MAE 140 - fall 2014

HW7 solution

10.5 10.7 10.9 10.15 10.18 10.22 10.27 10.32 10.58 10.68

$\boxed{10.5}$

a) $Z_{EQ} = (sL + \frac{1}{sC}) \,||\, R = \left(\frac{1+s^2LC}{sC}\right) \,||\, R$

$$= \frac{\frac{(1+s^2LC)R}{sC}}{\frac{1+s^2LC}{sC} + R} = \boxed{\frac{R + s^2LCR}{1 + sRC + s^2LC} = Z_{EQ}}$$

$$Z_{EQ} = \frac{s^2R + \frac{R}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$\rightarrow$ zeros: $\boxed{s_{1/2} = \pm j\sqrt{\frac{1}{LC}}}$

poles: $s_{1/2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

$$\boxed{s_{1/2} = -\frac{RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}}$$

b) $R = 2k\Omega$ and $C = 0.1\mu F$

want zeros at $\pm j\,5000$ rad/s

$$(5000) = \sqrt{\frac{1}{LC}}$$

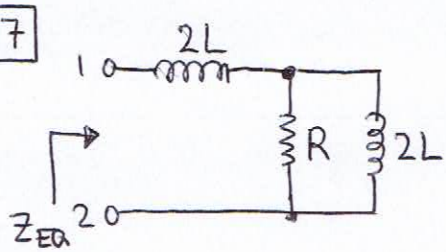
$$L = \frac{1}{C \cdot 5000^2} = \frac{1}{0.1 \cdot 10^{-6} \cdot 25 \cdot 10^6} = \boxed{0.4\ H = L}$$

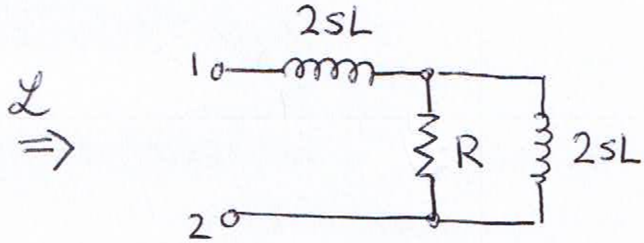
c) $s_{1/2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

$$= -\frac{2 \cdot 10^3}{2 \cdot 0.4} \pm \sqrt{\frac{\cancel{4} \cdot 10^6}{\cancel{4} \cdot 0.16} - 5000^2}$$

$$\boxed{s_{1/2} = -2.5 \cdot 10^3 \pm j\,4330.1}$$

$\boxed{10.7}$





a) $Z_{EQ} = 2sL + R \| (2sL) = 2sL + \dfrac{2sLR}{2sL+R}$

$Z_{EQ} = \dfrac{4sLR + 4s^2L^2}{2sL+R} = \boxed{\dfrac{4sL(R+sL)}{2sL+R} = Z_{EQ}}$

zeros: $\boxed{\begin{array}{l} s_1 = 0 \\ s_2 = -\dfrac{R}{L} \end{array}}$ poles: $\boxed{s = -\dfrac{R}{2L}}$

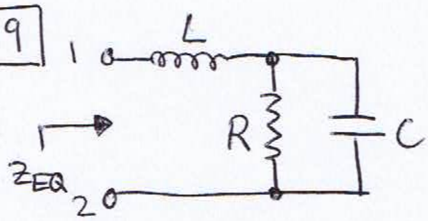
b) Want pole at -2000 rad/s

$2000 = \dfrac{R}{2L} \rightarrow L = \dfrac{R}{4000}$

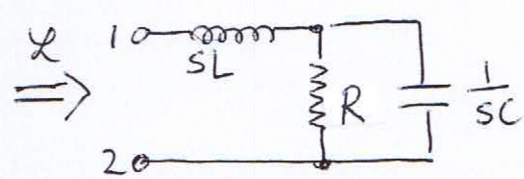
$\boxed{L(R = 1k\Omega) = 0.25\,H = 250\,mH}$

