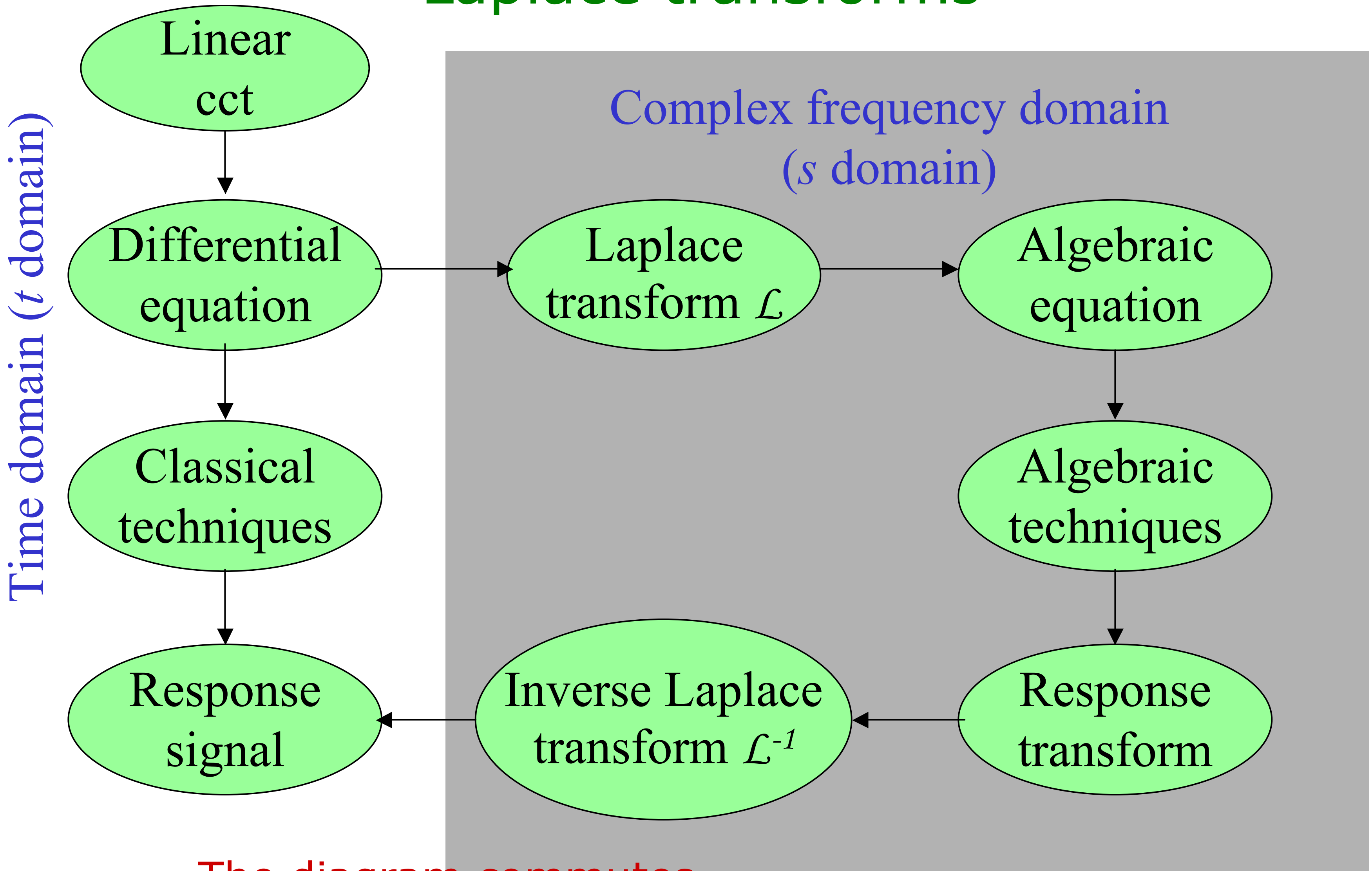
   Input and output signals $I_O(s)=Y(s)V_S(s)$ or $V_O(s)=Z(s)I_S(s)$

