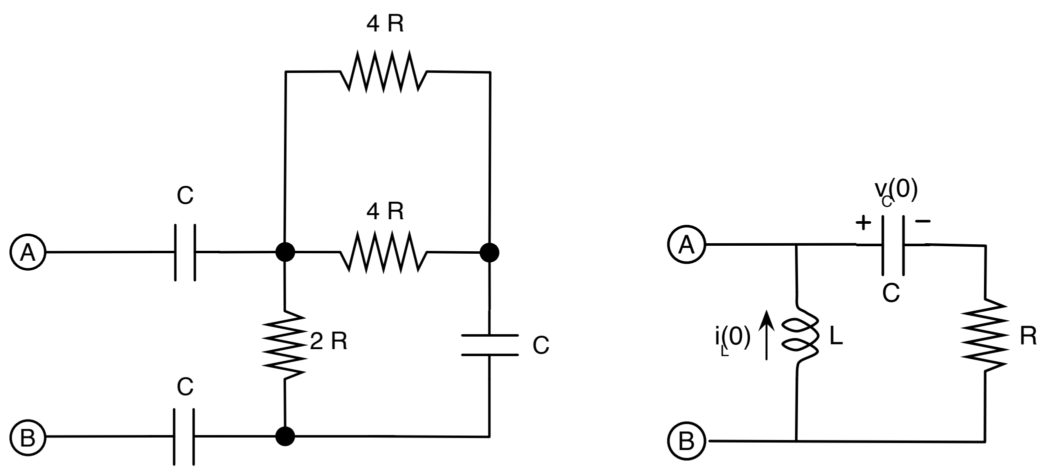
MAE140 – Linear Circuits – Winter 2009
Final, March 17

Instructions

1) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2) You have 170 minutes.
3) Write your name, student number and instructor.
4) On the questions for which we have given the answers, please provide detailed derivations.



(a) Part (a)

(b) Part (b)

Figure 1: Circuit for Question 1

Questions

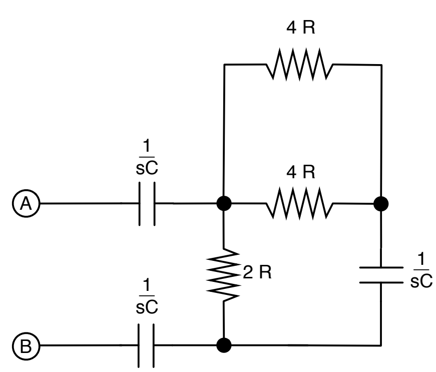
1. **Equivalent Circuits**

   (a) [5 points] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1(a) as seen from terminals A and B. The answer should be given as a ratio of two polynomials.

Solution:

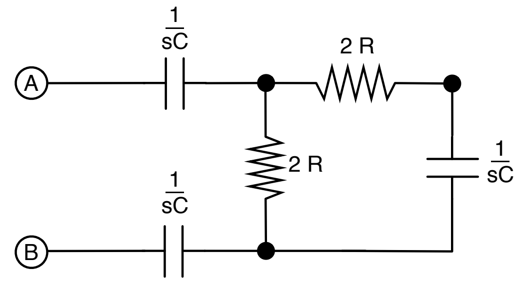
Since all initial conditions are zero, it is easy to transform the circuit to the s-domain.

[1 point]



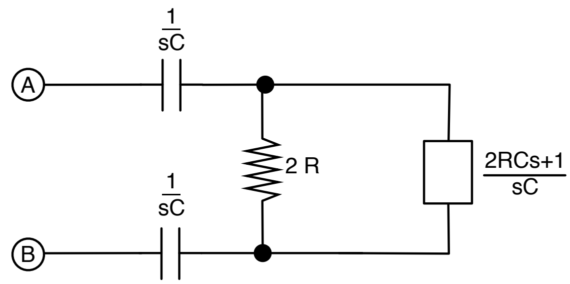
We now need to go through the different combinations of impedances carefully. We begin by combining the two resistances in parallel to get

[1 point]



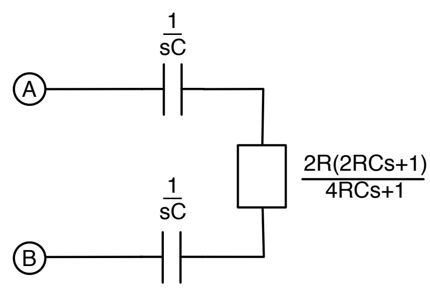
Now we combine the two impedances in series

[1 point]



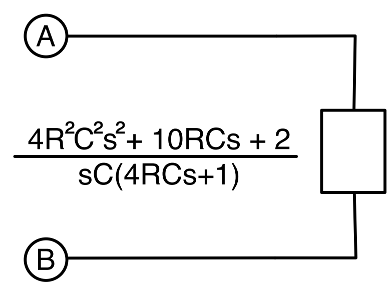
Now we combine the two impedances in parallel

[1 point]



Finally, we combine all the impedances in series

[1 point]



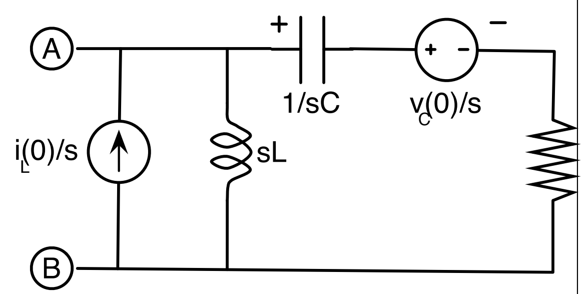
(b) [5 points] Assuming that the initial conditions of the inductor and capacitor are as indicated in the diagram, redraw the circuit shown in Figure 1(b) in the s-domain. Then use source transformations to find the s-domain Norton equivalent of this circuit as seen from terminals A and B. The transfer functions should be given as a ratio of two polynomials.