$\boxed{\text{zeros at} \quad s_1 = 0 \quad s_2 = -4000}$

$\boxed{10.9}$



$R = 1k\Omega$
$L = 2H$
$C = 0.5\mu F$



a) $Z_{EQ} = sL + R \| \dfrac{1}{sC} = sL + \dfrac{\frac{R}{sC}}{\frac{1}{sC}+R} = sL + \dfrac{R}{1+sCR} = \dfrac{R+sL+s^2LCR}{1+sCR}$

zeros: $s_{1/2} = -\dfrac{1}{2CR} \pm \sqrt{\dfrac{1}{4C^2R^2} - \dfrac{1}{LC}}$

$s_{1/2} = -1000 \pm \sqrt{10^6 - 10^6} \Rightarrow \boxed{s_1 = s_2 = -1000}$

poles: $s = -\dfrac{1}{CR} = \boxed{-2000 \text{ rad/s} = s}$

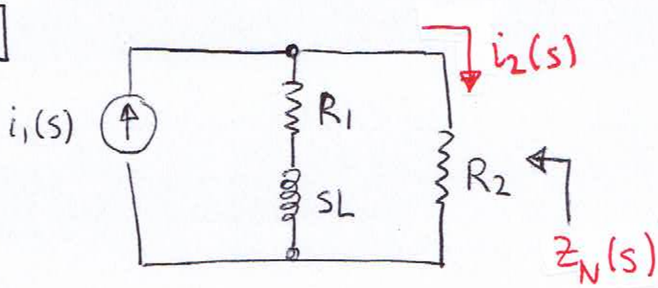
b) $L = 5H$

zeros $s_{1/2} = -1000 \pm \sqrt{10^6 - 0.4\,10^6} = -1000 \pm 774.6$

$\boxed{s_1 = -1774.6 \qquad s_2 = -225.4 \text{ rad/s}}$

poles $\boxed{s = -2000 \text{ rad/s}}$ (unchanged)

10.15

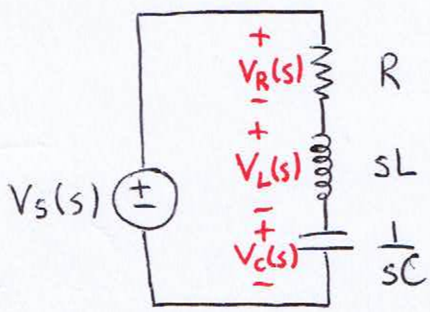


a) $$\frac{i_2(s)}{i_1(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_2} + \frac{1}{R_1 + sL}} = \frac{\frac{1}{R_2}}{\frac{R_1 + sL + R_2}{R_2(R_1 + sL)}} = \frac{sL + R_1}{R_1 + R_2 + sL}$$

$$\boxed{i_2(s) = \frac{sL + R_1}{sL + R_1 + R_2}\, i_1(s)}$$

b) $$Z_N(s) = R_2 \parallel (R_1 + sL) = \boxed{\frac{R_2(R_1 + sL)}{sL + R_1 + R_2} = Z_N(s)}$$

10.18



with input $V_s(t) = u(t)$
$\rightarrow \boxed{V_s(s) = \frac{1}{s}}$

a) Using voltage division we can write

$$V_R(s) = \frac{R}{R + sL + \frac{1}{sC}}\, V_s(s) = \frac{\frac{R}{L}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$V_L(s) = \frac{s^2}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \cdot \frac{1}{s} = \frac{s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$V_C(s) = \frac{\frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}} \cdot \frac{1}{s} = \frac{\frac{1}{LC}}{s\left(s^2 + s\frac{R}{L} + \frac{1}{LC}\right)}$$

b)

