MAE140 - Linear Circuits - Fall 13
Final, December 11

Instructions

(i) The exam is open book. You may use your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 180 minutes

(iii) Do not forget to write your name and student number

(iv) On the questions for which the answers are given, please provide detailed derivations

(v) The exam has 5 questions for a total of 50 points and 3 bonus points

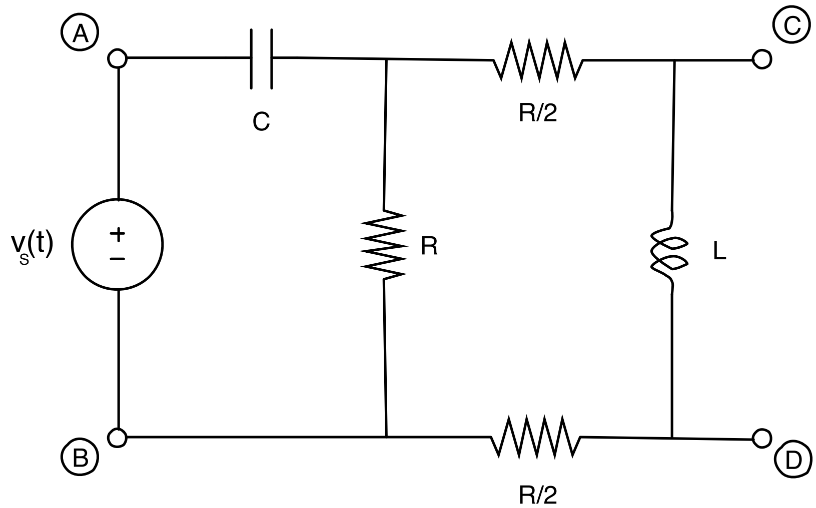


Figure 1: Circuit for Question 1.

1. **Equivalent Circuits**

Part I: [2 points] Assuming zero initial conditions, transform the circuit in Figure 1 into the s-domain.

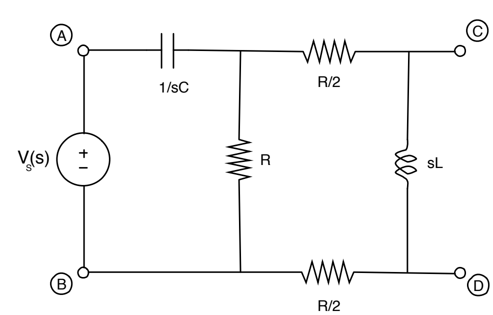
Part II: [4 points] Find the impedance equivalent in the circuit obtained in Part I as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Part III: [4 points] Use source transformations to find the s-domain Thévenin equivalent of the circuit obtained in Part I as seen from terminals C and D.

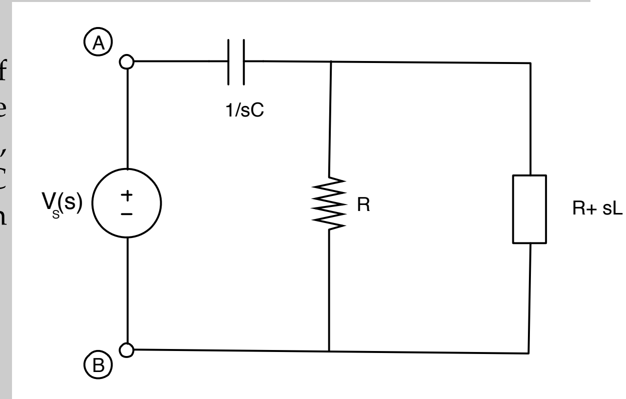
Solution: Part I:

Since all initial conditions are zero, it is easy to transform the circuit to the s-domain.

[2 points]

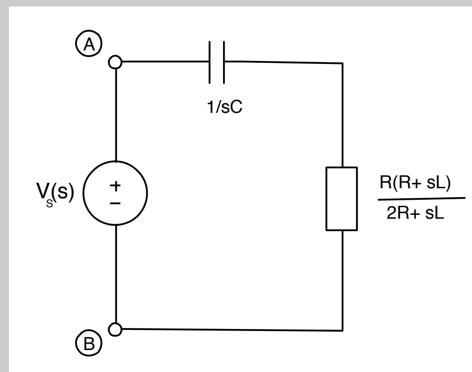


Part II:



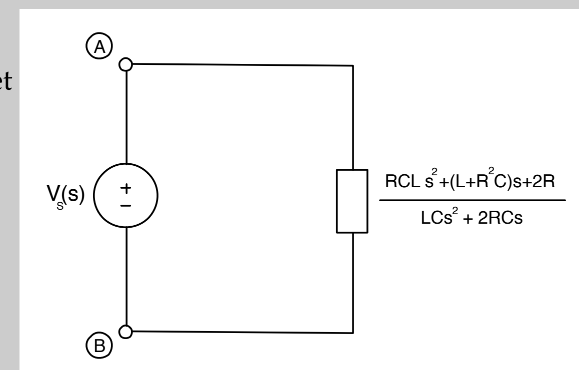
We need to go through the different combinations of impedances carefully. Since we have to compute the impedance equivalent as seen from terminals A and B, we can assume open circuit conditions at terminals C and D. Therefore, the two resistors and the inductor on the right are all in series, so we get

[2 points]



Now we combine the two admittances in parallel to get

[1 point]

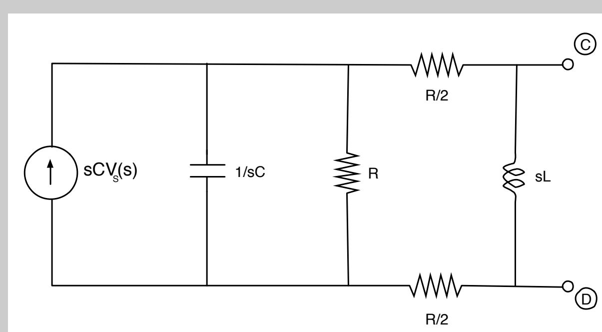


Finally, we combine the two impedances in series to get

$$Z(s) = \frac{RCLs^2 + (L + R^2C)s + 2R}{LCs^2 + 2RCs}$$

[1 point]