(*Hint: Use an equivalent model for the inductor in which the initial condition appears as a current source, and an equivalent model for the capacitor in which the initial condition appears as a voltage source.*)

Solution:

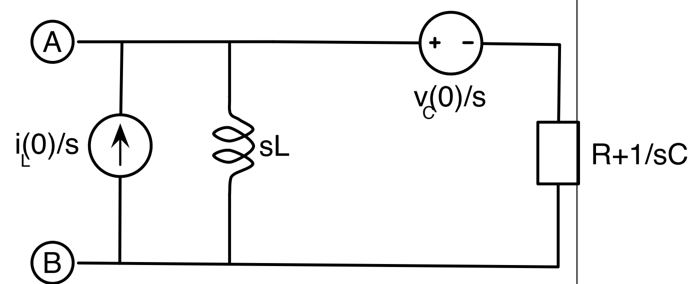
We begin by transforming the circuit into the s-domain. We use a current source for the initial condition of the inductor and a voltage source for the initial condition of the capacitor – this choice makes our life easier later.

[2 points]



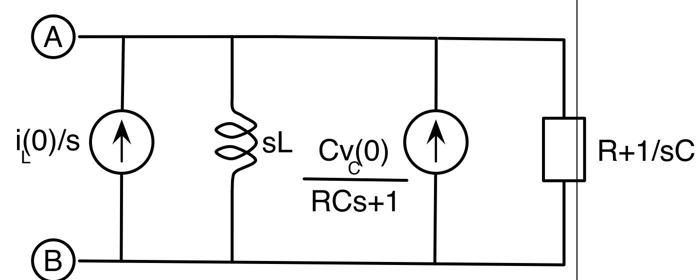
Now, we combine the capacitor and the resistor in series

[1 point]



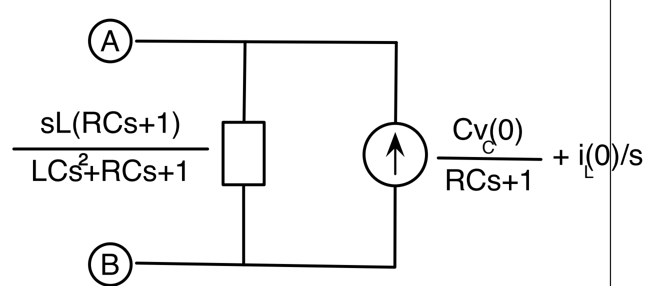
Next, we do a source transformation to turn the voltage source into a current source

[1 point]



Finally, we combine everything to get the Norton equivalent

[1 point]