	zeros	poles
$V_R(s)$	$s_1 = s_2 = \infty$	$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{L^2C^2}}$
$V_L(s)$	$s_1 = 0 \quad s_2 = \infty$	$s_{1/2} = -\frac{R}{2R} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{L^2C^2}}$
$V_C(s)$	$s_1 = s_2 = s_3 = \infty$	$s_{1/2} = -\frac{R}{2R} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{L^2C^2}}$ $s_3 = 0$

c) Initial and final value theorem

$$\lim_{t \to 0} v(t) = \lim_{s \to \infty} sV(s)$$

$$\lim_{t \to \infty} v(t) = \lim_{s \to 0} sV(s)$$

R)

$$\lim_{t \to 0} v_R(t) = \lim_{s \to \infty} sV_R(s) = \boxed{0\ V}$$ initial value

$$\lim_{t \to \infty} v_R(t) = \lim_{s \to 0} sV_R(s) = \boxed{0\ V}$$ final value

L)

$$\lim_{t \to 0} v_L(t) = \boxed{1\ V}$$ initial value

$$\lim_{t \to \infty} v_L(t) = \boxed{0 V}$$ final value

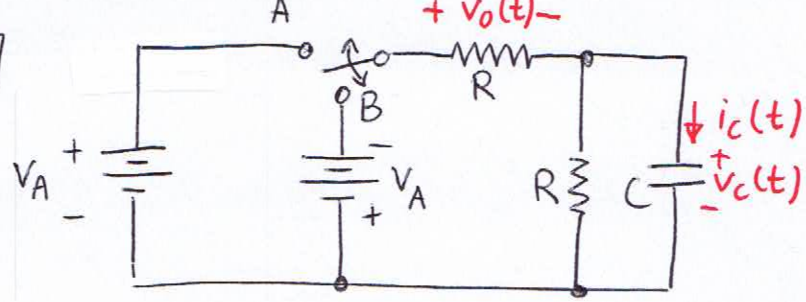
C)

$$\lim_{t \to 0} v_C(t) = \boxed{0\ V}$$ initial value

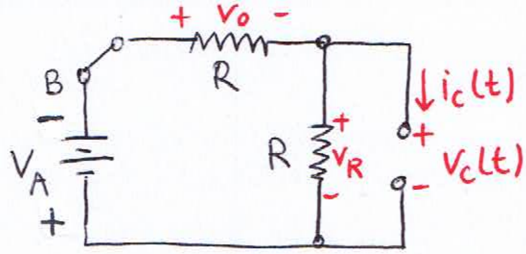
$$\lim_{t \to \infty} v_C(t) = \boxed{1\ V}$$ final value

With input voltage $v_s(t) = u(t)$ the entire voltage drops across L at $t=0$.
A $t = \infty$, the entire voltage drop is observed across the capacitor.
So depending on where you connect the terminals, you can use an RLC circuit to change the output voltage from 0 to 1, 1 to 0, or keep it at 0V.

$\boxed{10.22}$



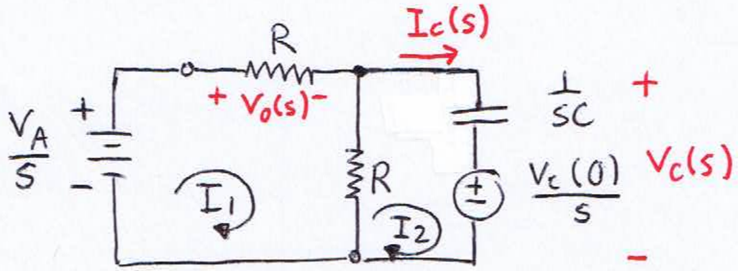
Initial condition (switch in position B)



For $t<0$ the circuit is in DC-steady-state condition and the capacitor acts like an open circuit. Consequently,

$$i_c(0) = 0$$
$$v_c(0) = v_R(t) = \frac{R}{R+R}(-V_A) = -\frac{1}{2}V_A$$
$$v_o(0) = -\frac{1}{2}V_A$$