Part III:

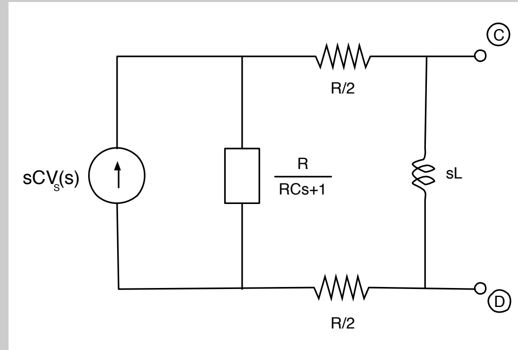


We begin with a source transformation.

[.5 point]

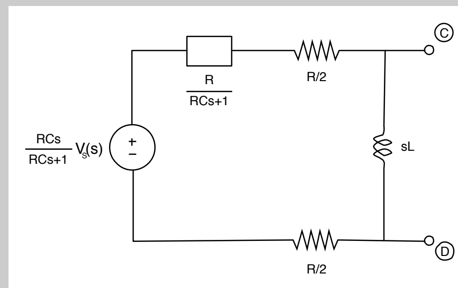
Now, we combine the capacitor and the resistor in parallel

[.5 point]



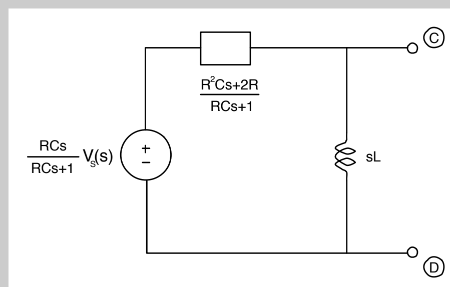
Next, we do a source transformation to turn the current source into a voltage source

[.5 point]



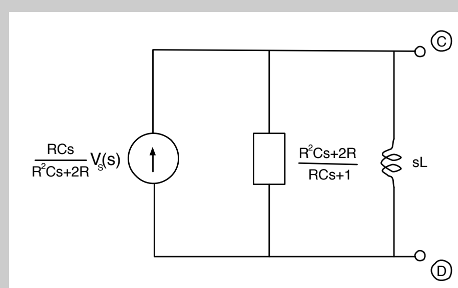
We next combine the impedance and the two resistors in series to get

[1 point]



Now, we transform the voltage source again into a current source

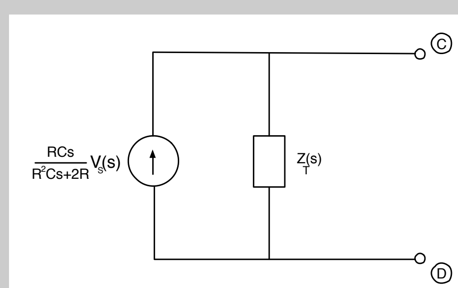
[.5 point]



We next combine the impedance and the inductor in parallel to get

$$Z_T(s) = \frac{R^2CLs^2 + 2RLs}{RCLs^2 + (L + R^2C)s + 2R}$$

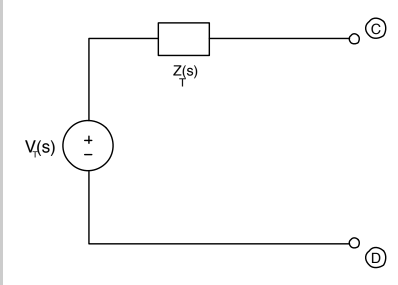
[.5 point]



Finally, we obtain the s-domain Thévenin equivalent with one last source transformation

$$V_T(s) = \frac{RCLs^2}{RCLs^2 + (L + R^2C)s + 2R} V_S(s)$$

[.5 point]



