

Laplace Transforms – recap for ccts

What's the big idea?

1. Look at initial condition responses of ccts due to capacitor voltages and inductor currents at time $t=0$

Mesh or nodal analysis with s -domain impedances (resistances) or admittances (conductances)

Solution of ODEs driven by their initial conditions

Done in the s -domain using Laplace Transforms

2. Look at forced response of ccts due to input ICSs and IVSs as functions of time

Input and output signals $I_O(s)=Y(s)V_S(s)$ or $V_O(s)=Z(s)I_S(s)$

The cct is a system which converts input signal to output signal

3. Linearity says we add up parts 1 and 2

The same as with ODEs

Why Laplace transforms?

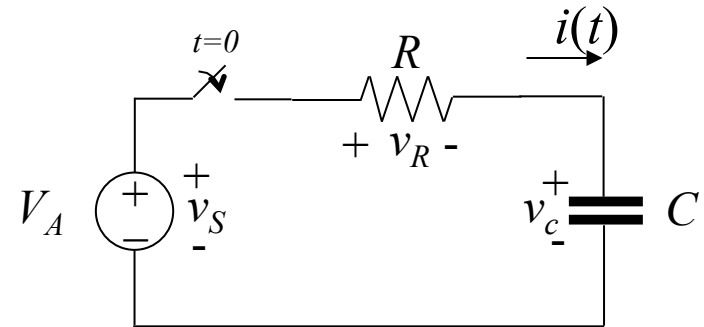
First-order RC cct

KVL $v_S(t) - v_R(t) - v_C(t) = 0$

instantaneous for each t

Substitute element relations

$$v_S(t) = V_A u(t), \quad v_R(t) = Ri(t), \quad i(t) = C \frac{dv_C(t)}{dt}$$



Ordinary differential equation in terms of capacitor voltage

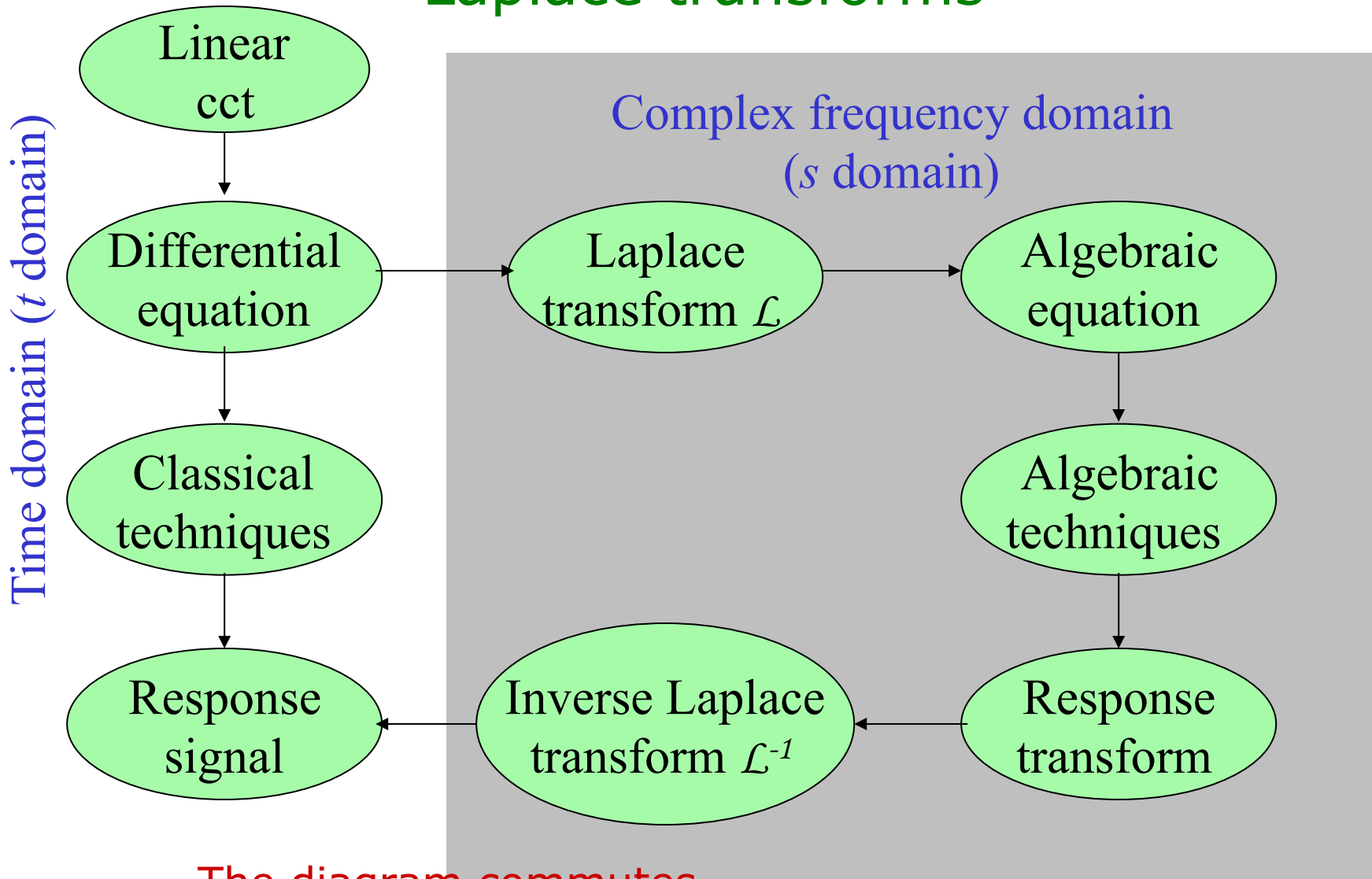
$$RC \frac{dv_C(t)}{dt} + v_C(t) = V_A u(t)$$