Now, transform the circuit into the s-domain for $t>0$ (switch in position A)



Using mesh-current analysis we can write

$$\underbrace{\begin{bmatrix} R+R & -R \\ -R & R+\frac{1}{sC} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \frac{V_A}{s} \\ -\frac{V_c(0)}{s} \end{bmatrix}}_{b} = \begin{bmatrix} \frac{V_A}{s} \\ \frac{V_A}{2s} \end{bmatrix}$$

$$A^{-1} = \frac{sC}{R(2+sRC)} \begin{bmatrix} \frac{1+sRC}{sC} & R \\ R & 2R \end{bmatrix}$$ (using MATLAB symbolic expressions)

$$x = A^{-1}b = \frac{sC}{R(2+sRC)} \begin{bmatrix} \frac{V_A}{s}\left(\frac{1+sRC}{sC} + \frac{R}{2}\right) \\ \frac{V_A}{s}\left(R + R\right) \end{bmatrix}$$

$$\Rightarrow I_1(s) = \frac{\cancel{sC}}{R(2+sRC)} \cdot \frac{V_A}{s} \cdot \frac{1+1.5sRC}{\cancel{sC}} = \frac{1+1.5sRC}{sR(2+sRC)} V_A$$

$$\Rightarrow \quad I_2(s) = \frac{\cancel{s}C}{\cancel{R}(2+sRC)} \cdot \frac{V_A}{\cancel{s}} \cdot 2\cancel{R} = \frac{2C}{2+sRC} V_A = \frac{\frac{2}{R}}{s+\frac{2}{RC}} V_A$$

$$I_2(s) = \frac{2}{R}\left(\frac{1}{s+\frac{2}{RC}}\right) V_A$$

Now we can solve for $V_o(s)$, $v_o(t)$, $V_c(s)$ and $v_c(t)$

$$\rightarrow \quad V_o(s) = R I_1(s) = \boxed{\frac{1+1.5sRC}{s(2+sRC)} V_A}$$

$$\rightarrow \quad v_o(t) = \mathcal{L}^{-1}\{V_o(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{k_1}{s} + \frac{k_2}{s+\frac{2}{RC}}\right) V_A\right\}$$

$$k_1 = sV_o(s)\Big|_{s=0} = \frac{1}{2}$$

$$k_2 = \left(s+\frac{2}{RC}\right)V_o(s)\Big|_{s=-\frac{2}{RC}} = 1$$

$$v_o(t) = \mathcal{L}^{-1}\left\{\left(\frac{0.5}{s} + \frac{1}{s+\frac{2}{RC}}\right) V_A\right\} = \boxed{\left(\frac{1}{2} + e^{-\frac{2t}{RC}}\right) V_A\, u(t)}$$

$$\rightarrow \quad V_c(s) = \frac{1}{sC} I_2(s) + \frac{V_c(0)}{s} = \frac{2}{RC}\left(\frac{1}{s(s+\frac{2}{RC})}\right) V_A - \frac{1}{2s} V_A$$

$$= \frac{2}{RC}\left[\frac{k_1}{s} + \frac{k_2}{s+\frac{2}{RC}}\right] V_A - \frac{1}{2s} V_A$$

$$k_1 = \frac{1}{s+\frac{2}{RC}}\Bigg|_{s=0} = \frac{RC}{2}$$

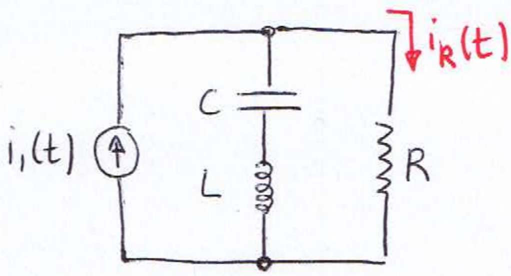
$$k_2 = \frac{1}{s}\Bigg|_{s=-\frac{2}{RC}} = -\frac{RC}{2}$$