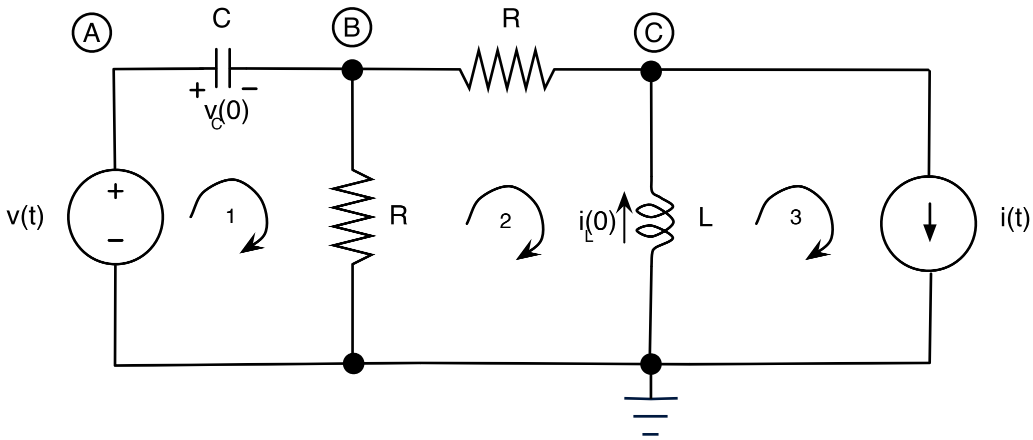


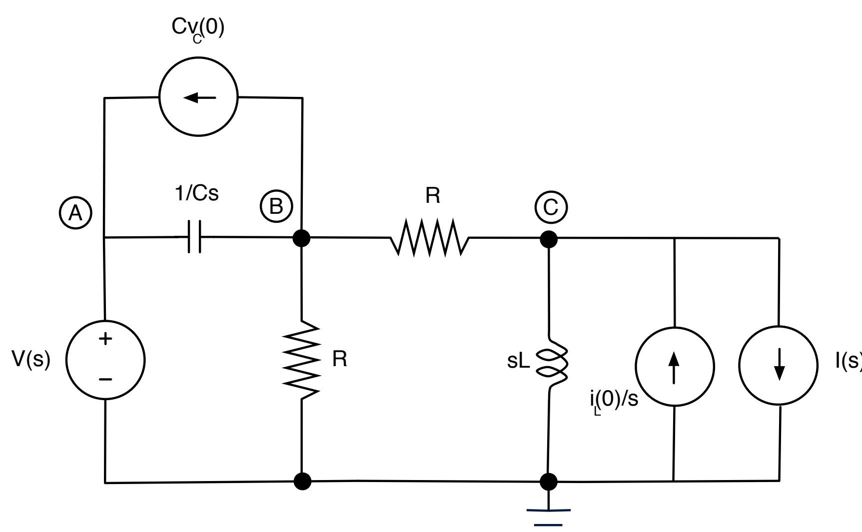
Figure 2: Nodal and Mesh Analysis Circuit

2. **Nodal and Mesh Analysis**

   In this question you do not need to solve any equations, but clearly indicate the equations and the unknown voltages or currents that you should solve for in order to determine the entire circuit.

   (a) [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Do not assume zero initial conditions! Transform initial conditions on the capacitor and on the inductor into current sources.

Solution:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity and current orientation.

[1 point for correct circuit]

[1 point for correct initial conditions]

The voltage source poses a problem. We can easily take care of it by realizing that

$$V_A = V(s).$$

[1 point]

Then, we only need to write node equations for nodes B and C. For node B, we have

$$Cs(V_B - V_A) + \frac{1}{R}(V_B) + \frac{1}{R}(V_B - V_C) + Cv_C(0) = 0$$

[1 point]

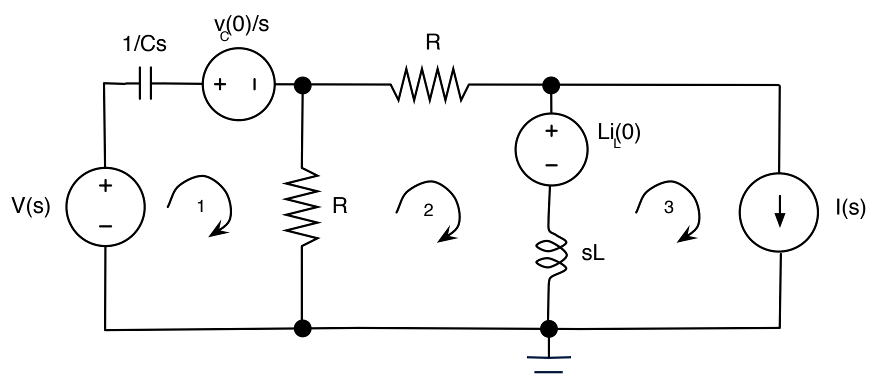
For node C, we have

$$\frac{1}{R}(V_C - V_B) + \frac{1}{sL}(V_C) - \frac{i_L(0)}{s} + I(s) = 0$$

[1 point]

(b) [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the currents shown in the figure. Do not assume zero initial conditions! Transform initial conditions on the capacitor and on the inductor into voltage sources.

Solution:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity and current orientation.

[1 point for correct circuit; 1 point for correct initial conditions]

The current source poses a problem. However, we can easily take care of it by realizing that it only belongs to the mesh 3, and therefore

$$i_3 = I(s)$$

[1 point]

Therefore, we only need to write mesh equations for meshes 1 and 2. For mesh 1, we have

$$R(i_1 - i_2) + \frac{1}{Cs}i_1 = V(s) - \frac{v_C(0)}{s}$$

[1 point]

For mesh 2, we have

$$R(i_2 - i_1) + Ri_2 + sL(i_2 - i_3) = -Li_L(0)$$

[1 point]