Laplace transform $RC[sV_C(s) - v_C(0)] + V_C(s) = \frac{1}{s} V_A$

Solve $V_C(s) = \frac{V_A / RC}{s(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$

Invert LT $v_C(t) = \left[V_A \left(1 - e^{-t/RC} \right) + v_C(0) e^{-t/RC} \right] u(t)$ Volts

Laplace transforms



The diagram commutes

Same answer whichever way you go

Laplace Transform - definition

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

0^- limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of s are $\text{sec}^{-1} = \text{Hz}$

A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)

Laplace transform examples

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925) $u(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

Delta (impulse) function $\delta(t)$

Laplace transform examples

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Delta (impulse) function $\delta(t)$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st} dt = 1 \text{ for all } s$$

Laplace Transform Pair Tables

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity – absolutely critical property

Follows from the integral definition

$$\mathcal{L}\{Af_1(t) + Bf_2(t)\} = A\mathcal{L}\{f_1(t)\} + B\mathcal{L}\{f_2(t)\} = AF_1(s) + BF_2(s)$$

Example

$$\mathcal{L}(A\cos(\beta t)) =$$

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Example

$$\begin{aligned}\mathcal{L}(A \cos(\beta t)) &= \mathcal{L}\left(\frac{A}{2}(e^{j\beta t} + e^{-j\beta t})\right) = \frac{A}{2}\mathcal{L}(e^{j\beta t}) + \frac{A}{2}\mathcal{L}(e^{-j\beta t}) \\ &= \frac{A}{2} \frac{1}{s - j\beta} + \frac{A}{2} \frac{1}{s + j\beta} \\ &= \frac{As}{s^2 + \beta^2}\end{aligned}$$

Laplace Transform Properties

Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^{\infty} \left[\int_0^t f(\tau)d\tau \right] e^{-st} dt$$

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Proof
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^{\infty} \left[\int_0^t f(\tau)d\tau\right] e^{-st} dt$$

Denote
$$x = \frac{-e^{-st}}{s}, \text{ and } y = \int_0^t f(\tau)d\tau$$

so
$$\frac{dx}{dt} = e^{-st}, \text{ and } \frac{dy}{dt} = f(t)$$

Integrate by parts
$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left[\frac{-e^{-st}}{s} \int_0^t f(\tau)d\tau\right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} f(t)e^{-st} dt$$

Laplace Transform Properties

Differentiation Property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

Proof via integration by parts again

$$\begin{aligned}\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[f(t)e^{-st} \right]_{0-}^{\infty} + s \int_{0-}^{\infty} f(t)e^{-st} dt \\ &= sF(s) - f(0-)\end{aligned}$$

Second derivative

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= \mathcal{L}\left\{\frac{d}{dt}\left[\frac{df(t)}{dt}\right]\right\} = s\mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df}{dt}(0-) \\ &= s^2 F(s) - sf(0-) - f'(0-)\end{aligned}$$