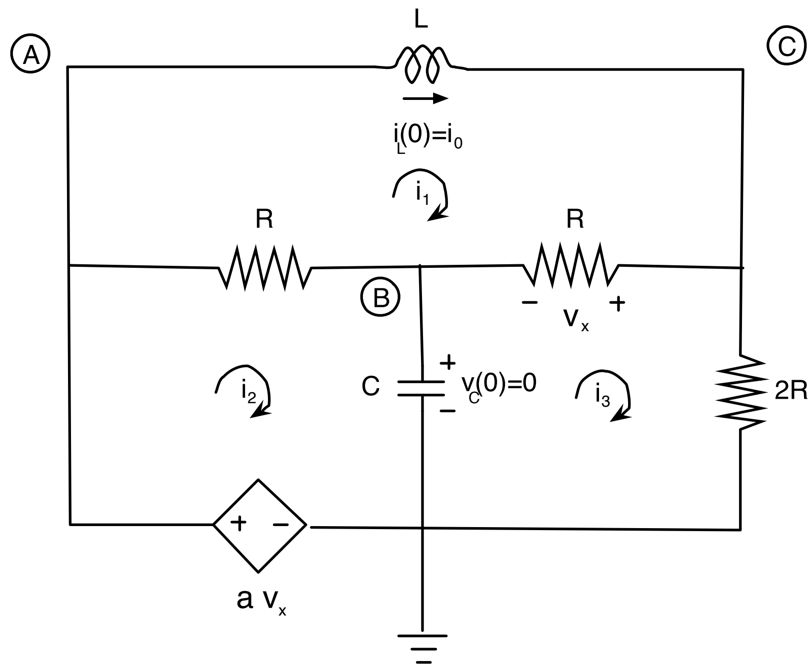


Figure 2: Nodal and Mesh Analysis Circuit

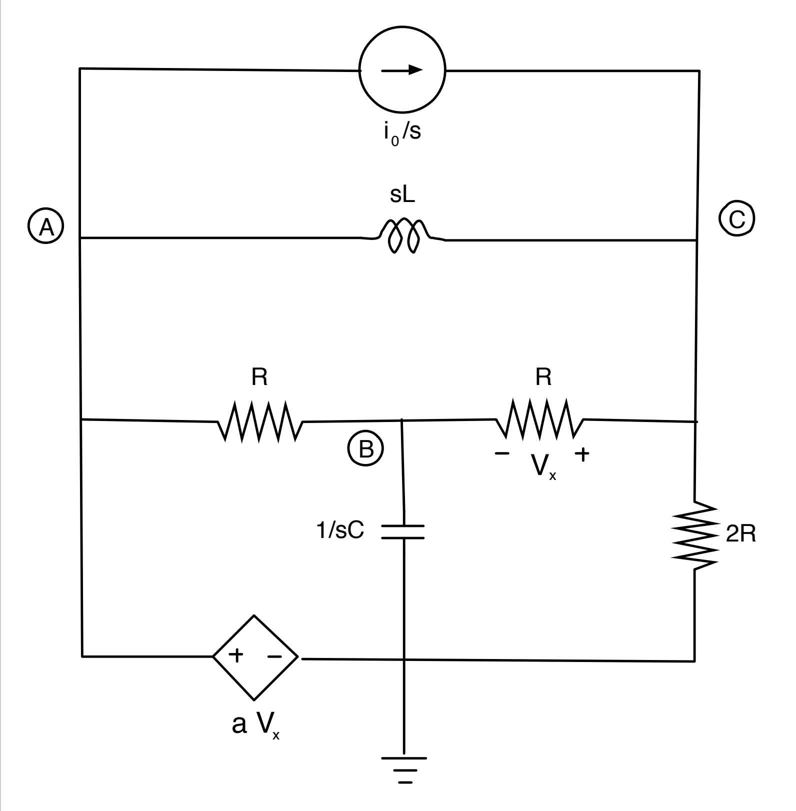
2. **Nodal and Mesh Analysis**

Part I: [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure and transform them into current sources. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

Part II: [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the currents shown in the figure. Use the initial conditions indicated in the figure and transform them into voltage sources. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

Part III: [1 bonus point] Express the transform $I_L(s)$ of the inductor current in terms of your unknown variables of Part I and also in terms of your unknown variables of Part II.

Solution: Part I:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the current orientation.

[.5 point for correct circuit; .5 point for correct initial conditions]

The voltage source poses a problem for nodal analysis. We can easily take care of it by realizing that (method #2)

$$V_A(s) = aV_x(s), \qquad \textbf{(1 point)}$$

and not writing KCL for node A.

Then, we only need to write KCL node equations for nodes B and C. For node B, we have

$$sC\,V_B(s) + \frac{1}{R}(V_B(s) - V_A(s)) + \frac{1}{R}(V_B(s) - V_C(s)) = 0 \qquad \textbf{(1 point)}$$

For node C, we have

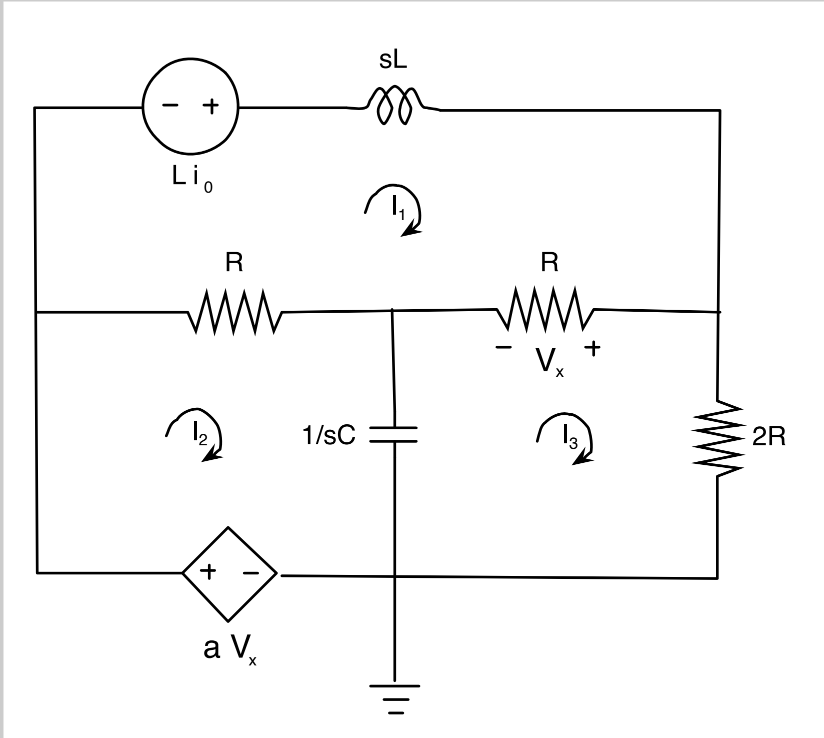
$$\frac{1}{sL}(V_C(s) - V_A(s)) + \frac{1}{R}(V_C(s) - V_B(s)) + \frac{1}{2R}V_C(s) = \frac{i_0}{s} \qquad \textbf{(1 point)}$$

Finally, because we have a dependent source, we need one more equation. This comes from realizing

$$V_x(s) = V_C(s) - V_B(s) \qquad \textbf{(1 point)}$$

This gives a total of 4 independent equations in 4 unknowns ($V_A(s)$, $V_B(s)$, $V_C(s)$, $V_x(s)$). Alternatively, one can take this last equation and substitute it in the first one to arrive at 3 independent equations in 3 unknowns ($V_A(s)$, $V_B(s)$, $V_C(s)$).

Part II:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the current orientation.

