MAE140 - Linear Circuits - Fall 14

Final, December 19

## Instructions

(i) The exam is open book. You may use your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 180 minutes

(iii) Do not forget to write your **name** and **student number**

(iv) On the questions for which the answers are given, please provide detailed derivations

(v) The exam has 5 questions for a total of 50 points and 3 bonus points

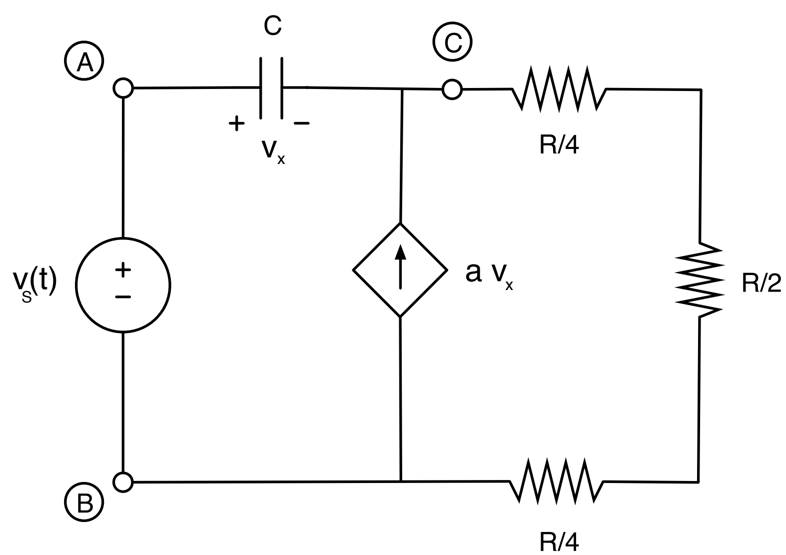



Figure 1: Circuit for Question 1.

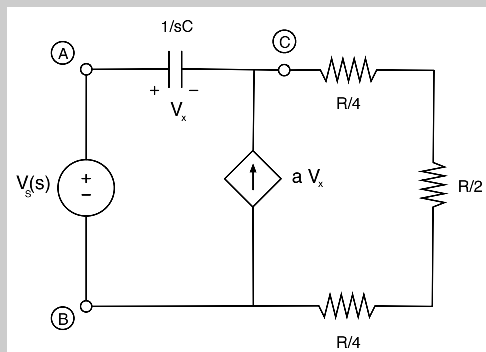
1. **Equivalent Circuits**

**Part I:** [2 points] Assuming zero initial conditions, transform the circuit in Figure 1 into the $s$-domain.

**Solution:**

Since all initial conditions are zero, it is easy to transform the circuit to the $s$-domain.

**[2 points]**

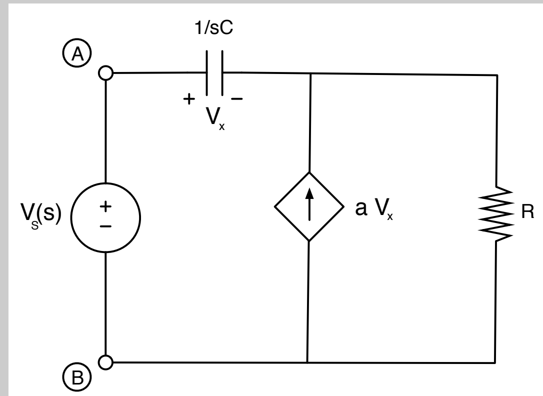


**Part II:** [4 points] Find the impedance equivalent in the circuit obtained in Part I as seen from terminals Ⓐ and Ⓑ. The answer should be given as a ratio of two polynomials.

**Solution:**

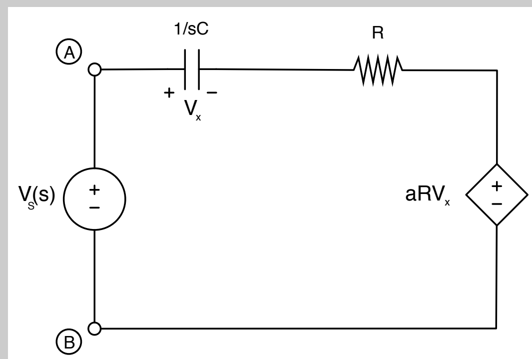
The three resistors on the right are in series when seen from terminals Ⓐ and Ⓑ, so we get

**[1 point]**




Now, we do source transformation with the dependent source in parallel with the resistor to get

**[1 point]**




By voltage division, we deduce that

$$V_x(s) = \frac{1/(sC)}{R + 1/sC}(V_S(s) - aRV_x(s)) \qquad \textbf{(1 point)}$$

and solving for $V_x(s)$, we obtain

$$V_x(s) = \frac{1}{1 + R(a + Cs)} V_S(s)$$

Therefore, the current through the capacitor is

$$I_x(s) = V_x(s)/(1/Cs) = \frac{Cs}{1 + R(a + Cs)} V_S(s)$$

Finally, the impedance equivalent as seen from terminals Ⓐ and Ⓑ is

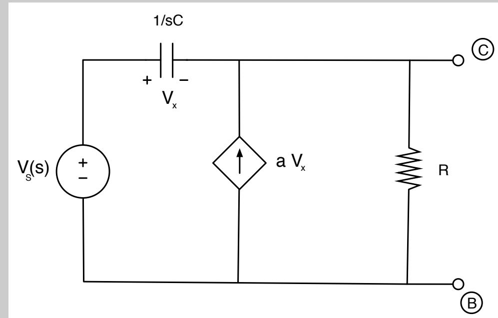
$$Z_{AB} = V_S(s)/I_x(s) = \frac{1 + R(a + Cs)}{Cs} \qquad \textbf{(1 point)}$$

**Part III:** [4 points] Find the $s$-domain Thévenin equivalent of the circuit obtained in Part I as seen from terminals Ⓒ and Ⓑ.