$$V_c(s) = \left[\frac{1}{s} - \frac{1}{s+\frac{2}{RC}}\right] V_A - \frac{1}{2s} V_A = \boxed{\left[\frac{1}{2s} - \frac{1}{s+\frac{2}{RC}}\right] V_A}$$

$$\rightarrow \quad v_c(t) = \mathcal{L}^{-1}\{V_c(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{2s} - \frac{1}{s+\frac{2}{RC}}\right) V_A\right\}$$

$$\boxed{v_c(t) = \frac{V_A}{2}\left(1 - 2e^{-\frac{2t}{RC}}\right) u(t)}$$

$\boxed{10.27}$



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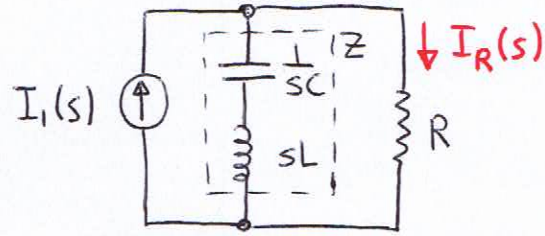


$$Z = \frac{1}{sC} + sL = \frac{1+s^2LC}{sC}$$

Using current division we can write

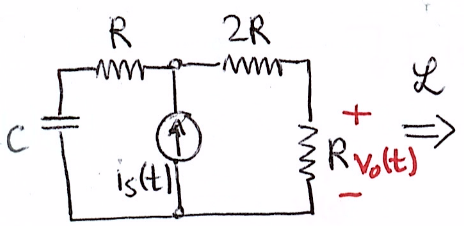
$$\frac{I_R(s)}{I_1(s)} = \frac{\frac{1}{R}}{\frac{1}{R}+\frac{1}{Z}} = \frac{Z}{R+Z} = \frac{\frac{1+s^2LC}{sC}}{\frac{1+s^2LC}{sC}+R} = \boxed{\frac{1+s^2LC}{1+sRC+s^2LC}}$$

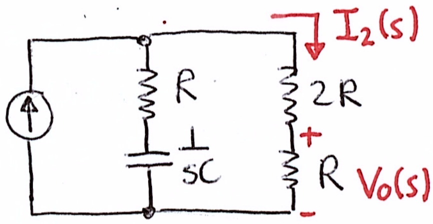
zeros: $\boxed{s = \pm j\sqrt{\frac{1}{LC}}}$

poles: $s^2 + s\frac{R}{L} + \frac{1}{LC} = 0$

$$\boxed{s_{1/2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}$$

10.32 Compute $v_o(t)$ using current division (no initial conditions)





$R = 5\,k\Omega$
$C = 0.2\,\mu F$

$i_s(t) = 5e^{-500t}u(t)\,mA$ $\qquad I(s) = \dfrac{0.005}{s+500}$

Using element voltage-current relation we know

$$V_o(s) = R I_2(s) \qquad (1)$$

Calculate $I_2(s)$ using current division

$$\frac{I_2(s)}{I_s(s)} = \frac{\frac{1}{3R}}{\frac{1}{3R} + \frac{1}{R+\frac{1}{sC}}} = \frac{R+\frac{1}{sC}}{3R + R + \frac{1}{sC}} = \frac{R+\frac{1}{sC}}{4R+\frac{1}{sC}} = \frac{s+\frac{1}{RC}}{s+\frac{1}{4RC}} \cdot \frac{1}{4}$$

