9.2) $f(t) = A\left[(2-\alpha t)e^{-\alpha t}\right]u(t)$

$$\mathcal{L}\{f(t)\} = A\left[\mathcal{L}\{2e^{-\alpha t}u(t)\} - \mathcal{L}\{\alpha t e^{-\alpha t}u(t)\}\right]$$

$$= A\left[\frac{2}{s+\alpha} - \frac{\alpha}{(s+\alpha)^2}\right]$$

$$= A\left[\frac{2s+\alpha}{(s+\alpha)^2}\right]$$

double poles at $s = -\alpha$

one zero at $s = -\alpha/2$

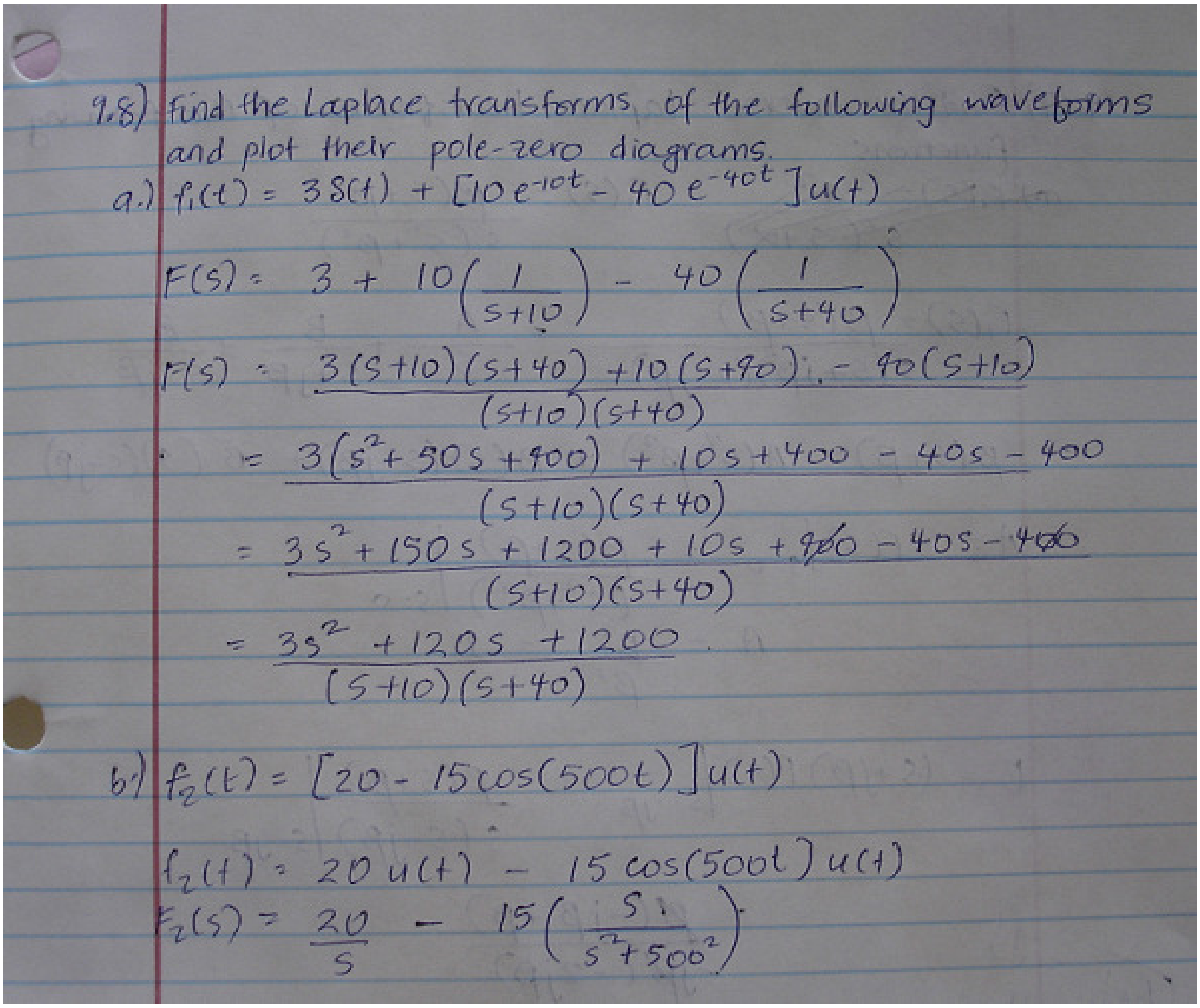
9.3 Find LT of $f(t) = A[1 - 2\cos(\beta t)]u(t)$.
Locate the poles and zeros of $F(s)$