   The cct is a system which converts input signal to output signal

3. Linearity says we add up parts 1 and 2

   The same as with ODEs

Laplace transforms



The diagram commutes

Same answer whichever way you go

Laplace Transform - definition

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$$

$0-$ limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

- There is a need to worry about regions of convergence of the integral

Units of s are sec^{-1}=Hz

- A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)

Laplace transform examples

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925)

$$u(t) = \begin{cases} 0, \text{ for } t < 0 \\ 1, \text{ for } t \geq 0 \end{cases}$$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

Delta (impulse) function $\delta(t)$

Laplace transform examples

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Delta (impulse) function $\delta(t)$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st}dt = 1 \text{ for all } s$$

Laplace Transform Pair Tables

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity – absolutely critical property

Follows from the integral definition

$$\mathcal{L}\left\{Af_1(t)+Bf_2(t)\right\} = A\mathcal{L}\left\{f_1(t)\right\} + B\mathcal{L}\left\{f_2(t)\right\} = AF_1(s)+BF_2(s)$$

Example

$$\mathcal{L}(A\cos(\beta t)) \;=$$

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$$\mathcal{L}\{Af_1(t)+Bf_2(t)\}=A\mathcal{L}\{f_1(t)\}+B\mathcal{L}\{f_2(t)\}=AF_1(s)+BF_2(s)$$

Example

$$\mathcal{L}(A\cos(\beta t))=\mathcal{L}\left(\frac{A}{2}\left(e^{j\beta t}+e^{-j\beta t}\right)\right)=\frac{A}{2}\mathcal{L}\left(e^{j\beta t}\right)+\frac{A}{2}\mathcal{L}\left(e^{-j\beta t}\right)$$

$$=\frac{A}{2}\frac{1}{s-j\beta}+\frac{A}{2}\frac{1}{s+j\beta}$$

$$=\frac{As}{s^2+\beta^2}$$

Laplace Transform Properties

Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right]e^{-st}dt$$

Laplace Transform Properties

Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof $$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right]e^{-st}dt$$

Denote $$x = \frac{-e^{-st}}{s}, \text{ and } y = \int_0^t f(\tau)d\tau$$

so $$\frac{dx}{dt} = e^{-st}, \text{ and } \frac{dy}{dt} = f(t)$$

Integrate by parts $$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left[\frac{-e^{-st}}{s}\int_0^t f(\tau)d\tau\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

Laplace Transform Properties

Differentiation Property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

Proof via integration by parts again

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[f(t)e^{-st}\right]_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)e^{-st} dt$$

$$= sF(s) - f(0-)$$

Second derivative

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left[\frac{df(t)}{dt}\right]\right\} = s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df}{dt}(0-)$$

$$= s^2 F(s) - sf(0-) - f'(0-)$$

Laplace Transform Properties

General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \cdots - f^{(m-1)}(0-)$$

Translation properties

s-domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$$

t-domain translation

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s) \quad \text{for} \quad a > 0$$

Laplace Transform Properties

Initial Value Property

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$$

Final Value Property

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Caveats:

- Laplace transform pairs do not always handle discontinuities properly
  - Often get the average value
- Initial value property no good with impulses
- Final value property no good with cos, sin etc

Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in s

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

A Little Complex Analysis

We are dealing with linear ccts

Our Laplace Transforms will consist of rational functions (ratios of polynomials in s) and exponentials like $e^{-s\tau}$

These arise from

- discrete component relations of capacitors and inductors
- the kinds of input signals we apply
  - Steps, impulses, exponentials, sinusoids, delayed versions of functions

Rational functions have a finite set of discrete *poles*

$e^{-s\tau}$ is an *entire function* and has no poles anywhere

To understand linear cct responses you need to look at the poles – they determine the exponential modes in the response circuit variables.