Laplace Transform Properties

General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \dots - f^{(m-1)}(0-)$$

Translation properties

s -domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s + \alpha)$$

t -domain translation

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as} F(s) \text{ for } a > 0$$

Laplace Transform Properties

Initial Value Property

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Final Value Property

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Caveats:

Laplace transform pairs do not always handle discontinuities properly

Often get the average value

Initial value property no good with impulses

Final value property no good with cos, sin etc

Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in s

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$
$$= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

A Little Complex Analysis

We are dealing with linear ccts

Our Laplace Transforms will consist of rational functions (ratios of polynomials in s) and exponentials like $e^{-s\tau}$

These arise from

- discrete component relations of capacitors and inductors
- the kinds of input signals we apply
 - Steps, impulses, exponentials, sinusoids, delayed versions of functions

Rational functions have a finite set of discrete *poles*

$e^{-s\tau}$ is an *entire function* and has no poles anywhere

To understand linear cct responses you need to look at the poles – they determine the exponential modes in the response circuit variables.

Two sources of poles: the cct – seen in the response to I_{cs}
the input signal LT poles – seen in the forced response

A Little More Complex Analysis

A complex function is *analytic* in regions where it has no poles

Rational functions are analytic everywhere except at a finite number of isolated points, where they have poles of finite order

Rational functions can be expanded in a Taylor Series about a point of analyticity

$$f(z) = f(a) + (z - a)f'(a) + \frac{1}{2!}(z - a)^2 f''(a) + \dots$$

They can also be expanded in a Laurent Series about an isolated pole

$$f(z) = \sum_{n=-N}^{-1} c_n (z - a)^n + \sum_{n=0}^{\infty} c_n (z - a)^n$$

General functions do not have N necessarily finite

Residues at poles

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

Simple pole: residue = $\lim_{s \rightarrow a} (s - a)F(s)$

Multiple pole: residue = $\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} ((s-a)^m F(s))$

The residue is the c_{-1} term in the Laurent Series

Cauchy Residue Theorem

The integral around a simple closed rectifiable positively oriented curve (*scroc*) is given by $2\pi j$ times the sum of residues at the poles inside

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Bundle complex conjugate pole pairs into second-order terms if you want

$$(s - \alpha - j\beta)(s - \alpha + j\beta) = \left[s^2 - 2\alpha s + (\alpha^2 + \beta^2) \right]$$

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials
For circuits the answers will be real

Inverse Laplace Transforms – the Bromwich Integral

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st} ds$$

This is a contour integral in the complex s -plane

α is chosen so that all singularities of $F(s)$ are to the left of
 $Re(s) = \alpha$