Substituting the expression for $I_2(s)$ in (1) yields

$$V_o(s) = \frac{R}{4} \cdot \frac{s+\frac{1}{RC}}{s+\frac{1}{4RC}} \cdot \frac{0.005}{s+500} \qquad \text{Note: } \frac{1}{RC} = \frac{1}{5\cdot 10^3 \cdot 0.2 \cdot 10^{-6}} = 10^3$$

$$V_o(s) = \frac{25}{4} \cdot \frac{s+1000}{(s+250)(s+500)} = \frac{k_1}{s+250} + \frac{k_2}{s+500}$$

$$k_1 = \lim_{s \to -250} \frac{25}{4} \cdot \frac{s+1000}{s+500} = \frac{25}{4} \cdot \frac{750}{250} = \frac{75}{4} = 18.75$$

$$k_2 = \lim_{s \to -500} \frac{25}{4} \cdot \frac{s+1000}{s+250} = \frac{25}{4} \cdot \frac{500}{-250} = -\frac{50}{4} = -12.5$$

$$V_o(s) = \frac{18.75}{s+250} - \frac{12.5}{s+500}$$

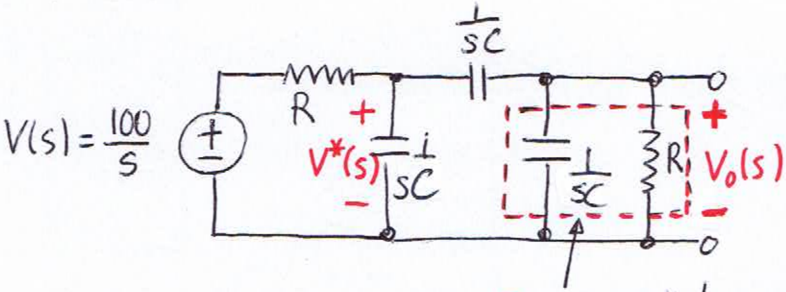
⟹ forced pole at $s = -500$ rad/s *
natural pole at $s = -250$ rad/s **

$$\boxed{v_o(t) = \left\{ 18.75e^{-250t} - 12.5e^{-500t} \right\} u(t)}$$

natural response $\qquad$ forced response

* forced by the input $I(s)$
** comes from the transfer function $T(S) = \dfrac{V_o(s)}{I(s)}$

$\boxed{10.58}$ Solve for $V_o(s)$ and $v_o(t)$ (no energy in the circuit)



$R = 1\,k\Omega$, $C = 0.2\,\mu F$ $\Big\}$ $RC = 0.2 \cdot 10^{-3}$, $\frac{1}{RC} = 5000$

$$Z_{EQ} = R \,\|\, \frac{1}{sC} = \frac{\frac{R}{sC}}{R + \frac{1}{sC}} = \frac{R}{1+sRC}$$

- $$V_0(s) = \frac{Z_{EQ}}{Z_{EQ} + \frac{1}{sC}} V^*(s) = \frac{\frac{R}{1+sRC}}{\frac{R}{1+sRC} + \frac{1}{sC}} V^*(s)$$

$$V_o(s) = \frac{sRC}{1+2sRC} V^*(s)$$

Note: You can also use source transformation (see next page)

- $$V^*(s) = \frac{\frac{1}{sC} \,\|\, \left(\frac{1}{sC} + Z_{EQ}\right)}{\frac{1}{sC} \,\|\, \left(\frac{1}{sC} + Z_{EQ}\right) + R} V(s)$$

$$\frac{1}{sC} \,\|\, \left(\frac{1}{sC} + \frac{R}{1+sRC}\right) = \frac{1}{sC} \,\|\, \frac{1+2sRC}{sC(1+sRC)} = \frac{\frac{1}{\cancel{sC}} \cdot \frac{1+2sRC}{sC(1+sRC)}}{\frac{1}{\cancel{sC}} + \frac{1+2sRC}{\cancel{sC}(1+sRC)}}$$

$$= \frac{\frac{1+2sRC}{sC(1+sRC)}}{1 + \frac{1+2sRC}{1+sRC}} = \frac{1+2sRC}{sC(1+sRC+1+2sRC)} = \frac{1+2sRC}{sC(2+3sRC)}$$