**Solution:**

We need to find $V_T$ and $Z_T$. The three resistors on the right are still in series when seen from terminals Ⓒ and Ⓑ, so that we get

**[.5 point]**



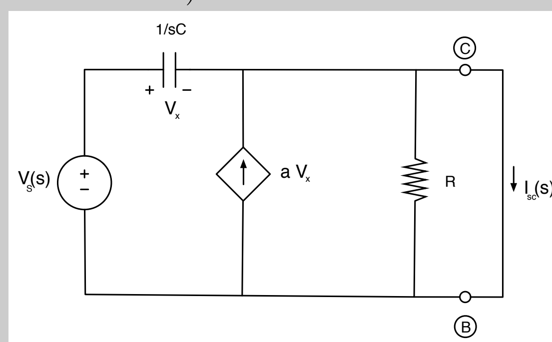

The Thévenin voltage is the open-circuit voltage at terminals Ⓒ and Ⓑ. From the picture, we see that this corresponds to the voltage seen by the resistor $R$, which is

$$V_T = -V_x(s) + V_S(s) = \frac{R(a + Cs)}{1 + R(a + Cs)} V_S(s) \qquad \textbf{(.5 point)}$$

(here, we have used the expression for $V_X(s)$ obtained in Part II).

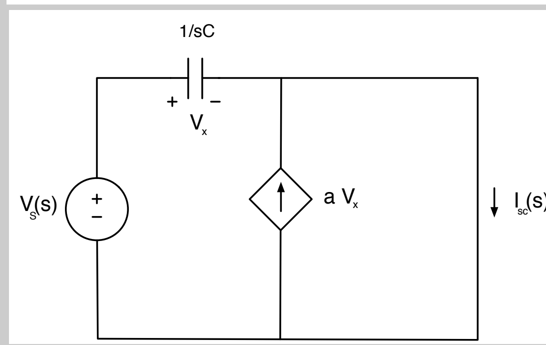
To find the Thévenin impedance, we need to compute the short-circuit current (turning off sources is not an option because of the presence of the dependent source)
**[1 point]**
The circuit then looks like the plot on the right

**[.5 point]**




The two terminals of the resistor are connected to the same node, so there is no voltage drop across it, and we get

**[.5 point]**




From the circuit, we see that $V_x(s) = V_S(s)$, and therefore, the current through the capacitor is

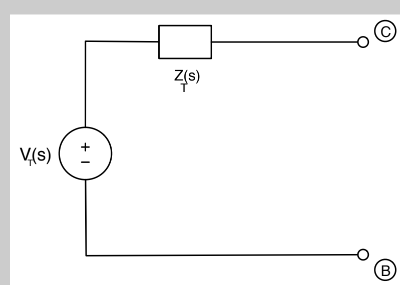
$$I_x(s) = sCV_S(s)$$

KCL finally yields

$$I_{sc}(s) = aV_x(s) + I_x(s) = aV_S(s) + sCV_S(s) = (a + sC)V_S(s) \qquad \textbf{(.5 point)}$$

Therefore, the Thévenin impedance is

$$Z_T(s) = \frac{V_T(s)}{I_{sc}(s)} = \frac{\frac{R(a+Cs)}{1+R(a+Cs)} V_S(s)}{(a + sC)V_S(s)} = \frac{R}{1 + R(a + Cs)}$$