Two sources of poles: the cct – seen in the response to Ics

the input signal LT poles – seen in the forced response

A Little More Complex Analysis

A complex function is *analytic* in regions where it has no poles

- Rational functions are analytic everywhere except at a finite number of isolated points, where they have poles of finite order
  - Rational functions can be expanded in a Taylor Series about a point of analyticity

$$f(z) = f(a) + (z-a)f'(a) + \frac{1}{2!}(z-a)^2 f''(a) + \ldots$$

  - They can also be expanded in a Laurent Series about an isolated pole

$$f(z) = \sum_{n=-N}^{-1} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n$$

    - General functions do not have N necessarily finite

Residues at poles

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

Simple pole: residue = $\lim_{s \to a} (s-a)F(s)$

Multiple pole: residue = $\dfrac{1}{(m-1)!} \lim_{s \to a} \dfrac{d^{m-1}}{ds^{m-1}} \left((s-a)^m F(s) \right)$

The residue is the c_{-1} term in the Laurent Series

Cauchy Residue Theorem

The integral around a simple closed rectifiable positively oriented curve (*scroc*) is given by $2\pi j$ times the sum of residues at the poles inside

Inverse Laplace Transforms – the Bromwich Integral

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}\,ds$$

This is a contour integral in the complex s-plane

- α is chosen so that all singularities of $F(s)$ are to the left of $Re(s)=\alpha$

It yields $f(t)$ for $t \geq 0$

- The inverse Laplace transform is always a *causal* function
  - For $t<0$ $f(t)=0$

Remember Cauchy's Integral Formula

- Counterclockwise contour integral =

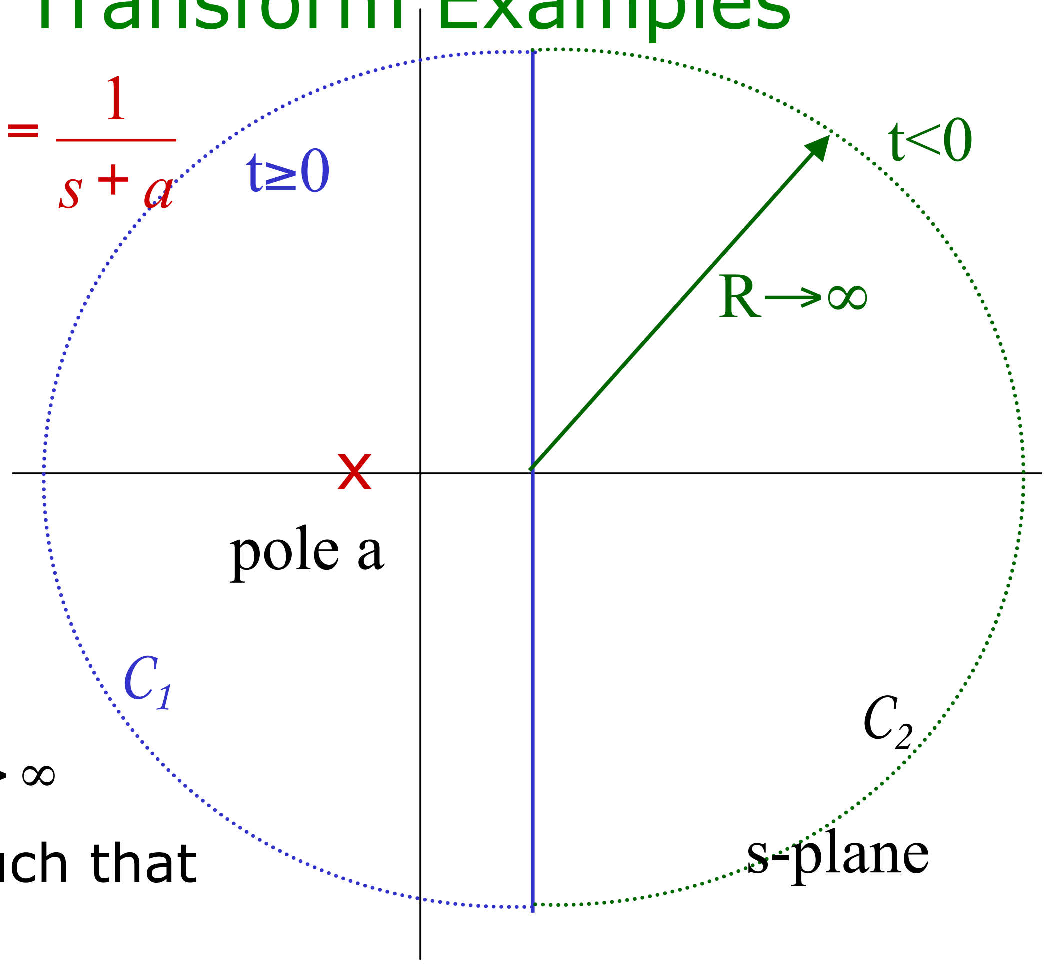
  $2\pi j\times$(sum of residues inside contour)