$$V^*(s) = \frac{\frac{1+2sRC}{sC(2+3sRC)}}{\frac{1+2sRC}{sC(2+3sRC)} + R} V(s) = \frac{1+2sRC}{sCR(2+3sRC) + 2sRC + 1} V(s)$$

$$\Rightarrow \quad V_o(s) = \frac{\cancel{sRC}}{\cancel{1+2sRC}} \cdot \frac{\cancel{1+2sRC}}{\cancel{sCR}\left[2 + 3sRC + 2 + \frac{1}{sCR}\right]} \frac{100}{s}$$

$$V_o(s) = \frac{1}{3sRC + 4 + \frac{1}{sRC}} \cdot \frac{100}{s} = \frac{\cancel{s}}{0.6 \cdot 10^{-3} s^2 + 4s + 5000} \frac{100}{\cancel{s}}$$

$$V_o(s) = \frac{10^5}{0.6s^2 + 4000s + 5 \cdot 10^6} = \frac{10^5}{\frac{3}{5}\left(s^2 + s\frac{20}{3}10^3 + \frac{25}{3}10^6\right)}$$

$$V_o(s) = \frac{5}{3}10^5\left[\frac{1}{s^2+\frac{20}{3}10^3 s+\frac{25}{3}10^6}\right]$$

$$s_{1/2} = -\frac{10}{3}10^3 \pm \sqrt{\frac{100}{9}-\frac{75}{9}}\;10^3$$

$$\Rightarrow s_1 = -\frac{15}{3}10^3 = -5000$$

$$\Rightarrow s_2 = -\frac{5}{3}10^3 = -1667$$

$$V_o(s) = \frac{5}{3}10^5\left[\frac{k_1}{(s+5000)}+\frac{k_2}{s+1667}\right]$$

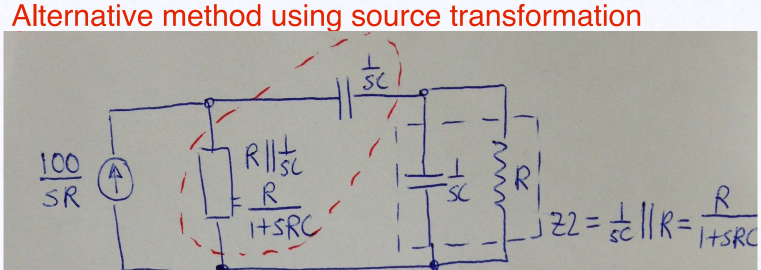
$$k_1 = \lim_{s\to -5000}\frac{1}{s+1667} = -30\cdot 10^{-5}$$

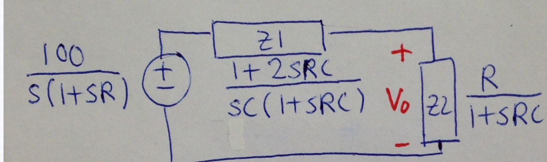
$$k_2 = \lim_{s\to -1667}\frac{1}{s+5000} = +30\cdot 10^{-5}$$

$$V_o(s) = 50\left[\frac{1}{s+1667}-\frac{1}{s+5000}\right]$$

$$\boxed{V_o(t) = 50\left[e^{-1667t}-e^{-5000t}\right]u(t)}$$

Alternative method using source transformation



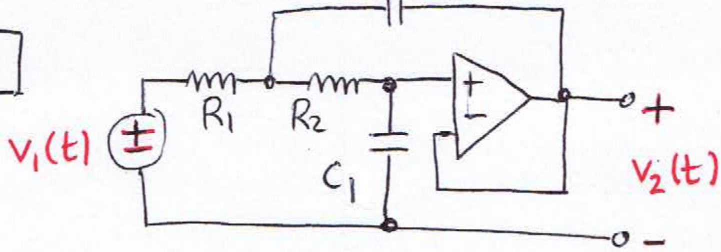


$$\Rightarrow V_o(s) = \frac{Z_2}{Z_1+Z_2}\cdot\frac{100}{s(1+sR)} = \frac{\frac{R}{\cancel{1+sRC}}}{\frac{R}{\cancel{1+sRC}}+\frac{1+2sRC}{sC(\cancel{1+sRC})}}\cdot\frac{100}{s(1+sRC)}$$