It yields $f(t)$ for $t \geq 0$

The inverse Laplace transform is always a *causal* function

For $t < 0$ $f(t) = 0$

Remember Cauchy's Integral Formula

Counterclockwise contour integral =

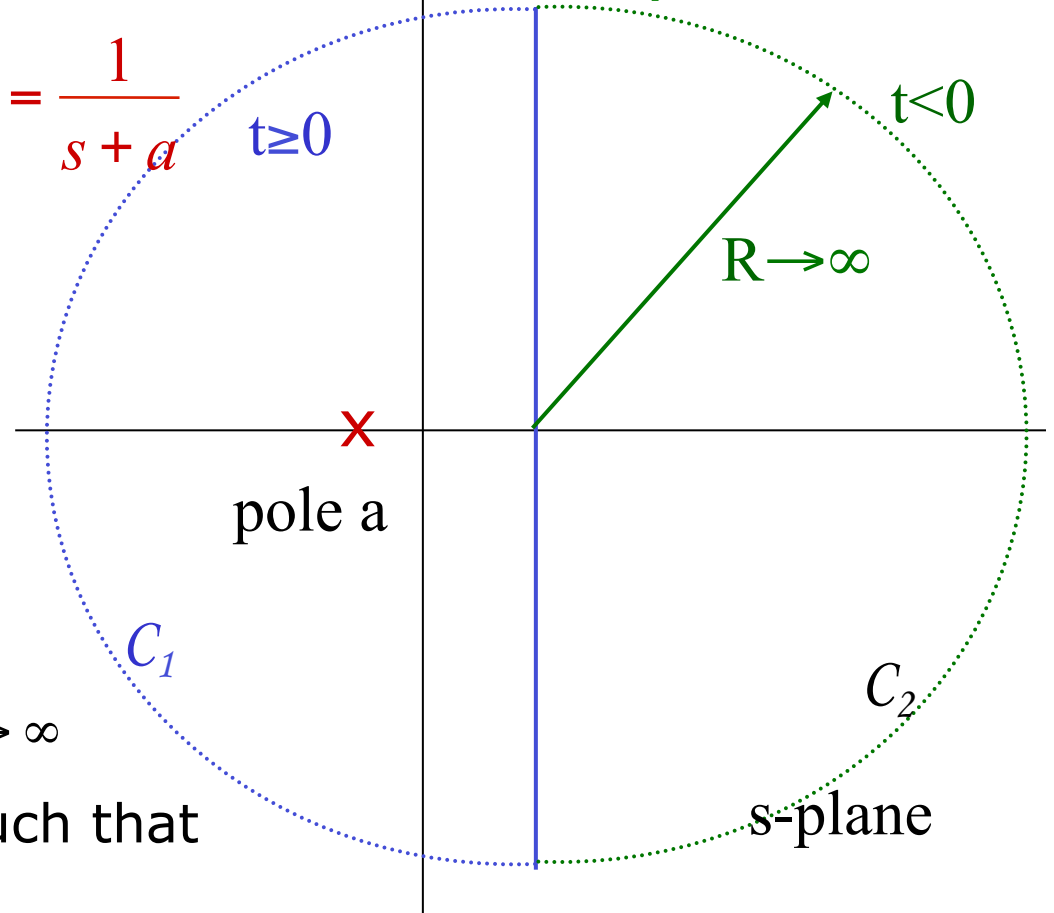
$2\pi j \times (\text{sum of residues inside contour})$

Inverse Laplace Transform Examples

Bromwich integral of $F(s) = \frac{1}{s+a}$ $t \geq 0$

$$f(t) = \int_{\alpha-j\infty}^{\alpha+j\infty} \frac{1}{s+a} e^{st} ds$$

$$= \begin{cases} e^{-at} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



On curve C_1

$$s = \alpha + re^{j\theta}, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \quad r \rightarrow \infty$$

For given θ there is $r \rightarrow \infty$ such that

$$\text{Re}(s) = \alpha + r \cos \theta < 0$$

$$e^{st} = e^{\text{Re}(s)t} e^{j \text{Im}(s)t} \rightarrow 0 \text{ as } r \rightarrow \infty \text{ for } t > 0$$

Integral disappears on C_1 for positive t

Inverting Laplace Transforms

Compute residues at the poles $\lim_{s \rightarrow a} (s - a)F(s)$

$$\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m F(s) \right]$$

Bundle complex conjugate pole pairs into second-order terms if you want

$$(s - \alpha - j\beta)(s - \alpha + j\beta) = \left[s^2 - 2\alpha s + (\alpha^2 + \beta^2) \right]$$

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Inverting Laplace Transforms in Practice

We have a table of inverse LTs

Write $F(s)$ as a partial fraction expansion

$$\begin{aligned} F(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \\ &= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \\ &= \frac{\alpha_1}{(s - p_1)} + \frac{\alpha_2}{(s - p_2)} + \frac{\alpha_{31}}{(s - p_3)} + \frac{\alpha_{32}}{(s - p_3)^2} + \frac{\alpha_{33}}{(s - p_3)^3} + \cdots + \frac{\alpha_q}{(s - p_q)} \end{aligned}$$

Now appeal to linearity to invert via the table

Surprise!

Computing the partial fraction expansion is best done by calculating the residues

Inverting Laplace Transforms

Compute residues at the poles $\lim_{s \rightarrow a} (s - a)F(s)$

$$\frac{1}{(m-1)!} \lim_{s \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^m F(s) \right]$$

Example $\frac{2s^2 + 5s}{(s+1)^3} = \frac{2(s+1)^2 + (s+1) - 3}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3}$

Inverting Laplace Transforms

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$$\lim_{s \rightarrow -1} \frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} = -3 \qquad \lim_{s \rightarrow -1} \frac{d}{ds} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 1$$

$$\frac{1}{2!} \lim_{s \rightarrow -1} \frac{d^2}{ds^2} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 2$$

$$\mathcal{L}^{-1} \left[\frac{2s^2 + 5s}{(s+1)^3} \right] = e^{-t}(2 + t - 3t^2)u(t)$$