Inverse Laplace Transform Examples

Bromwich integral of $F(s) = \dfrac{1}{s+a}$

$$f(t) = \int_{\alpha - j\infty}^{\alpha + j\infty} \frac{1}{s+a} e^{st} ds$$

$$= \begin{cases} e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



On curve C_1

$$s = \alpha + re^{j\theta}, \ \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \ r \to \infty$$

For given θ there is r→∞ such that

$$\text{Re}(s) = \alpha + r\cos\theta < 0$$

$$e^{st} = e^{\text{Re}(s)t} e^{j\,\text{Im}(s)t} \to 0 \ \text{as} \ r \to \infty \ \text{for} \ t > 0$$

Integral disappears on C_1 for positive t

Inverting Laplace Transforms

Compute residues at the poles $\lim_{s \to a} (s-a)F(s)$

$$\frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m F(s) \right]$$

Example $\dfrac{2s^2+5s}{(s+1)^3} = \dfrac{2(s+1)^2+(s+1)-3}{(s+1)^3} = \dfrac{2}{s+1} + \dfrac{1}{(s+1)^2} - \dfrac{3}{(s+1)^3}$

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$$\lim_{s\to -1}\frac{(s+1)^3(2s^2+5s)}{(s+1)^3} = -3 \qquad \lim_{s\to -1}\frac{d}{ds}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right] = 1$$

$$\frac{1}{2!}\lim_{s\to -1}\frac{d^2}{ds^2}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right] = 2$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+5s}{(s+1)^3}\right] = e^{-t}(2+t-3t^2)u(t)$$

Inverting Laplace Transforms

Compute residues at the poles $\lim_{s\to a}(s-a)F(s)$

$$\frac{1}{(m-1)!}\lim_{s\to a}\frac{d^{m-1}}{ds^{m-1}}\left[(s-a)^m F(s)\right]$$

Bundle complex conjugate pole pairs into second-order terms if you want

$$(s-\alpha-j\beta)(s-\alpha+j\beta)=\left[s^2-2\alpha s+\left(\alpha^2+\beta^2\right)\right]$$

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials

For circuits the answers will be real

Inverting Laplace Transforms in Practice

We have a table of inverse LTs

Write F(s) as a partial fraction expansion

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$
$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$
$$= \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \frac{\alpha_{31}}{(s-p_3)} + \frac{\alpha_{32}}{(s-p_3)^2} + \frac{\alpha_{33}}{(s-p_3)^3} + \ldots + \frac{\alpha_q}{(s-p_q)}$$

Now appeal to linearity to invert via the table

- Surprise!
- Computing the partial fraction expansion is best done by calculating the residues

T&R, 5th ed, Example 9-12

Find the inverse LT of $F(s) = \dfrac{20(s+3)}{(s+1)(s^2+2s+5)}$

T&R, 5th ed, Example 9-12

Find the inverse LT of $F(s) = \dfrac{20(s+3)}{(s+1)(s^2+2s+5)}$

$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1-j2} + \frac{k_2^*}{s+1+j2}$$

$$k_1 = \lim_{s \to -1} (s+1)F(s) = \left.\frac{20(s+3)}{s^2+2s+5}\right|_{s=-1} = 10$$

$$k_2 = \lim_{s \to -1+2j} (s+1-2j)F(s) = \left.\frac{20(s+3)}{(s+1)(s+1+2j)}\right|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$f(t) = \left[10e^{-t} + 5\sqrt{2}e^{(-1+j2)t+j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-j2)t-j\frac{5}{4}\pi}\right]u(t)$$

$$= \left[10e^{-t} + 10\sqrt{2}e^{-t}\cos(2t + \frac{5\pi}{4})\right]u(t)$$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \dfrac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \dfrac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$F(s) = s + 2 + \frac{s+2}{s^2 + 4s + 3}$$

$$= s + 2 + \frac{0.5}{s+1} + \frac{0.5}{s+3}$$

Invert as normal

$$f(t) = \left[\frac{d\delta(t)}{dt} + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t}\right]u(t)$$