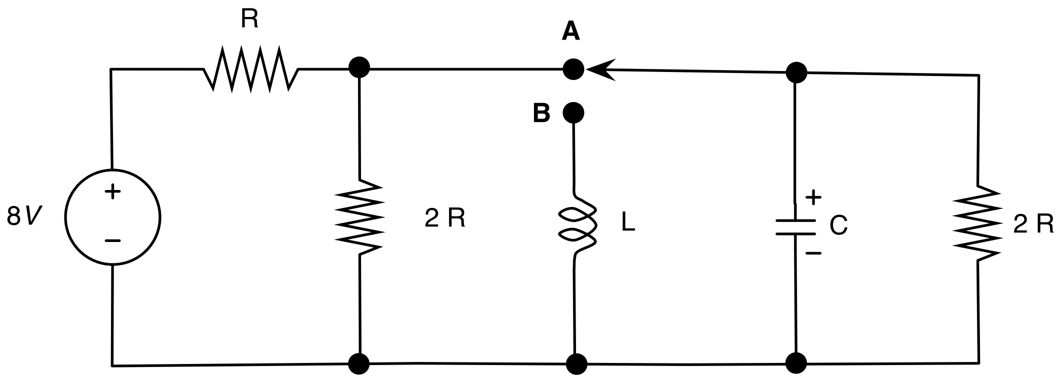


Figure 3: RCL circuit for Laplace Analysis

3. **Laplace Domain Circuit Analysis**

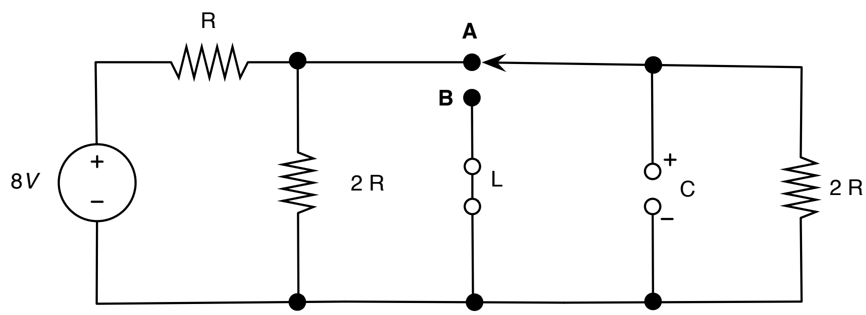
(a) [3 points] Consider the circuit depicted in Figure 3. The voltage source is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial capacitor voltage and inductor currents are given by

$$v_C(0^-) = 4\,\mathrm{V}, \quad i_L(0^-) = 0\,\mathrm{A}.$$

[Show your work]

Solution:

To find the initial conditions, we substitute the inductor by a short circuit and the capacitor by an open circuit.



[1 point]

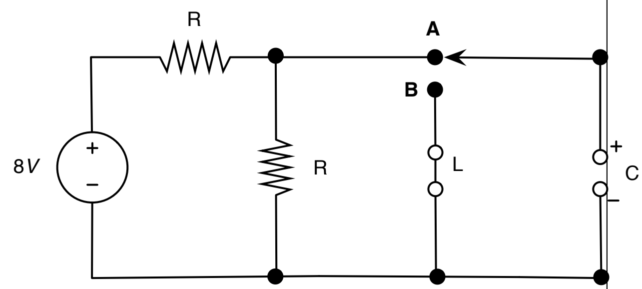
Then it is clear from the circuit that

$$i_L(0^-) = 0$$

[1 point]

Combining the two resistances in parallel as in the figure, we find by voltage division that

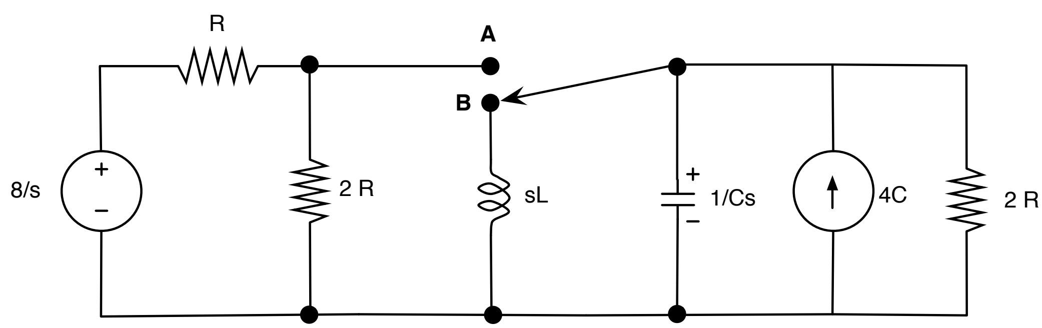
$$v_C(0^-) = \frac{R}{R+R}8 = 4V.$$



[1 point]

(b) [2 points] Use these initial conditions to transform the circuit into the s-domain for $t \geq 0$. Use equivalent models for the capacitor and the inductor in which the initial conditions appear as current sources.
[Show your work]

Solution:
Since there is no initial current through the inductor, we do not need to add an independent current source for it. **[1 point]**
We add one current source in parallel for the capacitor to take care of its initial condition.



[1 point]

Off course you also get all points if you only transform the part of the circuit to the right of the switch for $t \geq 0$.

(c) [5 points] Use s-domain circuit analysis and inverse Laplace transforms to show that the capacitor voltage is

$$v_C(t) = (8e^{-2t} - 4e^{-t})u(t)\,\text{V}.$$

when $C = \frac{1}{6}$ F, $L = 3$ H, and $R = 1\ \Omega$.

Solution:

We can find the current through the impedance $1/Cs$ using current division,

$$I_{\frac{1}{Cs}}(s) = \frac{Cs}{1/sL + Cs + 1/2R}4C = \frac{8RLC^2s^2}{2RCLs^2 + sL + 2R}$$

[1 point]

Therefore, the voltage across the impedance $1/Cs$, which is also transform of the voltage across the capacitor, is

$$V_C(s) = \frac{1}{Cs} I_{\frac{1}{Cs}}(s) = \frac{8RLCs}{2RCLs^2 + sL + 2R} = \frac{4s}{s^2 + 3s + 2}$$

[1 point]

To find the capacitor voltage, we need to compute the inverse Laplace transform. Using partial fractions, we set

$$V_C(s) = \frac{4s}{s^2 + 3s + 2} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

You can use your preferred method to find k_1 and k_2. We use here the cover-up method

$$k_2 = \lim_{s \to -1} (s+1) V_C(s) = \lim_{s \to -1} \frac{4s}{s+2} = -4$$
$$k_1 = \lim_{s \to -2} (s+2) V_C(s) = \lim_{s \to -2} \frac{4s}{s+1} = 8$$