[.5 point for correct circuit; .5 point for correct initial conditions]

We need to write mesh equations for meshes 1, 2, 3. For mesh 1, we have

$$R(I_1(s) - I_2(s)) + R(I_1(s) - I_3(s)) + sLI_1(s) = Li_0 \qquad \textbf{(1 point)}$$

For mesh 2, we have

$$R(I_2(s) - I_1(s)) + \frac{1}{sC}(I_2(s) - I_3(s)) = aV_x(s) \qquad \textbf{(1 point)}$$

For mesh 3, we have

$$\frac{1}{sC}(I_3(s) - I_2(s)) + R(I_3(s) - I_1(s)) + 2RI_3(s) = 0 \qquad \textbf{(1 point)}$$

Finally, because we have a dependent source, we need one more equation. This comes from realizing

$$V_x(s) = -RI_3(s) \qquad \textbf{(1 point)}$$

This gives a total of 4 independent equations in 4 unknowns ($I_1(s)$, $I_2(s)$, $I_3(s)$, $V_x(s)$). Alternatively, one can take this last equation and substitute it in the second one to arrive at 3 independent equations in 3 unknowns ($I_1(s)$, $I_2(s)$, $I_3(s)$).

Part III:

We just need to be careful to not lose track of the transform of the inductor current. In the case of Part I, because we use a current source to account for the initial condition, we actually have

$$I_L(s) = i_0/s + (V_A(s) - V_C(s))/(sL)$$ **(.5 bonus point)**

In the case of Part II, because we use a voltage source to account for the initial condition, we have

$$I_L(s) = I_1(s)$$ **(.5 bonus point)**

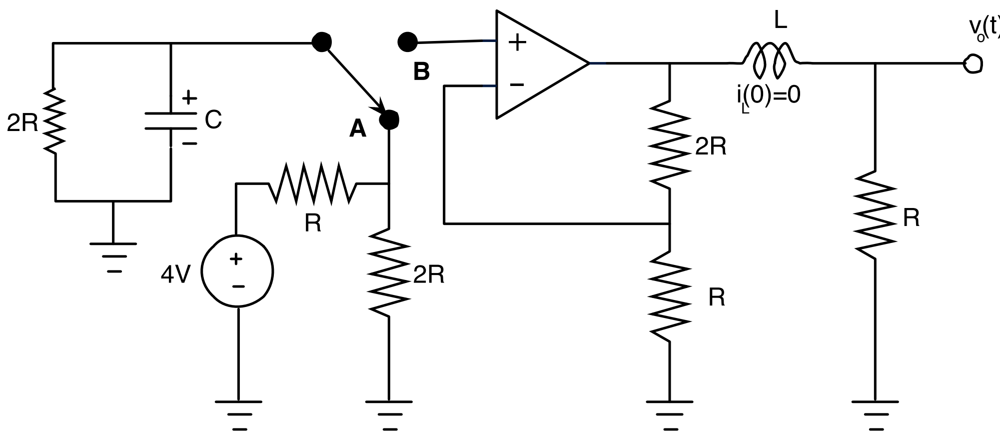


Figure 3: RC circuit for Laplace Analysis

3. **Laplace Domain Circuit Analysis**

Part I: [2 points] Consider the circuit depicted in Figure 3. The voltage source is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial capacitor voltage is given by

$$v_C(0^-) = 2V.$$

[Show your work]

Part II: [3 points] Use this initial condition to transform the circuit into the s-domain for $t \geq 0$. Use an equivalent model for the capacitor in which the initial condition appears as a voltage source. Find the transfer function of the circuit.
[Show your work]

Part III: [5 points] Use domain circuit analysis and inverse Laplace transforms to show that the output voltage $v_o(t)$ when $C = \frac{1}{6}F$, $L = 1H$, and $R = 3\Omega$ is

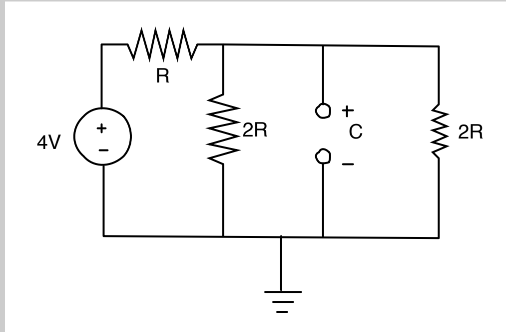
$$v_o(t) = 9(e^{-t} - e^{-3t})u(t).$$

Solution:

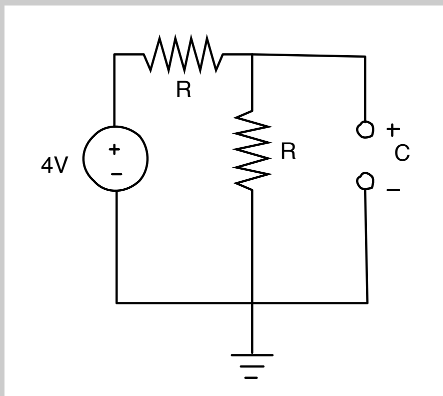
Part I:

To find the initial conditions, we substitute the capacitor by an open circuit.

[.5 point for correct circuit; .5 point for substituting capacitor by open circuit]



Combining the two resistors in parallel, we get the circuit



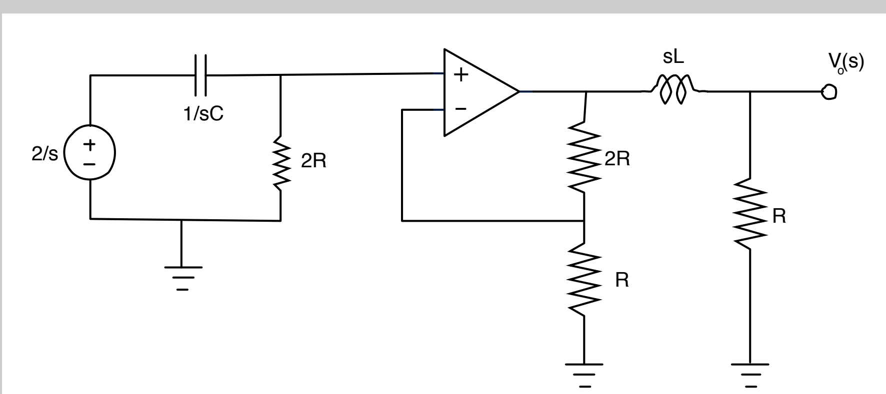
[.5 point]

Using voltage division, we find that

$$v_C(0^-) = \frac{R}{R+R}4 = 2V. \qquad \textbf{(.5 point)}$$

Part II:

We add one voltage source in series for the capacitor to take care of its initial condition, paying special attention to the polarities.