T&R, 5th ed, Example 9-12

Find the inverse LT of $F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$

T&R, 5th ed, Example 9-12

Find the inverse LT of $F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$

$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1-j2} + \frac{k_2^*}{s+1+j2}$$

$$k_1 = \lim_{s \rightarrow -1} (s+1)F(s) = \frac{20(s+3)}{s^2+2s+5} \Big|_{s=-1} = 10$$

$$k_2 = \lim_{s \rightarrow -1+2j} (s+1-2j)F(s) = \frac{20(s+3)}{(s+1)(s+1+2j)} \Big|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$f(t) = \left[10e^{-t} + 5\sqrt{2}e^{(-1+j2)t+j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-j2)t-j\frac{5}{4}\pi} \right] u(t)$$

$$= \left[10e^{-t} + 10\sqrt{2}e^{-t} \cos\left(2t + \frac{5\pi}{4}\right) \right] u(t)$$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$\begin{aligned} F(s) &= s + 2 + \frac{s + 2}{s^2 + 4s + 3} \\ &= s + 2 + \frac{0.5}{s + 1} + \frac{0.5}{s + 3} \end{aligned}$$

Invert as normal

$$f(t) = \left[\frac{d\delta(t)}{dt} + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t} \right] u(t)$$

Multiple Poles

Look for partial fraction decomposition

$$F(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)^2} = \frac{k_1}{s - p_1} + \frac{k_{21}}{s - p_2} + \frac{k_{22}}{(s - p_2)^2}$$

$$Ks - Kz_1 = k_1(s - p_2)^2 + k_{21}(s - p_1)(s - p_2) + k_{22}(s - p_1)$$

Equate like powers of s to find coefficients

$$k_1 + k_{21} = 0$$

$$-2k_1p_2 - 2k_{21}(p_1 + p_2) + k_{22} = K$$

$$k_1p_2^2 + k_{21}p_1p_2 - k_{22}p_1 = Kz_1$$

Solve

Recall motivating example for LT

First-order RC cct

$$\text{KVL } v_S(t) - v_R(t) - v_C(t) = 0$$

instantaneous for each t

Substitute element relations

$$v_S(t) = V_A u(t), \quad v_R(t) = Ri(t), \quad i(t) = C \frac{dv_C(t)}{dt}$$

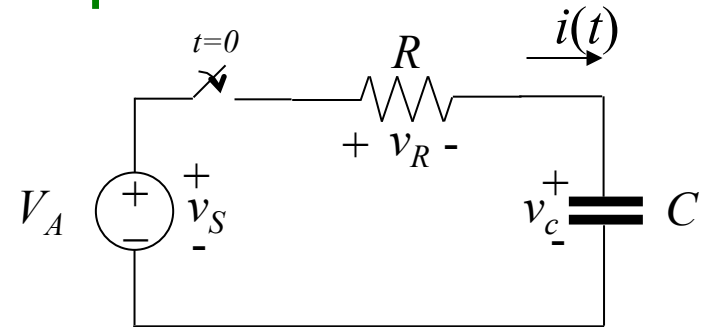
Ordinary differential equation in terms of capacitor voltage

$$RC \frac{dv_C(t)}{dt} + v_C(t) = V_A u(t)$$

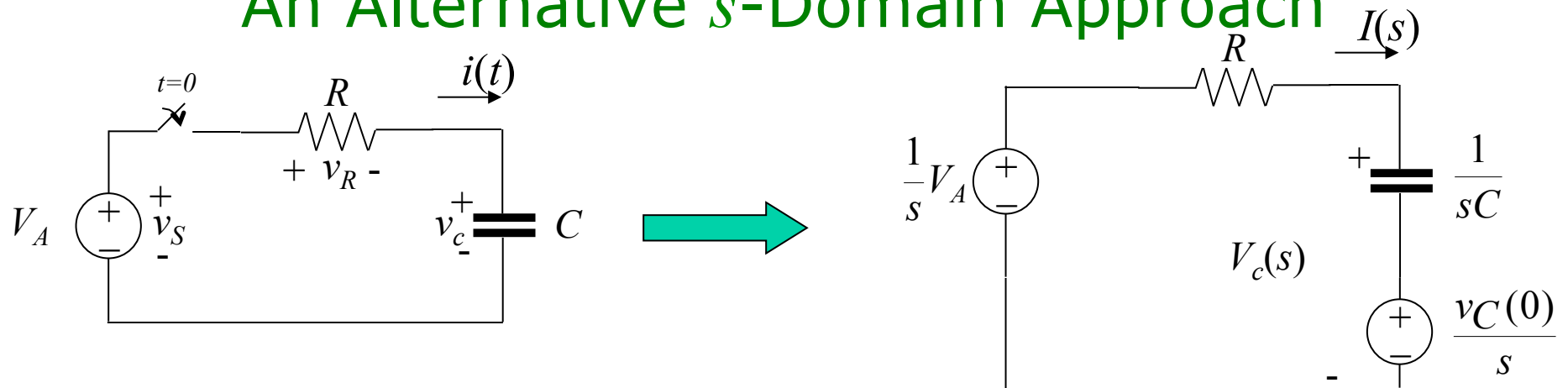
$$\text{Laplace transform } RC[sV_C(s) - v_C(0)] + V_C(s) = \frac{1}{s} V_A$$