Therefore, we have

$$V_C(s) = -\frac{4}{s+1} + \frac{8}{s+2}$$

[2 points]

The capacitor voltage is then

$$v_C(t) = (-4e^{-t} + 8e^{-2t}) u(t)$$

[1 point]

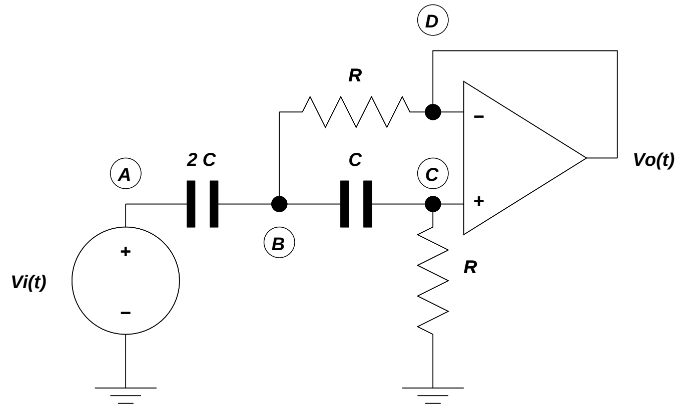


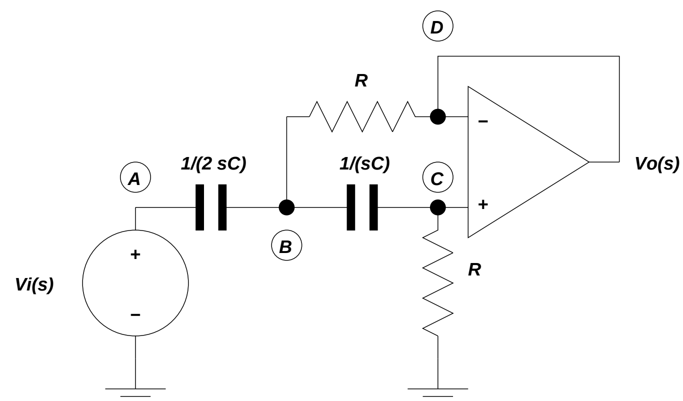
Figure 4: Active Filter Circuit

4. **Active Filter Analysis**

In this question assume zero initial conditions.

(a) [1 point] Transform the circuit in Figure 4 to the s-domain.

Solution:
Here is the solution:



(b) [3 points] Formulate node-voltage equations in the s-domain that would allow one to compute the transfer function from $V_i(s)$ to $V_0(s)$. You do not need to solve any equations, but clearly indicate the equations that need to be solved for and the unknowns. Tell how you can find $V_0(s)/V_i(s)$ from the unknowns. Use the labels for the nodes provided in Figure 4.

Solution:
Write node-voltage equations by inspection at nodes B and C:

$$\begin{bmatrix} -2Cs & Cs + 2Cs + \frac{1}{R} & -Cs & -\frac{1}{R} \\ 0 & -Cs & Cs + \frac{1}{R} & 0 \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_C(s) \\ V_D(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[1 point]

These equations should be solved along with

$$V_A(s) = V_i(s), \qquad V_D(s) = V_C(s).$$

on the unknowns $V_A(s)$, $V_B(s)$, $V_C(s)$ and $V_D(s)$. Any simplified version would also give you all points.

[1 point]

You would find $V_0(s)/V_i(s)$ by dividing $V_D(s) = V_C(s) = V_0(s)$ by $V_A(s) = V_i(s)$.

[1 point]

(c) [3 points (bonus)] Solve the node-voltage equations you formulated in part (b) to show that the transfer function from $V_i(s)$ to $V_0(s)$ is

$$T(s) = \frac{V_0(s)}{V_i(s)} = \frac{s^2}{s^2 + (3\lambda/2)s + \lambda^2/2}$$

where $\lambda = 1/(RC)$.

Solution:

Continuing the previous item, we obtain the following reduced set of equations after substituting for $V_A(s)$ and $V_C(s)$:

$$\begin{bmatrix} 3Cs + \frac{1}{R} & -Cs - \frac{1}{R} \\ -Cs & Cs + \frac{1}{R} \end{bmatrix} \begin{pmatrix} V_B(s) \\ V_D(s) \end{pmatrix} = \begin{pmatrix} 2CsV_i(s) \\ 0 \end{pmatrix}$$

Summing the two equations we obtain

$$(2Cs + \frac{1}{R})V_B(s) = 2CsV_i(s),$$

from where

$$V_B(s) = \frac{2Cs}{2Cs + \frac{1}{R}} V_i(s) = \frac{s}{s + \frac{1}{2RC}} V_i(s).$$

[1 point]

Substituting in the second equation

$$(Cs + \frac{1}{R})V_D(s) = CsV_B(s) = Cs \frac{s}{s + \frac{1}{2RC}} V_i(s)$$

from where

$$V_D(s) = \frac{Cs}{Cs + \frac{1}{R}} \frac{s}{s + \frac{1}{2RC}} V_i(s) = \frac{s}{\left(s + \frac{1}{RC}\right)} \frac{s}{\left(s + \frac{1}{2RC}\right)} V_i(s)$$

[1 point]

Hence, for $\lambda = 1/(RC)$.

$$T(s) = V_0(s)/V_i(s) = \frac{s^2}{(s+\lambda)(s+\lambda/2)} = \frac{s^2}{s^2 + (3\lambda/2)s + \lambda^2/2}$$

which is the expression in the statement.