(1 point)

The transfer function can be easily found by realizing that the circuit is the composition of a voltage divider, a non-inverting op-amp, and a voltage divider. Thanks to the non-inverting op-amp, the chain rule applies.

(1 point)

Given the above, the transfer function is simply

$$T(s) = \frac{R}{R+sL} \times \frac{2R+R}{R} \times \frac{2R}{2R+\frac{1}{sC}} = \frac{6R^2Cs}{(Ls+R)(2RCs+1)} \qquad \textbf{(1 point)}$$

Part III:

From our answer to Part II, the Laplace transform of the output voltage is

$$V_o(s) = T(s)\frac{2}{s} = \frac{12R^2C}{(Ls+R)(2RCs+1)} \qquad \textbf{(1 point)}$$

Substituting the RLC values, we get

$$V_o(s) = \frac{18}{(s+3)(s+1)}$$

To find the output voltage, we need to compute the inverse Laplace transform. Using partial fractions, we set

$$V_o(s) = \frac{k_1}{s+1} + \frac{k_2}{s+3} \qquad \textbf{(1 point)}$$

You can use your preferred method to find k_1 and k_2. We use here the cover-up or residue method

$$k_1 = \lim_{s\to-1}(s+1)V_o(s) = \lim_{s\to-1}\frac{18}{s+3} = 9$$
$$k_2 = \lim_{s\to-3}(s+3)V_o(s) = \lim_{s\to-3}\frac{18}{s+1} = -9$$

Therefore, we have

$$V_o(s) = \frac{9}{s+1} - \frac{9}{s+3} \qquad \textbf{(2 points)}$$

The output voltage is then

$$v_o(t) = 9(e^{-t} - e^{-3t})u(t) \qquad \textbf{(1 point)}$$

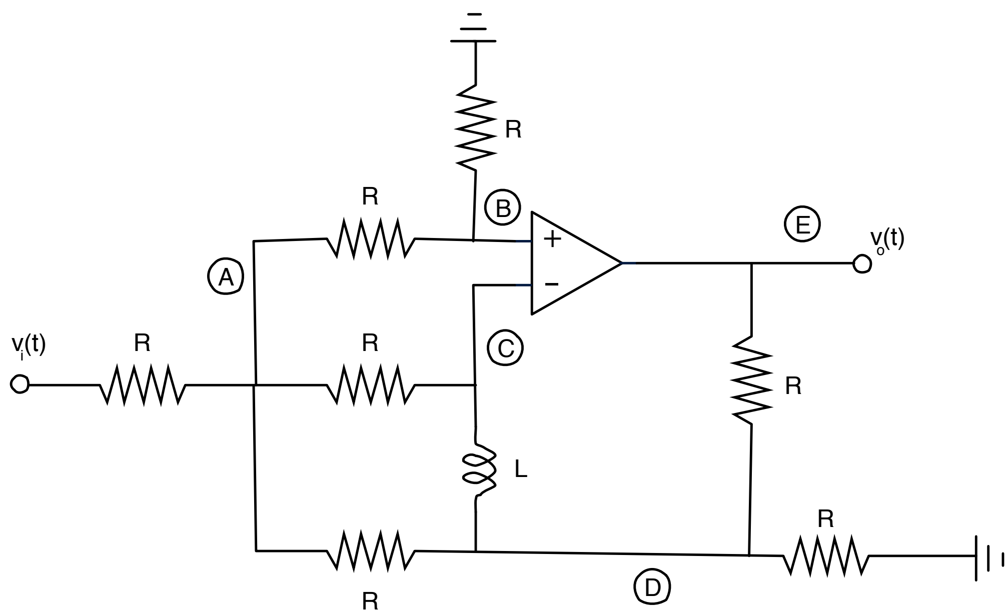
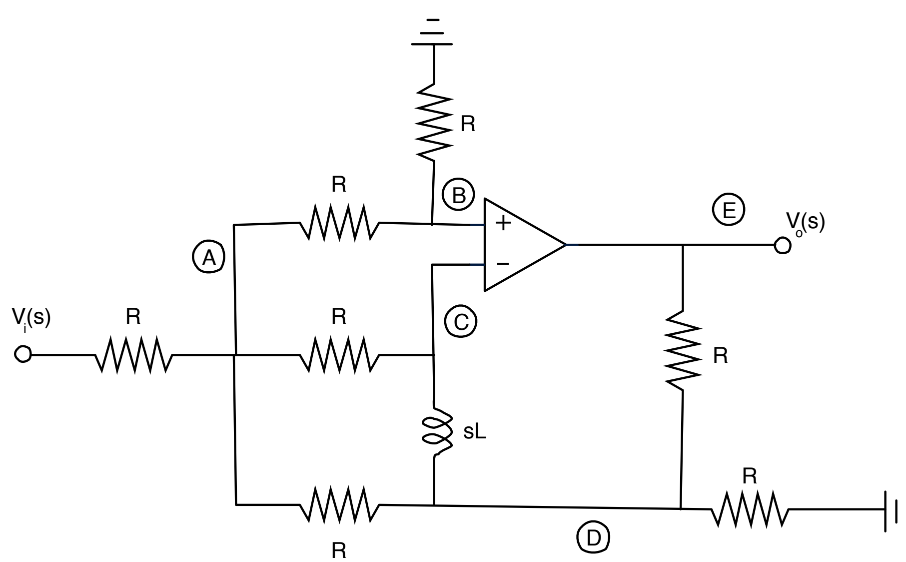


Figure 4: Frequency Response Analysis.