$$\text{Solve } V_C(s) = \frac{V_A / RC}{s(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$$

$$\text{Invert LT } v_C(t) = \left[V_A \left(1 - e^{-t/RC} \right) + v_C(0) e^{-t/RC} \right] u(t) \quad \text{Volts}$$



An Alternative s -Domain Approach



Transform the cct element relations

Work in s -domain directly

$$V_C(s) = \frac{1}{Cs} I_C(s) + \frac{v_C(0)}{s}$$

$$I_C(s) = sCV_C(s) - Cv_C(0)$$

OK since \mathcal{L} is linear

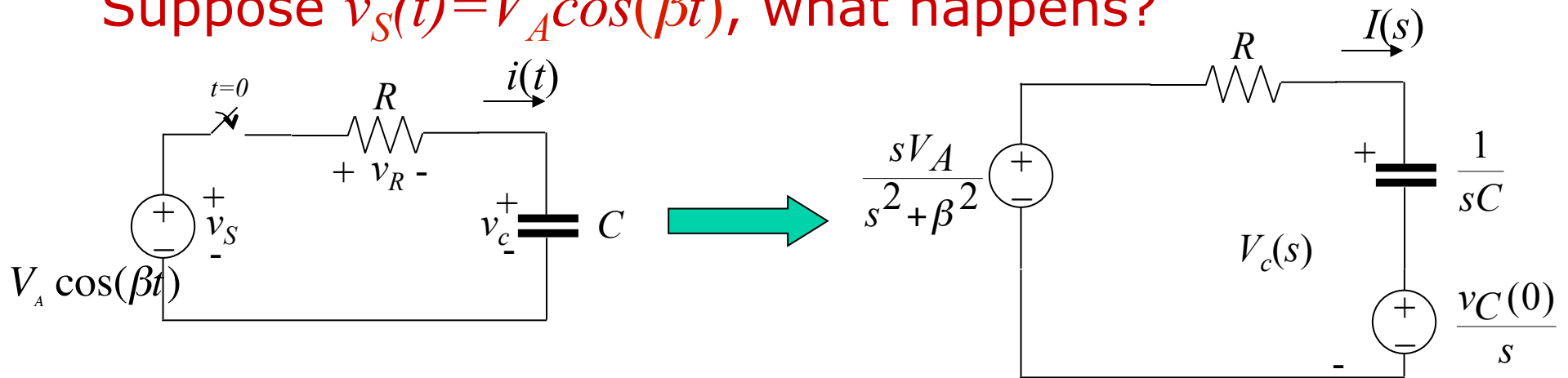
Impedance + source

Admittance + source

KVL in s -Domain $sCRV_C(s) - CRv_C(0) + V_C(s) = \frac{1}{s}V_A$

Time-varying inputs

Suppose $v_S(t) = V_A \cos(\beta t)$, what happens?



KVL as before $(RCs + 1)V_C(s) - RCv_C(0) = \frac{sV_A}{s^2 + \beta^2}$

$$V_C(s) = \frac{sV_A / RC}{(s^2 + \beta^2)(s + 1/RC)} + \frac{v_C(0)}{s + 1/RC}$$

Solve $v_C(t) = \left[\frac{V_A}{\sqrt{1 + (\beta RC)^2}} \cos(\beta t + \theta) - \frac{V_A}{1 + (\beta RC)^2} e^{-t/RC} + v_C(0) e^{-t/RC} \right] u(t)$