[1 point]

(d) [3 points] What is the gain and phase at $\omega = 0$ and $\omega \to \infty$? What is the gain at $\omega = \lambda$?

Solution:
At $s \to j0$

$$T(j0) = \lim_{\epsilon \to 0} \frac{-\epsilon^2}{(j\epsilon + \lambda)(j\epsilon + \lambda/2)} = \lim_{\epsilon \to 0} -\epsilon^2 = 0\angle\pi$$

[1 point (1/2 point if the phase is right but the gain is wrong or vice versa)]

At $s \to j\infty$

$$T(j\infty) = \lim_{r \to 0} \frac{-1}{(j + r\lambda)(j + r\lambda/2)} = 1\angle 0$$

[1 point (1/2 point if the phase is right but gain is wrong or vice versa)]

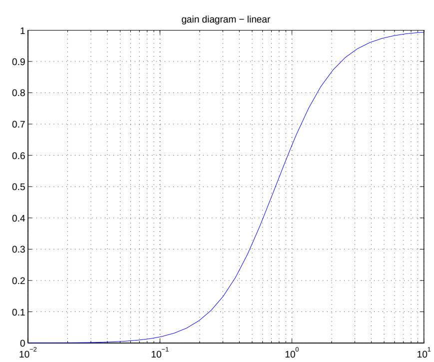
At $s = j\lambda$

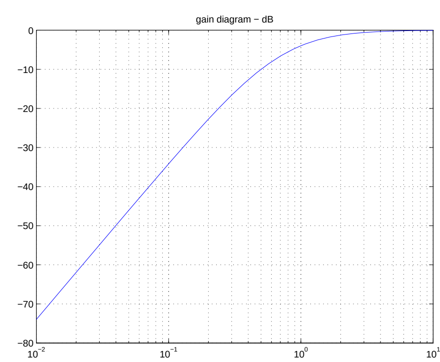
$$|T(j\lambda)| = \frac{\lambda^2}{|j\lambda + \lambda||j\lambda + \lambda/2|} = \frac{1}{\sqrt{1+1}\sqrt{1+1/4}} = \frac{1}{\sqrt{5/2}} \approx 0.63$$

[1 point]

(e) [3 points] Sketch the gain versus frequency diagram. What type of filter is this circuit? Is the cut-off frequency of the circuit equal, less or greater than λ rad/s?

Solution:
Here are the gain diagram in linear scale and dB:





[1 point]

This is a high-pass filter (see the gain diagram).

[1 point]

The cut-off frequency is greater than λ because $\lambda \approx 0.63 < 0.71$.

[1 point]

5. **Active Filter Design**

Consider the transfer function

$$T(s) = \frac{V_0(s)}{V_i(s)} = \frac{s^2}{s^2 + (3\lambda/2)s + \lambda^2/2}$$

of the filter studied in the previous question where the parameter λ is now to be specified by the user. In this question, always assume zero initial conditions.

(a) [3 points] Show that the transfer function $T(s)$ can be realized as a product of two first-order high-pass filters of the form

$$T_1(s) = \frac{\pm s}{s + \omega_1}, \qquad T_2(s) = \frac{\pm s}{s + \omega_2}$$

that is, $T(s) = T_1(s) \times T_2(s)$. What is the cut-off frequency and gain of $T_1(s)$ and $T_2(s)$ in terms of λ?

Solution:
First factor the denominator as

$$s^2 + (3\lambda/2)s + \lambda^2/2 = (s + \lambda)(s + \lambda/2)$$

[1 point]

Then

$$T(s) = T_1(s)T_2(s), \qquad T_1(s) = \frac{\pm s}{s + \lambda}, \qquad T_2(s) = \frac{\pm s}{s + \lambda/2}$$

[1 point]

The cut-off frequencies of each filter are

$$\omega_1 = \lambda, \qquad \omega_2 = \lambda/2$$

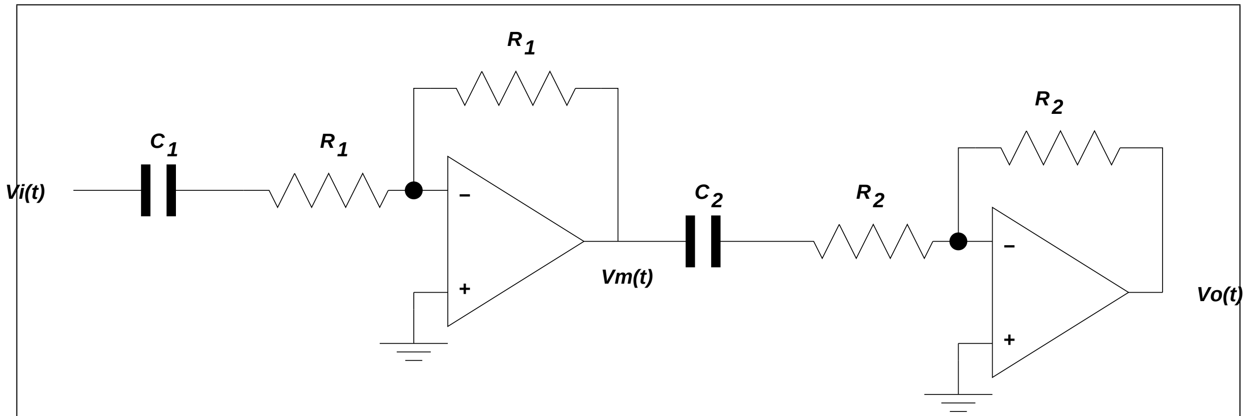
and the gains are calculated at $s = j\infty$

$$|T_1(j\infty)| = |T_2(j\infty)| = 1.$$

[1 point]