**[.5 point]**

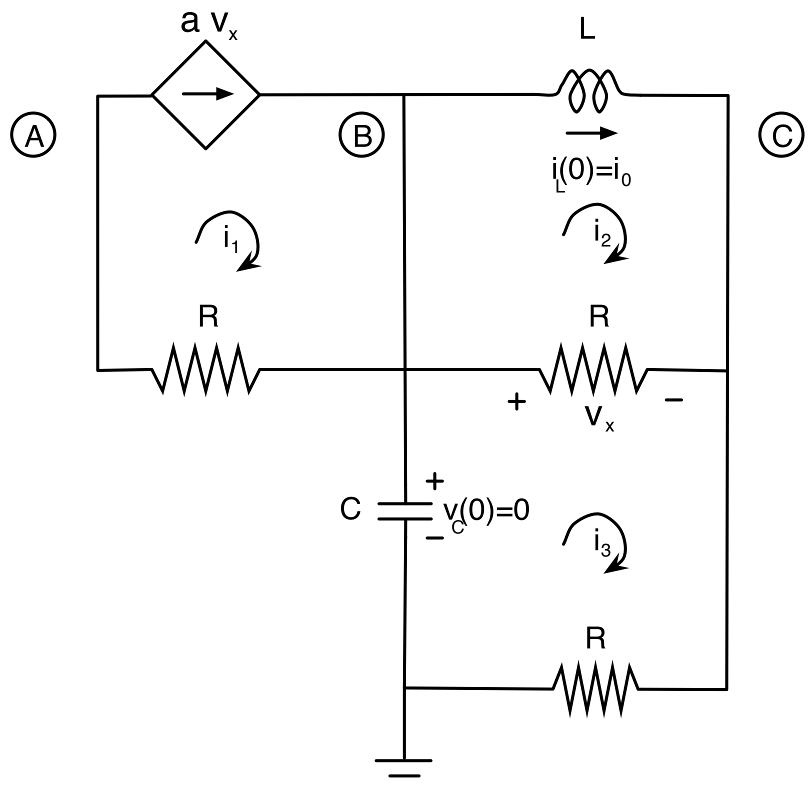
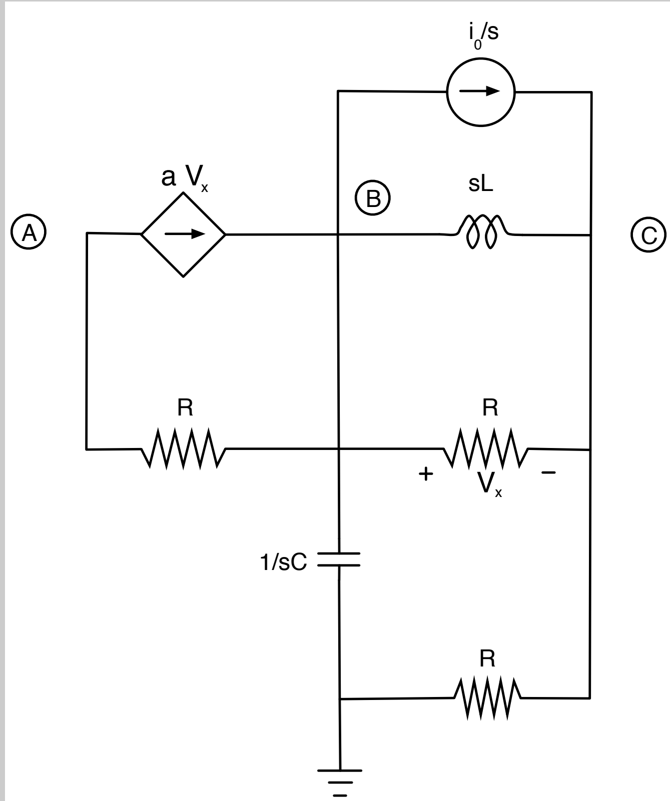



Figure 2: Nodal and Mesh Analysis Circuit

2. **Nodal and Mesh Analysis**

**Part I:** [5 points] Formulate node-voltage equations in the $s$-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure and transform them into current sources. Explain how you deal with the presence of the dependent source. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

**Solution:**




In the above figure, we have transformed the circuit into the $s$-domain, taking good care of respecting the current orientation. Since the initial condition of the capacitor is zero, no need to worry about adding a source for it.

**[.5 point for correct circuit; .5 point for correct initial conditions]**

We next write KCL node equations for nodes A, B and C. For node A, we have

$$\frac{1}{R}(V_A(s) - V_B(s)) = -aV_x(s) \qquad \textbf{(1 point)}$$

For node B, we have

$$\frac{1}{R}(V_B(s) - V_A(s)) + sC\,V_B(s) + \frac{1}{R}(V_B(s) - V_C(s)) + \frac{1}{sL}(V_B(s) - V_C(s)) = aV_x(s) - \frac{i_0}{s}$$

**(1 point)**

For node C, we have

$$\frac{1}{sL}(V_C(s) - V_B(s)) + \frac{1}{R}(V_C(s) - V_B(s)) + \frac{1}{R}V_C(s) = \frac{i_0}{s} \qquad \textbf{(1 point)}$$

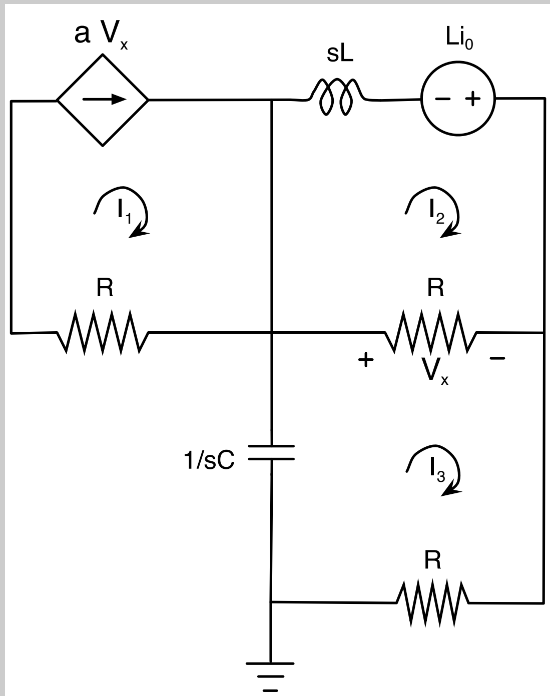
Finally, because we have a dependent source, we need one more equation. This comes from realizing

$$V_x(s) = V_B(s) - V_C(s) \qquad \textbf{(1 point)}$$

This gives a total of 4 independent equations in 4 unknowns ($V_A(s)$, $V_B(s)$, $V_C(s)$, $V_x(s)$). Alternatively, one can take this last equation and substitute it in the first one to arrive at 3 independent equations in 3 unknowns ($V_A(s)$, $V_B(s)$, $V_C(s)$).