Multiple Poles

Look for partial fraction decomposition

$$F(s)=\frac{K(s-z_1)}{(s-p_1)(s-p_2)^2}=\frac{k_1}{s-p_1}+\frac{k_{21}}{s-p_2}+\frac{k_{22}}{(s-p_2)^2}$$

$$Ks-Kz_1=k_1(s-p_2)^2+k_{21}(s-p_1)(s-p_2)+k_{22}(s-p_1)$$

Equate like powers of s to find coefficients

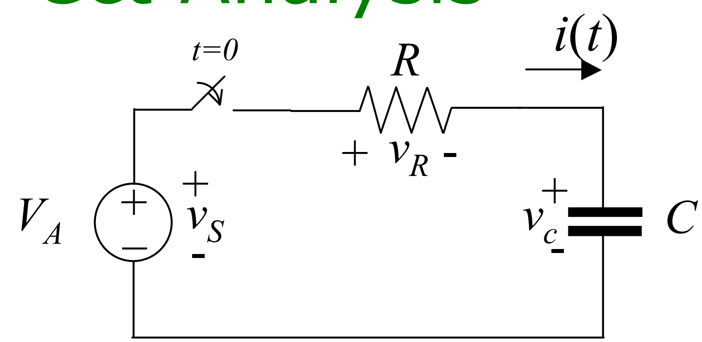
$$k_1+k_{21}=0$$

$$-2k_1p_2-2k_{21}(p_1+p_2)+k_{22}=K$$

$$k_1p_2^2+k_{12}p_1p_2-k_{22}p_1=Kz_1$$

Solve

Introductory s-Domain Cct Analysis



First-order RC cct

KVL $v_S(t) - v_R(t) - v_C(t) = 0$

instantaneous for each t

Substitute element relations

$$v_S(t) = V_A u(t), \quad v_R(t) = Ri(t), \quad i(t) = C\frac{dv_C(t)}{dt}$$

Ordinary differential equation in terms of capacitor voltage

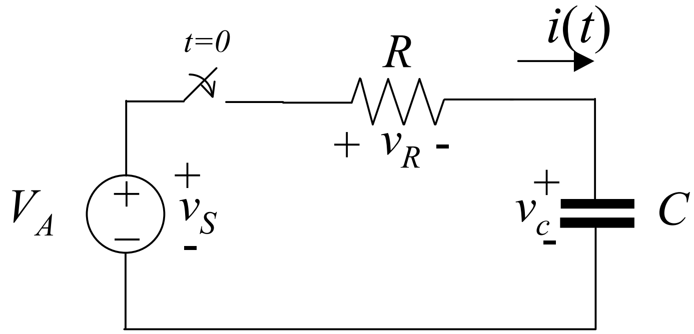
$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_A u(t)$$

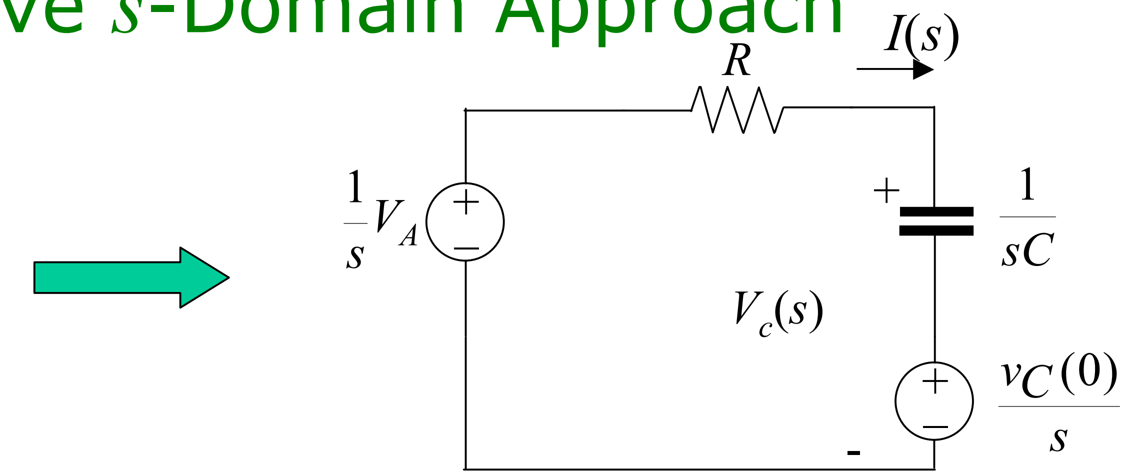
Laplace transform $RC[sV_C(s) - v_C(0)] + V_C(s) = \frac{1}{s}V_A$