4. **Frequency Response Analysis**

Part I: [1 point] Assuming zero initial conditions, transform the circuit in Figure 4 into the s-domain.

Solution: Since all initial conditions are zero, there is no need to add an independent source for the capacitors. Therefore, the circuit in the s-domain looks like



[1 point]

Part II: [3 points] Show that the transfer function from $V_i(s)$ to $V_o(s)$ is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{-3Ls}{5R + Ls}.$$

[Show your work]
Hint: use node voltage analysis

Solution: Since we cannot recognize any of the basic building blocks of OpAmps, we resort to nodal analysis. Nodal analysis at node A gives

$$\frac{1}{R}(V_A(s) - V_i(s)) + \frac{1}{R}(V_A(s) - V_B(s)) + \frac{1}{R}(V_A(s) - V_C(s)) + \frac{1}{R}(V_A(s) - V_D(s)) = 0$$

(.5 point)

Nodal analysis at node B gives

$$\frac{1}{R}(V_B(s) - V_A(s)) + \frac{1}{R}V_B(s) = 0$$ **(.5 point)**

Nodal analysis at node C gives

$$\frac{1}{R}(V_C(s) - V_A(s)) + \frac{1}{sL}(V_C(s) - V_D(s)) = 0$$ **(.5 point)**

Nodal analysis at node D gives

$$\frac{1}{R}(V_D(s) - V_A(s)) + \frac{1}{sL}(V_D(s) - V_C(s)) + \frac{1}{R}V_D(s) + \frac{1}{R}(V_D(s) - V_E(s)) = 0$$ **(.5 point)**

Additionally, the ideal OpAmp conditions give

$$V_B(s) = V_C(s)$$ **(.5 point)**

Finally, we have that $V_o(s) = V_E(s)$. **(.5 point)**
Solving the above system of equations, we get

$$V_o(s) = \frac{-3Ls}{5R + Ls}V_i(s).$$

from which the answer follows.

Part III [4 points] Let $R = 100$ mΩ, $L = 10$ mH. Compute the gain and phase functions of $T(s)$. What are the DC gain and the ∞-freq gain? What is the cut-off frequency ω_c? Use these values to sketch the magnitude of the frequency response of the circuit. Is this circuit a low-pass, high-pass, or band-pass filter?
[Explain your answer]

Solution: If $R = 100$ mΩ, $L = 10$ mH, the transfer function takes the form

$$T(s) = \frac{-3s}{50 + s}$$

The frequency response is then the complex function

$$T(j\omega) = \frac{-3j\omega}{50 + j\omega}, \quad \omega \geq 0$$

Its magnitude is the gain function,

$$|T(j\omega)| = \frac{3\omega}{|50 + j\omega|} = \frac{3\omega}{\sqrt{2500 + \omega^2}}$$ **(.5 point)**

And its phase is

$$\angle T(j\omega) = \angle(-3j\omega) - \angle(50 + j\omega) = \frac{3\pi}{2} - \arctan\left(\frac{\omega}{50}\right)$$ **(.5 point)**

At $\omega = 0$, we obtain

$$|T(j0)| = 0, \quad \angle T(j0) = \frac{3\pi}{2}$$ **(correct DC-gain gets .5 point)**

At $\omega = \infty$, we obtain

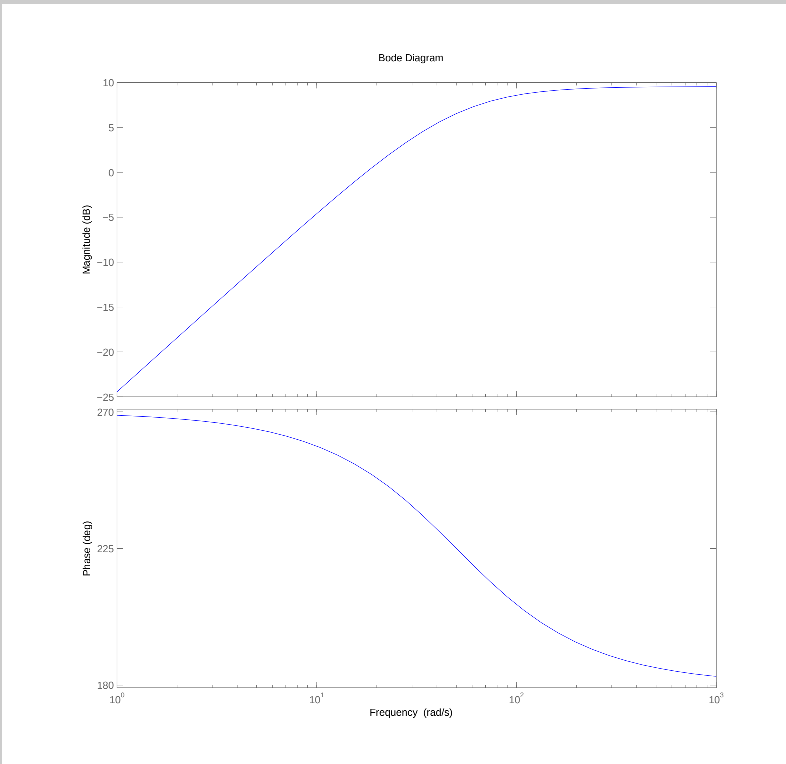
$$|T(j\infty)| = 3, \quad \angle T(j\infty) = \pi$$ **(correct ∞-freq gain gets .5 point)**

The cut-off frequency is defined by

$$|T(j\omega_c)| = \frac{T_{\max}}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Solving for it, we find $\omega_c = 50$ rad/s. **(.5 point)**

With the values obtained above, you can sketch the magnitude of the frequency response as



(only top plot required, the other one here for completeness) **(.5 point)**

This circuit is a high-pass filter. **(1 point)**

Part IV [2 points] Using what you know about frequency response, compute the steady state response $v_o^{SS}(t)$ of this circuit when $v_i(t) = -2\cos(50t - \frac{\pi}{4})$ using the same values of R and L as in Part III.