**Part II:** [5 points] Formulate mesh-current equations in the $s$-domain for the circuit in Figure 2. Use the currents shown in the figure. Use the initial conditions indicated in the figure and transform them into voltage sources. Explain how you deal with the presence of the dependent source. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

**Solution:**




In the above figure, we have transformed the circuit into the $s$-domain, taking good care of respecting the current orientation. Again, no need to worry about the initial condition of the capacitor because it is zero.

[**.5 point for correct circuit; .5 point for correct initial conditions**]

For mesh-current analysis, the presence of the current source is a problem that must be dealt with. In this case, since it only belongs to one mesh, we deal with it by simply setting

$$I_1(s) = aV_x(s) \qquad \textbf{(1 point)}$$

(this is method #2).

We need to write mesh equations for meshes 2 and 3. For mesh 2, we have

$$R(I_2(s) - I_3(s)) + sLI_2(s) = Li_0 \qquad \textbf{(1 point)}$$

For mesh 3, we have

$$\frac{1}{sC}I_3(s) + R(I_3(s) - I_2(s)) + RI_3(s) = 0 \qquad \textbf{(1 point)}$$

Finally, because we have a dependent source, we need one more equation. This comes from realizing

$$V_x(s) = R(I_3(s) - I_2(s)) \qquad \textbf{(1 point)}$$

This gives a total of 4 independent equations in 4 unknowns ($I_1(s)$, $I_2(s)$, $I_3(s)$, $V_x(s)$). Alternatively, one can take this last equation and substitute it in the second one to arrive at 3 independent equations in 3 unknowns ($I_1(s)$, $I_2(s)$, $I_3(s)$).

**Part III:** [1 bonus point] Express the transform $I_L(s)$ of the inductor current in terms of your unknown variables of Part I and also in terms of your unknown variables of Part II.

**Solution:**

We just need to be careful to not lose track of the transform of the inductor current. In the case of Part I, because we use a current source to account for the initial condition, we actually have

$$I_L(s) = i_0/s + (V_B(s) - V_C(s))/(sL) \qquad \textbf{(.5 bonus point)}$$

In the case of Part II, because we use a voltage source to account for the initial condition, we have

$$I_L(s) = I_2(s) \qquad \textbf{(.5 bonus point)}$$

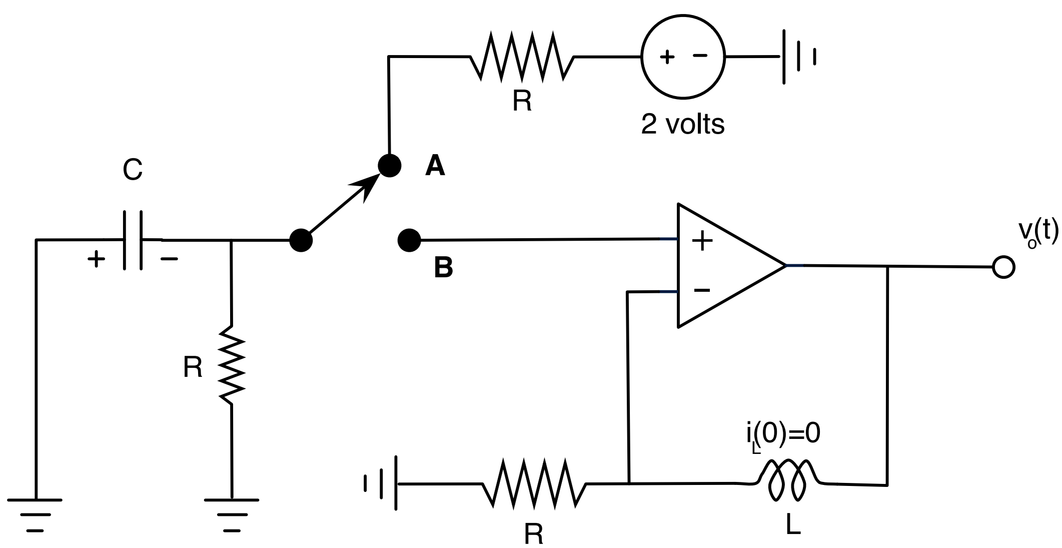



Figure 3: RCL circuit for Laplace Analysis

3. **Laplace Domain Circuit Analysis**

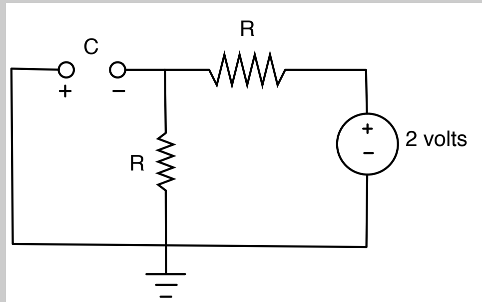
**Part I:** [2 points] Consider the circuit depicted in Figure 3. The voltage source is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial capacitor voltage is given by

$$v_C(0^-) = -1V.$$

[Show your work]