(b) [4 points] Design a circuit that implements $T(s)$ as the product of the two filters $T_1(s)$ and $T_2(s)$ using no more than 2 OpAmps.

Solution:
Here is one possible solution:



The transfer function from $V_i(s)$ to $V_m(s)$ is

$$T_1(s) = \frac{-R_1}{\frac{1}{sC_1} + R_1} = \frac{-s}{s + \frac{1}{R_1 C_1}}$$

Likewise, the transfer function from $V_m(s)$ to $V_o(s)$ is

$$T_2(s) = \frac{-R_2}{\frac{1}{sC_2} + R_2} = \frac{-s}{s + \frac{1}{R_2 C_2}}$$

[4 points]

(c) [3 points] Find values of the components in your design so that $\lambda = 1000$ Hz.

Solution:
For the circuit designed in the previous item we have

$$\omega_1 = \frac{1}{R_1 C_1} = \lambda, \qquad \omega_2 = \frac{1}{R_2 C_2} = \frac{\lambda}{2}$$

Converting λ to rad/s we have $\lambda = 2000\pi$ rad/s.

[1 point]

The set of components is then any combination of R_1, R_2, C_1 and C_2 satisfying

$$R_1 C_1 = \frac{1}{2000\pi}, \qquad R_2 C_2 = \frac{1}{1000\pi}$$

[1 point]

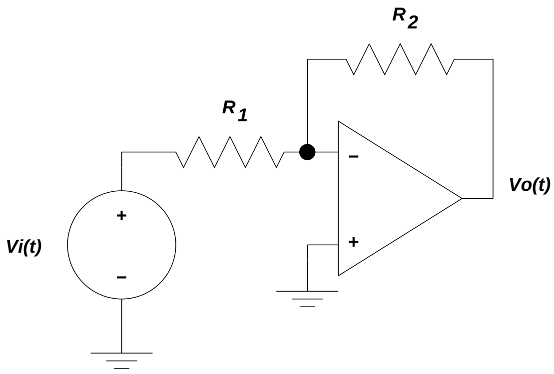
For instance, if

$$C_1 = C_2 = \frac{1}{\pi} \times 10^{-9} \approx 318 \times 10^{-12} = 318\text{pF}$$

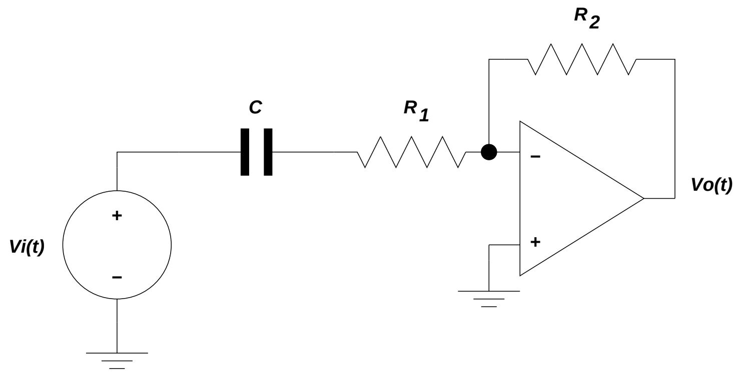
then

$$R_1 = \frac{1}{2000\pi C_1} = \frac{1}{2000 \times 10^{-9}} = 0.5 \times 10^6 = 500\,\mathrm{K}\,\Omega,$$
$$R_2 = \frac{1}{1000\pi C_2} = \frac{1}{1000 \times 10^{-9}} = 10^6 = 1\,\mathrm{M}\,\Omega,$$

[1 point]
[-1 point if you forget to convert λ from Hz to rad/s]



(a) Circuit I



(b) Circuit II

Figure 5: OpAmp Circuits for AC Amplifier

6. **AC Amplifier**

A group of former MAE 140 students needs to build an OpAmp circuit to amplify telephone signals. Because of reasons that were relevant some hundred years ago, your telephone jack has a DC 48 V voltage superimposed to the voice signal. To make things simple in this exam, think now of a "test" signal coming through your telephone line with a fixed frequency of $\omega = 1500$ rad/s. On your telephone jack you would measure a voltage

$$V_i(t) = 48 + 12\cos(\omega\, t) \text{ V}.$$

In order to solve this problem the students have selected two circuits, shown in Figure 5. The power supplies to the OpAmps (not shown in the figure) are $V_{CC} = +12$ V and $-V_{CC} = -12$ V. For impedance matching set $R_1 = 600\ \Omega$.