Solution: To compute the steady-state response to the input $v_i(t) = -2\cos(50t - \frac{\pi}{4})$, we use the frequency response values for $\omega = 50$. In this way,

$$v_o^{SS}(t) = -2\,|T\,(j50)|\cos\left(50t - \frac{\pi}{4} + \angle T(j50)\right) \qquad \text{([1 point for correct expression])}$$

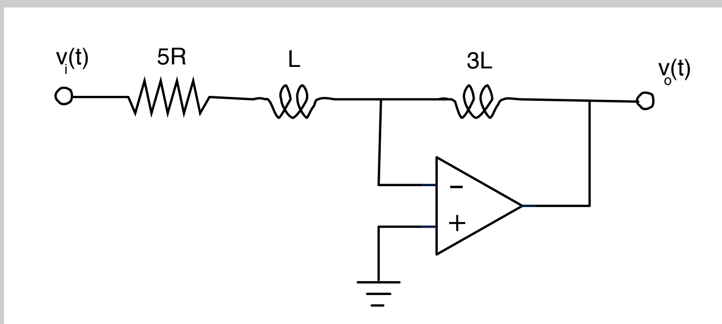
$$= -2\frac{3}{\sqrt{2}}\cos\left(50t - \frac{\pi}{4} + \frac{5\pi}{4}\right) = -3\sqrt{2}\cos\left(50t + \pi\right) = 3\sqrt{2}\cos\left(50t\right)$$

(1 point for correct values [either of the last two answers is valid])

Part V: [2 bonus points] Design an inverting OpAmp circuit that has transfer function $T(s)$. What design would you recommend, your design or the one in Figure 4? Why?

Solution:

We can easily accomplish the same transfer function with an inverting OpAmp with an inductor to generate $Z_2(s)$ and one inductor and one resistor in series to generate $Z_1(s)$. Something like this



(1 bonus point)

Both this design and the one in Figure 3 have zero output impedance. The noninverting OpAmp design seems preferable because of its simplicity and fewer number of components.

(1 bonus point)

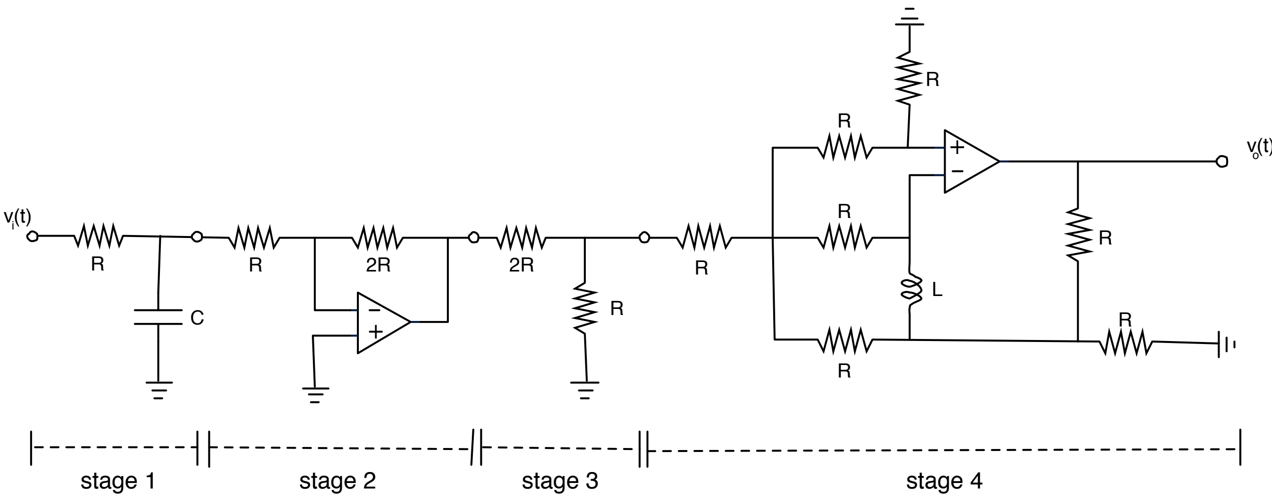


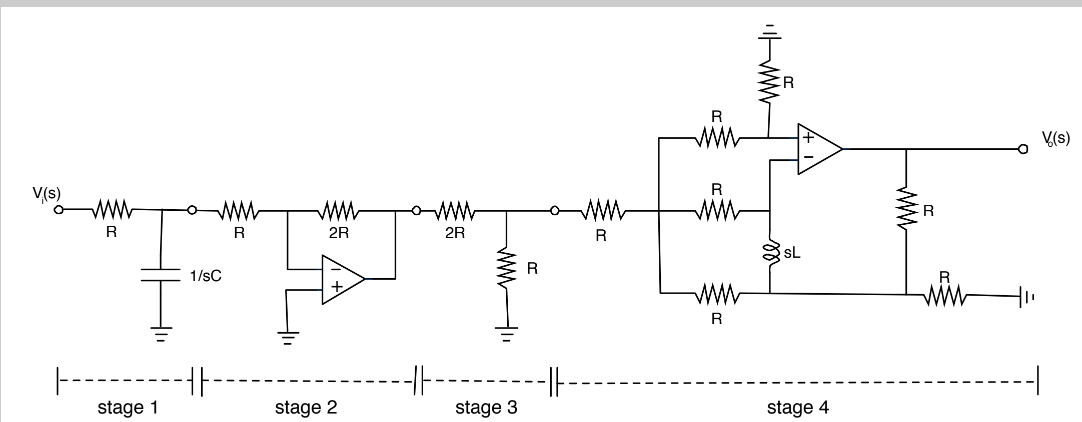
Figure 5: Circuit for Question 5. Stage 4 is the circuit of Question 4.