**Solution:** To find the initial conditions, we substitute the capacitor by an open circuit.

**[.5 point for correct circuit; .5 point for substituting capacitor by open circuit]**

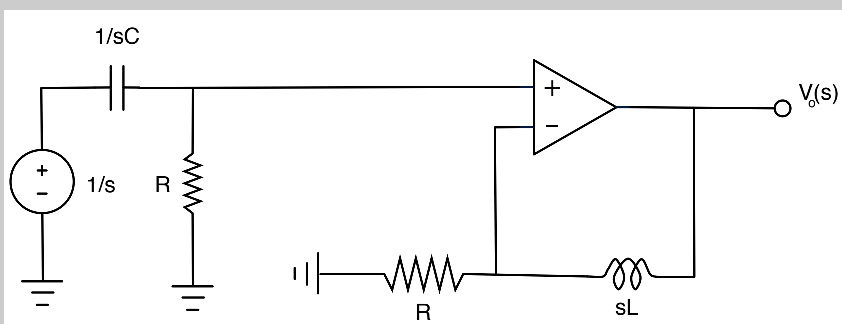



Using voltage division, we find that

$$v_C(0^-) = -\frac{R}{R+R}2 = -1V. \qquad \textbf{(1 point)}$$

**Part II:** [4 points] Use this initial condition to transform the circuit into the $s$-domain for $t \geq 0$. Use an equivalent model for the capacitor in which the initial condition appears as a voltage source. Find the transfer function of the circuit.
[Show your work]

**Solution:** We add one voltage source in series for the capacitor to take care of its initial condition, paying special attention to the polarities.




**(1 point for correct circuit; 1 point for correct polarity)**

The transfer function can be easily found by realizing that the circuit is the composition of a voltage divider and a non-inverting op-amp. Thanks to the non-inverting op-amp, the chain rule applies.

**(1 point)**

Given the above, the transfer function is simply

$$T(s) = \frac{R}{R+\frac{1}{sC}} \times \frac{sL+R}{R} = \frac{Cs(sL+R)}{RCs+1} \qquad \textbf{(1 point)}$$

**Part III:** [4 points] Use domain circuit analysis and inverse Laplace transforms to show that the output voltage $v_o(t)$ when $C = 1000\,\mu F$, $L = 1000\,\mu H$, and $R = 1$ k$\Omega$ is

$$v_o(t) = 10^{-6}\delta(t) + (1 - 10^{-6})e^{-t}u(t).$$

**Solution:** From our answer to Part II, the Laplace transform of the output voltage is

$$V_o(s) = T(s)\frac{1}{s} = \frac{C(sL+R)}{RCs+1} \qquad \textbf{(1 point)}$$

Substituting the RLC values, we get

$$V_o(s) = \frac{10^{-6}s+1}{s+1}$$

Using long division, we can express this as

$$V_o(s) = 10^{-6} + \frac{1-10^{-6}}{s+1} \qquad \textbf{(2 points)}$$

The output voltage is then

$$v_o(t) = 10^{-6}\delta(t) + (1-10^{-6})e^{-t}u(t) \qquad \textbf{(1 point)}$$

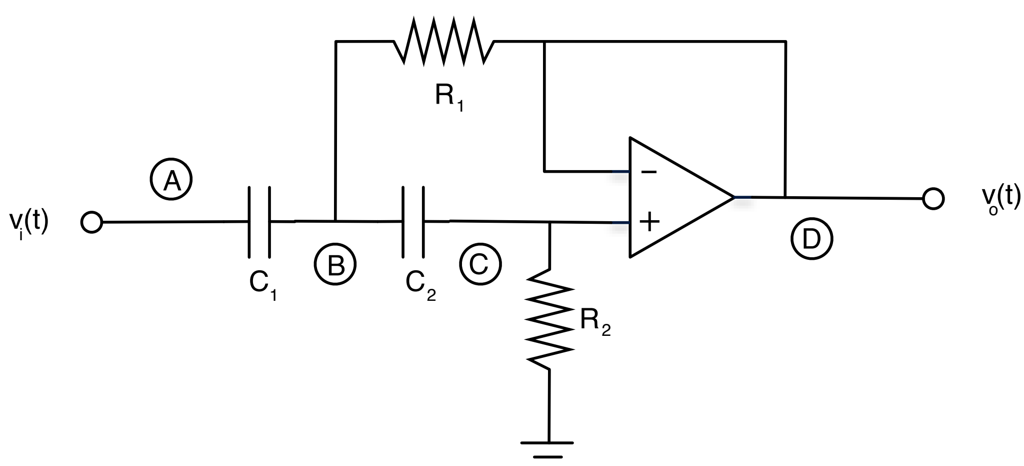
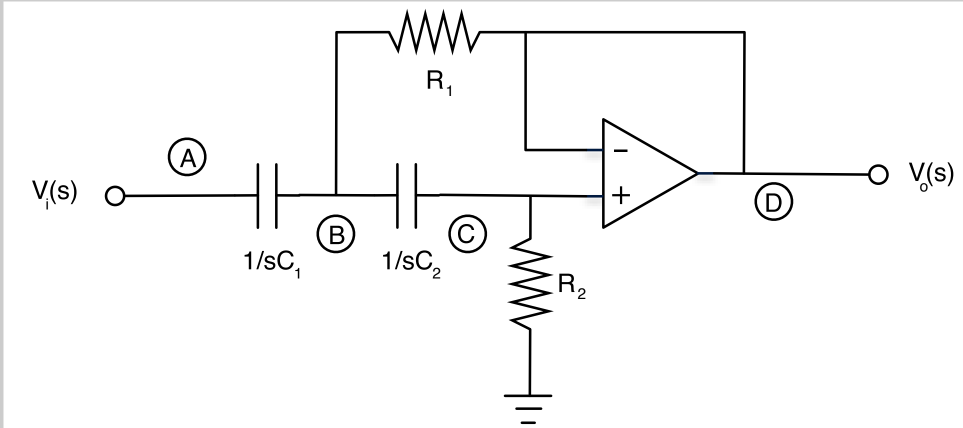



Figure 4: Frequency Response Analysis.