(a) [2 points] The students built Circuit I with $R_2 = R_1 = 600\ \Omega$ but found out to their surprise that they could not hear the $\omega = 1500$ rad/s tone after they plugged $V_0(t)$ into a speaker. Why that happened?
(*Hint: Recall the role played by V_{CC} and $-V_{CC}$ in an OpAmp!*)

Solution: The gain in this problem is $-R_2/R_1 = -1$, hence $V_0(t)$ would be

$$V_0(t) = -V_i(t) = -48 - 12\cos(\omega\, t) \text{ V}$$

if the OpAmp were operating in its linear range, that is $-12 \le V_0(t) \le 12$.

[1 point]

The problem here is that $-48 - 12\cos(\omega\, t) < -V_{CC} = -12$ so that the output of the OpAmp is actually $V_0(t) = -V_{CC} = -12$ V, hence no tone can be heard!

[1 point]

(b) [3 points] By replacing R_2 with a potentiometer, they eventually found a value of resistance R_2 below which they could hear the tone *without any distortion*. What is this critical value of R_2 and what is the corresponding amplitude of the tone fed to the speaker?

Solution: When R_2 is replaced by a potentiometer the gain become $-R_2/R_1 = -R_2/600$, hence

$$V_0(t) = -(R_2/600)V_i(t) = -(R_2/600)48 - (R_2/600)12\cos(\omega\, t) \text{ V}$$

if the OpAmp is operating in its linear range.

[1 point]

For no distortion we should have $-12 \le V_0(t) \le 12$ hence

$$|(R_2/600)48 + (R_2/600)12\cos(\omega\, t)| \le 12$$

for all t. This condition can be restated as

$$|(R_2/600)48 + (R_2/600)12\cos(\omega\, t)| \le |(R_2/600)60| \le 12$$

which implies that for any resistance below

$$R_2 \le 600 \times 12/60 = 120\ \Omega$$

you can hear the tone withou any distortion.

[1 point]

The maximum amplitude of undistorted signal is when the circuit has its maximum possible gain $R_2/600$, which happens at $R_2 = 120$. Hence

$$V_0(t) = -(120/600)V_i(t) = -(1/5)V_i(t) = -9.6 - 2.4\cos(\omega\, t)\text{V}$$

in which the amplitude of the tone is just 2.4 V.

[1 point]

(c) [3 points] One student suggested that all they needed to do to fix the problem was to add a capacitor to Circuit I, obtaining Circuit II. In order to explain why this new circuit solves the problem, compute the steady state response $V_0(t)$ as a function of R_1, R_2 and C. Why does this circuit fix the problem?
(*Hint: Because the circuit is linear you can compute the steady-state response to the input $V_i(t)$ as the sum of the steady-state response to the cosine tone plus the steady-state response to the constant 48 V.*)

Solution:
In order to compute the steady-state responses we compute

$$V_0(s) = -\frac{R_2}{R_1 + \dfrac{1}{Cs}}V_i(s) \quad \Rightarrow \quad T(s) = \frac{V_0(s)}{V_i(s)} = \frac{-s\dfrac{R_2}{R_1}}{s + \dfrac{1}{R_1 C}}$$

With $R_2 = R_1 = 600\ \Omega$

$$T(s) = \frac{-s}{s + \dfrac{1}{600C}}$$

Then the steady-state response to the constant input 48 V is

$$V_0^{SS}(t)^{48} = 48|T(0)|\cos(\angle T(0)) = 0$$

because $T(0)$ is zero!

[1 point]

Consequently, , the steady-state response is simply the response to the tone, that is

$$V_0^{SS}(t) = 12|T(j\omega)|\cos(\omega t + \angle T(j\omega)) = 12\frac{\omega}{\sqrt{\omega^2 + \frac{1}{(600C)^2}}}\cos(\omega t + \pi - \angle 600C\omega)$$

[1 point]

For any $\omega > 0$ we have

$$\frac{\omega}{\sqrt{\omega^2 + \frac{1}{(600C)^2}}} \leq 1$$

so that

$$|V_0^{SS}(t)| \leq 12$$

That is the OpAmp never saturates, always operating within the linear regime, and the circuit operates as expected, except for some blocking of low pass frequencies due to the high pass nature of $T(s)$. The capacitor effectively "blocks" the DC component of the input!

[1 point]

(d) [2 points] Find values of R_2 and C so that the amplitude of the steady-state cosine function in $V_0(t)$ be 12 V for large ω ($\rightarrow \infty$) and at least $12/\sqrt{2}$ V for $\omega = 1500$ rad/s.

Solution:

For large ω we have $T(j\infty) \rightarrow -R_2/R_1$ which is the standard OpAmp gain. Therefore, because the amplitude of the tone is 12 V, we need to have $R_2/R_1 = 1$ which leaves $R_2 = R_1 = 600\ \Omega$.

[1 point]

For the requirement that the amplitude be $12/\sqrt{2}$ at $\omega = 1500$ rad/s, recall that $1/\sqrt{2}$ is exactly the attenuation at the cut-off frequency. Hence we just need to set the cut-off frequency to 1500 rad/s. That is

$$\omega_c = \frac{1}{600C} = 1500 \quad \Longrightarrow \quad C = \frac{1}{600 \times 1500} \approx 10^{-6} = 1.1\,\mu\text{F}.$$

[1 point]