5. **Chain Rule and Circuit Design**

   Consider the circuit in Figure 5 (note that stage 4 is the circuit of Question 4). You can assume zero initial conditions.

   Part I: [3 points] Redraw the circuit of Figure 5 in the s-domain and compute the transfer functions $T_1(s)$, $T_2(s)$, $T_3(s)$, $T_4(s)$ of each one of the stages.

Solution: With no initial conditions, the circuit in the s-domain simply looks like



(1 point)

Stage 1 is a voltage divider, hence

$$T_1(s) = \frac{1/sC}{R + \frac{1}{sC}} = \frac{1}{RCs + 1}$$ **(.5 point)**

Stage 2 is an inverting OpAmp, hence

$$T_2(s) = -\frac{2R}{R} = -2$$ **(.5 point)**

Stage 3 is a voltage divider, hence

$$T_3(s) = \frac{R}{R+2R} = \frac{1}{3} \qquad \textbf{(.5 point)}$$

Stage 4 is the circuit studied in Question 4, which has transfer function

$$T_4(s) = -\frac{3sL}{5R+sL} \qquad \textbf{(.5 point)}$$

Part II: [2 points] Somebody with a rusty recollection of linear circuits analyzed the circuit in Figure 5 and concluded that the transfer function $T(s)$ from $V_i(s)$ to $V_o(s)$ is equal to the product of the transfer functions

$$\begin{aligned}\widetilde{T}(s) = T_1(s) \times T_2(s) \times T_3(s) \times T_4(s) &= \frac{1}{RCs+1}(-2)\frac{1}{3}\left(\frac{-3sL}{5R+sL}\right) \\ &= \frac{2sL}{RCLs^2 + (5R^2C + L)s + 5R}\end{aligned}$$

of the 4 stages depicted in the plot. Identify two problems that invalidate this conclusion.

Solution: The two problems with the conclusion is that

(i) stage 2 is loading stage 1, **(1 point)**

(ii) stage 4 is loading stage 3, **(1 point)**

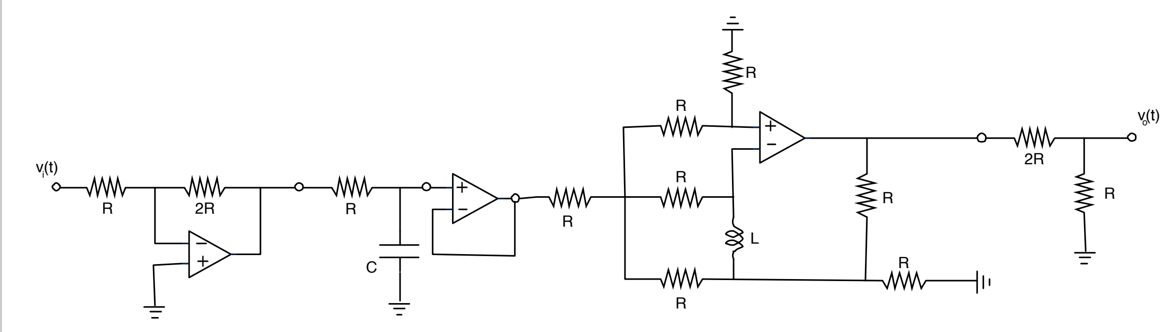
and hence the chain rule does not apply. In both cases, there is current flowing through the input resistors.

Part III: [2 points] Modify Figure 5, keeping all 4 stages but possibly re-ordering them, so that the resulting circuit does have transfer function $\widetilde{T}(s)$ by adding at most 1 OpAmp.
[Justify your answer]

Solution:
Given the two problems identified in Part II, the easiest way to do this would be by adding two OpAmps in a voltage follower configuration, one between stages 1 and 2, and one between stages 3 and 4. This would make the chain rule valid. However, this also requires 2 OpAmps, and they ask us to only use 1 OpAmp. Therefore, we can only use one OpAmp in a voltage follower configuration and we need to re-order the stages to make use of the fact that stage 4 has zero output impedance.

(1 point)

There are multiple ways to do this so that no stage $i+1$ would load stage i. A possible design is the following



(1 point)

Part IV: [3 points] Use stage 1, a noninverting OpAmp, and a voltage divider to design a circuit whose transfer function is $\widetilde{T}(s)$.
[Provide reasons that justify how you arrived at your design]

Solution:
Stage 1 has transfer function $1/(RCs + 1)$. From Part II, the product $T_2(s) \times T_3(s) \times T_4(s)$ simplifies to

$$(-2)\frac{1}{3}\left(\frac{-3sL}{5R+sL}\right) = \frac{2sL}{5R+sL}$$

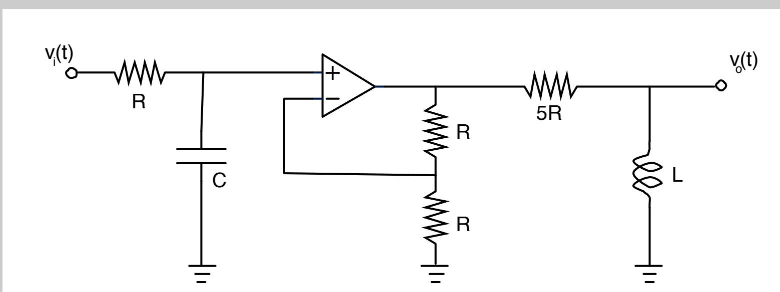
We can also obtain this transfer function with a noninverting OpAmp with gain 2 and a voltage divider involving an inductor L and a resistor $5R$.

(1 point)

Additionally, if we put the noninverting OpAmp between stage 1 and the voltage divider involving the inductor and the resistor, then there is no load from the OpAmp onto stage 1, or from the voltage divider onto the OpAmp, and hence the chain rule applies.

(1 point)

Such a design would yield



(1 point)