4. **Frequency Response Analysis**

**Part I:** [1 point] Assuming zero initial conditions, transform the circuit in Figure 4 into the $s$-domain.

**Solution:** Since all initial conditions are zero, there is no need to add an independent source for the capacitors. Therefore, the circuit in the $s$-domain looks like




**[1 point]**

**Part II:** [3 points] Show that the transfer function from $V_i(s)$ to $V_o(s)$ is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{R_1R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + R_1(C_1+C_2)s + 1}.$$

[Show your work]
*Hint:* use node voltage analysis

**Solution:** Since we cannot recognize any of the basic building blocks of OpAmps, we resort to nodal analysis. Right away, we see that $V_A = V_i(s)$ and $V_D = V_o(s)$. **(1 point)**
Therefore, we just need to write KCL equations for nodes B and C. Nodal analysis at node B gives

$$sC_1(V_B(s) - V_A(s)) + sC_2(V_B(s) - V_C(s)) + \frac{1}{R_1}(V_B(s) - V_D(s)) = 0 \qquad \text{(.5 point)}$$

Nodal analysis at node C gives

$$sC_2(V_C(s) - V_B(s)) + \frac{1}{R_2}V_C(s) = 0 \qquad \text{(.5 point)}$$

Additionally, the ideal OpAmp conditions give

$$V_C(s) = V_D(s) \qquad \text{(.5 point)}$$

Solving the above system of equations, we get

$$V_o(s) = \frac{R_1R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + R_1(C_1+C_2)s + 1}V_i(s). \qquad \text{(.5 point)}$$

from which the answer follows.

**Part III** [4 points] Let $R_1 = R_2 = 10\ \text{k}\Omega$, $C_1 = C_2 = 220$ nF. Compute the gain and phase functions of $T(s)$. What are the DC gain and the $\infty$-freq gain? What is the cut-off frequency $\omega_c$? Use these values to sketch the magnitude of the frequency response of the circuit. Is this circuit a low-pass,

high-pass, or band-pass filter?
[Explain your answer]

**Solution:** For $R_1 = R_2 = 10$ kΩ, $C_1 = C_2 = 220$ nF, the transfer function takes the form

$$T(s) = \frac{484 * 10^{-8} s^2}{484 * 10^{-8} s^2 + 440 * 10^{-5} s + 1} = \frac{s^2}{s^2 + \frac{440}{484} * 10^3 s + \frac{1}{484} 10^8}$$

For convenience, we write as

$$T(s) = \frac{s^2}{s^2 + a_1 s + a_0}$$

with $a_1 = \frac{440}{484} * 10^3$ and $a_0 = \frac{1}{484} 10^8$.
The frequency response is then the complex function

$$T(j\omega) = \frac{-\omega^2}{-\omega^2 + a_1 j\omega + a_0}, \quad \omega \geq 0$$

Its magnitude is the gain function,

$$|T(j\omega)| = \frac{\omega^2}{\sqrt{(a_0 - \omega^2)^2 + a_1^2 \omega^2}} \qquad \textbf{(.5 point)}$$

And its phase is

$$\angle T(j\omega) = \angle(-\omega^2) - \angle(a_0 - \omega^2 + a_1 \omega j) = \pi - \arctan\left(\frac{a_1 \omega}{a_0 - \omega^2}\right) \qquad \textbf{(.5 point)}$$

At $\omega = 0$, we obtain

$$|T(j0)| = 0, \quad \angle T(j0) = \pi \qquad \textbf{(correct DC-gain gets .5 point)}$$

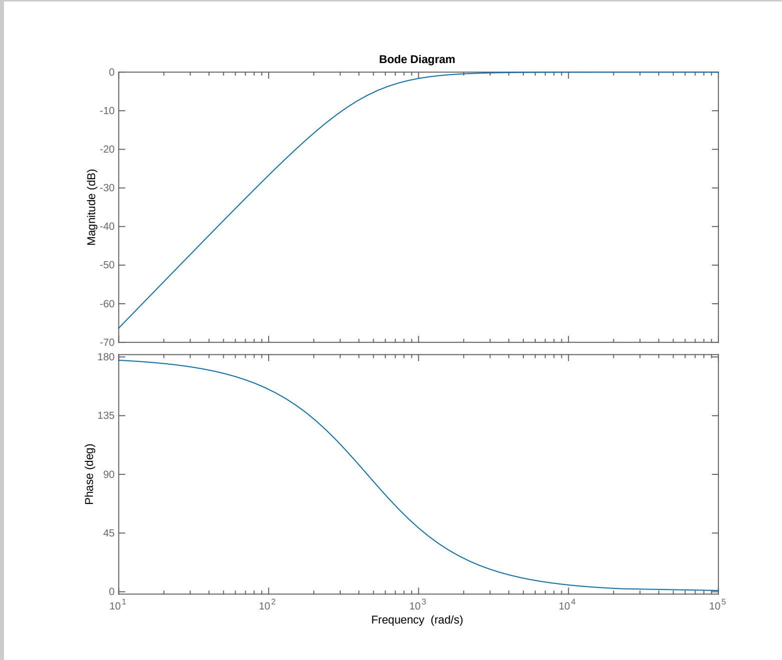
At $\omega = \infty$, we obtain

$$|T(j\infty)| = 1, \quad \angle T(j\infty) = 0 \qquad \textbf{(correct } \infty\textbf{-freq gain gets .5 point)}$$