Solve $V_C(s) = \dfrac{V_A / RC}{s(s+1/RC)} + \dfrac{v_C(0)}{s+1/RC}$

Invert LT $v_C(t) = \left[V_A\left(1 - e^{-t/RC}\right) + v_C(0)e^{-t/RC}\right]u(t) \quad \text{Volts}$

An Alternative s-Domain Approach





Transform the cct element relations

Work in s-domain directly — OK since $\mathcal{L}$ is linear

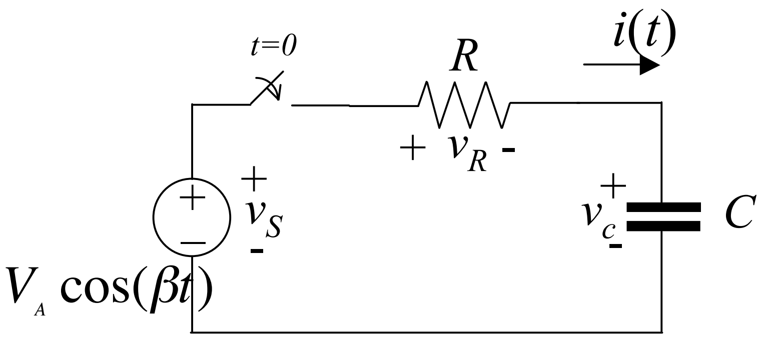
$$V_C(s) = \frac{1}{Cs} I_C(s) + \frac{v_C(0)}{s}$$ Impedance + source

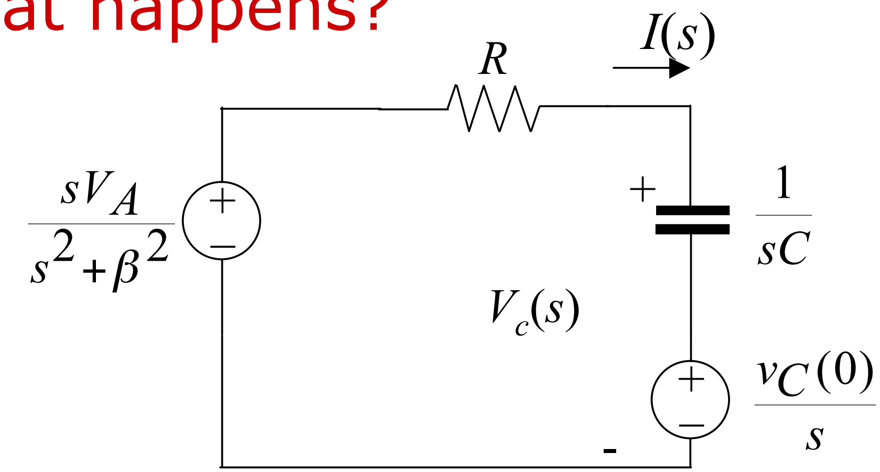
$$I_C(s) = sCV_C(s) - Cv_C(0)$$ Admittance + source

KVL in s-Domain $$sCRV_C(s) - CRv_C(0) + V_C(s) = \frac{1}{s} V_A$$

Time-varying inputs

Suppose $v_S(t) = V_A\cos(\beta t)$, what happens?





KVL as before $\quad (RCs+1)V_C(s) - RCv_C(0) = \dfrac{sV_A}{s^2+\beta^2}$

$$V_C(s) = \frac{sV_A/RC}{(s^2+\beta^2)(s+1/RC)} + \frac{v_C(0)}{s+1/RC}$$

Solve $\quad v_C(t) = \left[\dfrac{V_A}{\sqrt{1+(\beta RC)^2}}\cos(\beta t+\theta) - \dfrac{V_A}{1+(\beta RC)^2}e^{-t/RC} + v_C(0)e^{-t/RC}\right]u(t)$