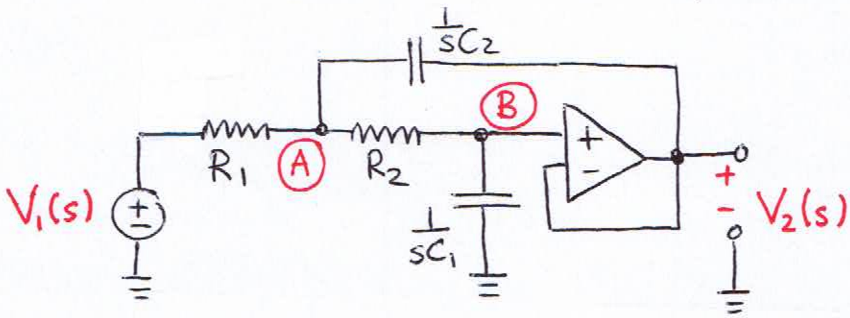
$$= \frac{\cancel{s}RC}{(sRC+1+2sRC)}\cdot\frac{100}{\cancel{s}(1+sRC)}$$

$$= \frac{100}{3RC}\;\frac{1}{\left(s+\frac{1}{3RC}\right)\left(s+\frac{1}{RC}\right)} = \frac{5}{3}10^5\frac{1}{(s+1667)(s+5000)}$$

10.68



Transforming into s-domain yields



Note that
$i^+ = i^- = 0$ and thus
$V^+ = V^- = V_2(s)$

Using KCL at node A yields

$$\frac{V_1(s) - V_A(s)}{R_1} = \frac{V_A(s) - V_2(s)}{\frac{1}{sC_2}} + \frac{V_A(s) - 0V}{R_2 + \frac{1}{sC_1}}$$

$$\frac{1}{R_1}V_1(s) + sC_2V_2(s) = \left[\frac{1}{R_1} + sC_2 + \frac{sC_1}{1 + sC_1R_2}\right] V_A \qquad (1)$$

Using KCL at node B yields

$$\frac{V_A(s) - V^+(s)}{R_2} = \frac{V^+(s) - 0V}{\frac{1}{sC_1}} \qquad V^+ = V^- = V_2(s)$$

$$V_A(s) = (1 + sC_1R_2)V_2(s) \qquad (2)$$

Substitute (2) in (1)

$$\frac{1}{R_1}V_1(s) + sC_2V_2(s) = \left[\frac{1 + sR_1C_2}{R_1} + \frac{sC_1}{1 + sR_2C_1}\right](1 + sR_2C_1)V_2(s)$$

$$\frac{1}{R_1}V_1(s) = \left[\frac{1}{R_1}(1 + sR_1C_2)(1 + sR_2C_1) + sC_1 - sC_2\right]V_2(s)$$

$$\frac{V_1(s)}{V_2(s)} = 1 + s^2R_1R_2C_1C_2 + s(\cancel{R_1C_2} + R_2C_1) + sR_1(C_1 \cancel{- C_2})$$

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{s^2R_1R_2C_1C_2 + sC_1(R_1 + R_2) + 1}$$

$$\boxed{\frac{V_2(s)}{V_1(s)} = \frac{(R_1R_2C_1C_2)^{-1}}{s^2 + s\left(\frac{1}{C_2R_2} + \frac{1}{C_2R_1}\right) + \frac{1}{R_1R_2C_1C_2}}}$$