The cut-off frequency is defined by

$$|T(j\omega_c)| = \frac{T_{\max}}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Solving for it, we find $\omega_c = 706.26$ rad/s. **(.5 point)**
With the values obtained above, you can sketch the magnitude of the frequency response as




**(only top plot required, the other one here for completeness)** **(.5 point)**

This circuit is a high-pass filter. **(1 point)**

**Part IV** [2 points] Using what you know about frequency response, compute the steady state response $v_o^{SS}(t)$ of this circuit when $v_i(t) = 2\cos(706.26t - \frac{\pi}{2})$ using the same values of $R_1$, $R_2$, $C_1$, and $C_2$ as in Part III.

**Solution:** To compute the steady-state response to the input $v_i(t) = 2\cos(706.26t - \frac{\pi}{2})$, we use the frequency response values for $\omega_c = 706.26$, $|T(j\omega_c)| = 1/\sqrt{2}$ and $\angle T(j\omega_c) = 1.14372$. In this way,

$$v_o^{SS}(t) = 2\,|T\,(j706.26)|\cos\Big(706.26t - \frac{\pi}{2} + \angle T(j706.26)\Big) \quad \textbf{(1 point for correct expression)}$$
$$= 2\frac{1}{\sqrt{2}}\cos\Big(706.26t - \frac{\pi}{2} + 1.14372\Big) = \sqrt{2}\cos\left(706.26t - 0.427077\right)$$

**(1 point for correct values)**

5. **Op-Amps and Loading**

Consider the circuit in Figure 5

**Part I:** [3 points] Considering stage 1 and stage 2 separately, use nodal analysis to obtain the output voltages $v_C$ and $v_D$ of the Op-amps in stage 1.

**Solution:** We only write KCL equations for nodes A and B, since nodes C and D are output nodes of op-amps.

**(.5 point)**

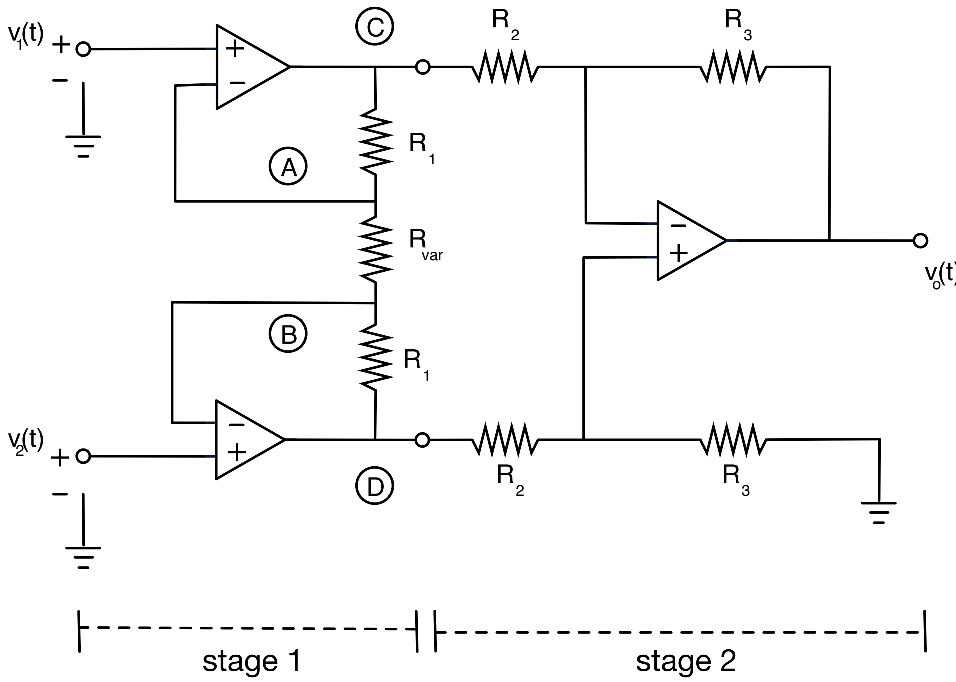



Figure 5: Circuit for Question 5.

KCL for node (A) takes the form

$$\frac{1}{R_1}(v_A - v_C) + \frac{1}{R_{var}}(v_A - v_B) = 0 \qquad \textbf{(.5 point)}$$

KCL for node (B) takes the form

$$\frac{1}{R_1}(v_B - v_D) + \frac{1}{R_{var}}(v_B - v_A) = 0 \qquad \textbf{(.5 point)}$$

Moreover, from the ideal op-amp equations, we have $v_A = v_1$ and $v_B = v_2$.

**(.5 point)**

Solving for $v_C$ and $v_D$, we obtain

$$v_C = v_1 - \frac{R_1}{R_{var}}(v_2 - v_1) \qquad \textbf{(.5 point)}$$

$$v_D = v_2 + \frac{R_1}{R_{var}}(v_2 - v_1) \qquad \textbf{(.5 point)}$$

**Part II:** [2 points] Do you recognize stage 2 as any of the basic op-amp circuits? If so, specify which one and provide an explicit expression for the output voltage $v_o$.

**Solution:** Yes, stage 2, when viewed independently, is an inverting differential amplifier.

**(1 point)**

Accordingly, if we denote the input voltages to the second stage by $v_{i,1}$ (top) and $v_{i,2}$ (bottom), we obtain the expression for the output voltage

$$v_o = -\frac{R_3}{R_2}v_{i,1} + \frac{R_3}{R_2+R_3}\frac{R_2+R_3}{R_2}v_{i,2} = \frac{R_3}{R_2}(v_{i,2}-v_{i,1}) \qquad \textbf{(1 point)}$$

**Part III:** [2 points] Consider now the connection of stages 1 and 2 depicted in Figure 5. Is there loading? Justify your answer.

**Solution:** There is no loading because the output impedance of the op-amps on the left is zero. So even if there is still current flowing through the $R_2$ resistors, the voltages $v_C$ and $v_D$ do not change when the two stages are connected.

**(2 points)**

**Part IV:** [1 point] For the circuit in Figure 5, show that the output voltage $v_o$ as a function of the input voltages $v_1$ and $v_2$ is

$$v_o(t) = \frac{R_3}{R_2}\left(1+\frac{2R_1}{R_{var}}\right)(v_2(t)-v_1(t))$$

[Justify your work]

[Hint: use your answer to Part III]

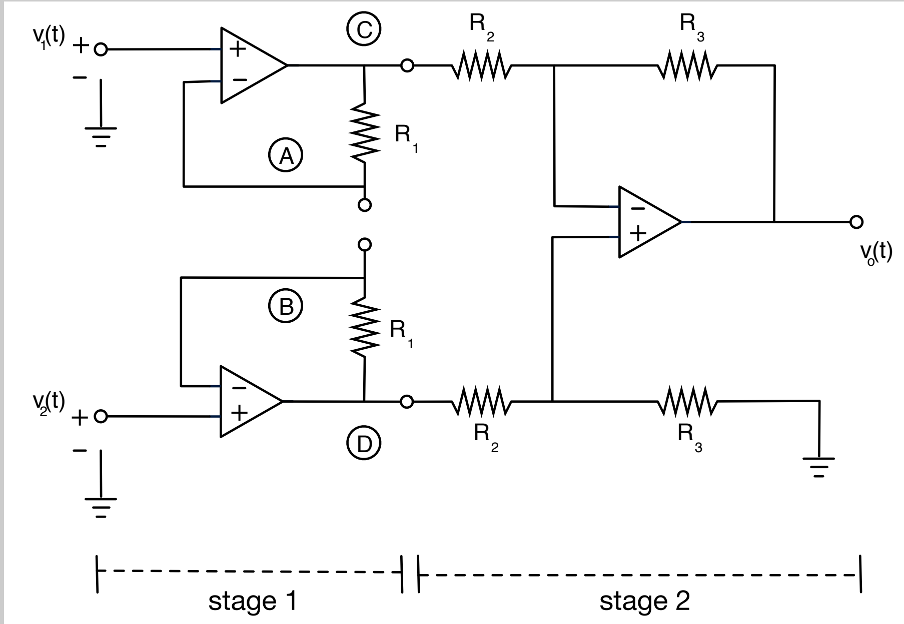
**Solution:** Given that there is no loading (cf. Part III), we can simply combine our answers to Parts I and II to obtain

$$\begin{aligned} v_o(t) &= \frac{R_3}{R_2}(v_{i,2}-v_{i,1}) = \frac{R_3}{R_2}(v_D - v_C) \\ &= \frac{R_3}{R_2}\left(v_2(t) + \frac{R_1}{R_{var}}(v_2(t)-v_1(t)) - v_1(t) + \frac{R_1}{R_{var}}(v_2(t)-v_1(t))\right) \\ &= \frac{R_3}{R_2}\left(1+\frac{2R_1}{R_{var}}\right)(v_2(t)-v_1(t)) \end{aligned}$$

**(1 point)**

**Part V:** [2 points] Consider the case when $R_{var}$ is removed and substituted by an open circuit. How would you describe stage 1 then? What would the expression for the output voltage $v_o$ in this case?

**Solution:** When $R_{var}$ is removed and substituted by an open circuit, the circuit looks like




In this case, stage 1 just looks like two (independent) unity-gain voltage followers. **(1 point)**
The output voltage can be readily obtained from Part IV by substituting $R_{var} = \infty$,

$$v_o(t) = \frac{R_3}{R_2}(v_2(t) - v_1(t)) \qquad \textbf{(1 point)}$$

**Part VI:** [2 bonus points] The circuit is called *instrumentation amplifier*. To compute the difference between the voltages produced by two different instruments, it would seem that simply using stage 2 would do. Do you agree with this? What do you think the role of stage 1 is? Finally, can you explain what the role of $R_{var}$ is in this design?

**Solution:** Using just stage 2 with two different instruments connected directly to a differential amplifier is not a good idea, because in general there would be loading, and hence the output voltages of the instruments (which are the things we want to compute the difference of in the first place) would be changed when connected. Preventing this is precisely the role of stage 1. The top and bottom op-amps have infinite input impedance, and hence the circuit in Figure 5 will not load the instruments.

**(1 extra point)**

This whole effect could be achieved without the resistor $R_{var}$ (as we discussed in Part V). However, the inclusion of this resistor is convenient to further amplify the difference between the two instrument voltages, as can be seen in the expression for the output voltage obtained in Part IV. **(1 extra point)**