$$\Rightarrow \mathcal{L}\{f(t)\} = A\left[\frac{1}{s} - \frac{2s}{s^2+\beta^2}\right]$$

$$F(s) = \frac{A(\beta^2 - s^2)}{s(s^2+\beta^2)}$$

Poles: $0, \pm j\beta$

Zeros: $\pm \beta$



9.8) Find the Laplace transforms of the following waveforms and plot their pole-zero diagrams.

a.) $f_1(t) = 3\delta(t) + [10e^{-10t} - 40e^{-40t}]u(t)$

$$F(s) = 3 + 10\left(\frac{1}{s+10}\right) - 40\left(\frac{1}{s+40}\right)$$

$$F(s) = \frac{3(s+10)(s+40) + 10(s+40) - 40(s+10)}{(s+10)(s+40)}$$

$$= \frac{3(s^2+50s+400) + 10s+400 - 40s - 400}{(s+10)(s+40)}$$

$$= \frac{3s^2+150s+1200+10s+\cancel{400}-40s-\cancel{400}}{(s+10)(s+40)}$$

$$= \frac{3s^2+120s+1200}{(s+10)(s+40)}$$

b.) $f_2(t) = [20 - 15\cos(500t)]u(t)$

$f_2(t) = 20u(t) - 15\cos(500t)u(t)$

$$F_2(s) = \frac{20}{s} - 15\left(\frac{s}{s^2+500^2}\right)$$

9.10) Remember:

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \quad \text{for } a>0$$

a) $f_1(t) = 2\delta(t-2)$

$$\mathcal{L}\{2\delta(t-2)\} = 2e^{-2s}$$

b) $f(t) = e^{-50(t-1)}u(t-1)$

$$\mathcal{L}\{e^{-50(t-1)}u(t-1)\} = e^{-s}\frac{1}{s+50}$$

c) $f(t) = e^{-50(t-2)}u(t-2)$

$$\mathcal{L}\{e^{-50(t-2)}u(t-2)\} = e^{-2s}\frac{1}{s+50}$$

9.16 find inverse LT of:

a) $F_1(s) = \dfrac{s+20}{s(s+10)}$

b) $F_2(s) = \dfrac{s^2+10s+10}{s(s+10)}$

⇒

a) $f(t) = \mathcal{L}^{-1}\{F_1(s)\}$

$$F_1(s) = \frac{A}{s} + \frac{B}{s+10} = \frac{s+20}{s(s+10)}$$

$$\frac{A(s+10) + Bs}{s(s+10)} = \frac{s+20}{s(s+10)}$$

If $s=0$,

$$A(10) + B(0) = 20$$

$$A = 2$$

If $s = -10$,

$$A(-10+10) + B(-10) = 10$$

$$B = -1$$

$$\therefore F_1(s) = \frac{2}{s} + \frac{-1}{s+10}$$

$$\mathcal{L}^{-1}\{F_1(s)\} = 2u(t) - e^{-10t}u(t)$$

$$\boxed{\therefore f_1(t) = [2 - e^{-10t}]u(t)}$$

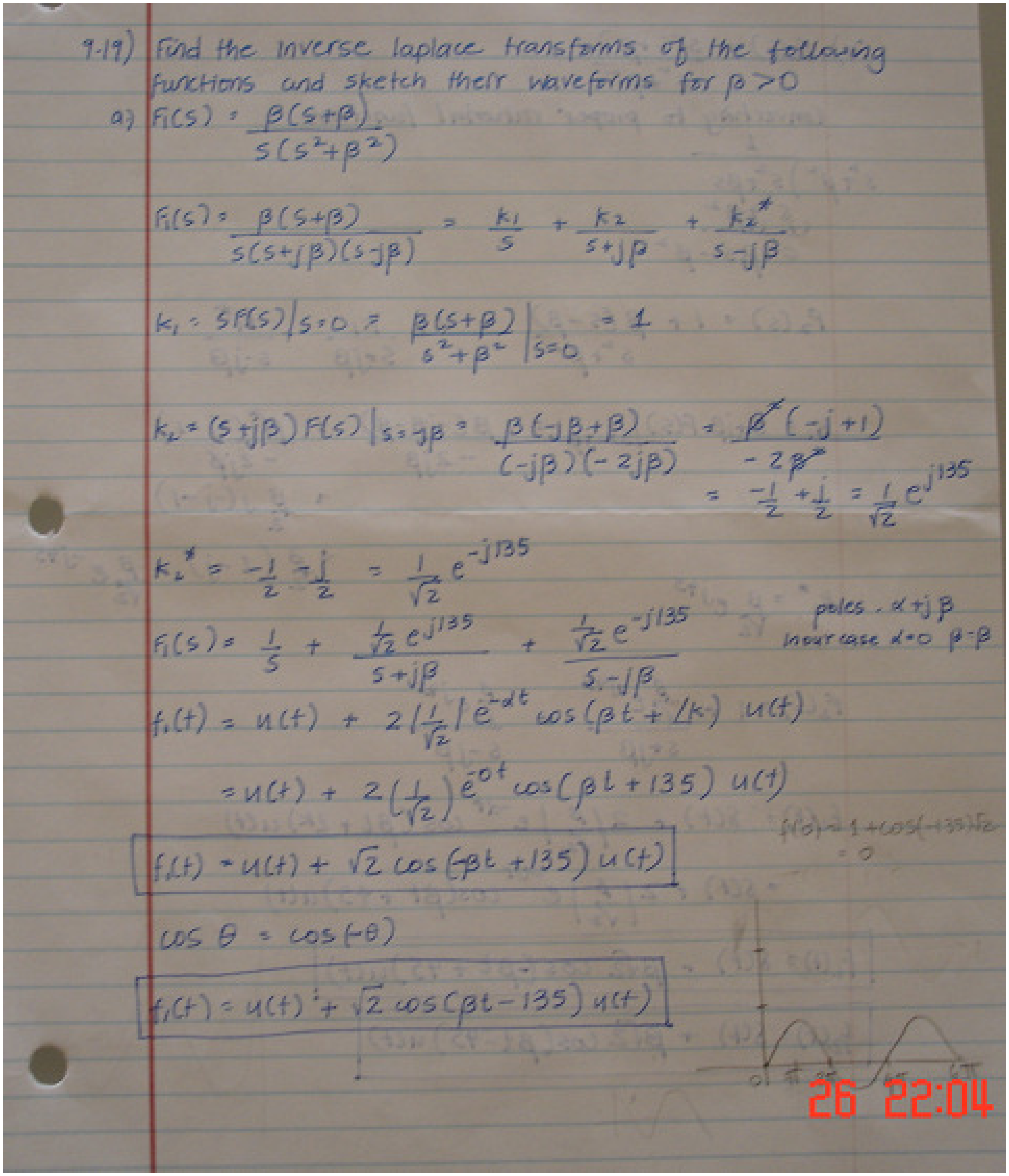
9.16 contd.

b) $$F_2(s) = \frac{s^2+10s+10}{s(s+10)} = 1 + \frac{10}{s(s+10)}$$

$$= 1 + \frac{1}{s} - \frac{1}{s+10}$$

$$f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = \delta(t) + u(t) - e^{-10t}u(t)$$

$$\boxed{\therefore f_2(t) = \delta(t) + [1 - e^{-10t}]u(t)}$$



9.19) Find the inverse laplace transforms of the following functions and sketch their waveforms for $\beta > 0$

a) $F_1(s) = \dfrac{\beta(s+\beta)}{s(s^2+\beta^2)}$

$$F_1(s) = \frac{\beta(s+\beta)}{s(s+j\beta)(s-j\beta)} = \frac{k_1}{s} + \frac{k_2}{s+j\beta} + \frac{k_2^*}{s-j\beta}$$

$$k_1 = sF(s)\Big|_{s=0} = \frac{\beta(s+\beta)}{s^2+\beta^2}\Big|_{s=0} = 1$$

$$k_2 = (s+j\beta)F(s)\Big|_{s=-j\beta} = \frac{\beta(-j\beta+\beta)}{(-j\beta)(-2j\beta)} = \frac{\cancel{\beta}(-j+1)}{-2\cancel{\beta}}$$

$$= \frac{-1}{2} + \frac{j}{2} = \frac{1}{\sqrt{2}}e^{j135}$$

$$k_2^* = \frac{-1}{2} - \frac{j}{2} = \frac{1}{\sqrt{2}}e^{-j135}$$

poles . $\alpha + j\beta$

In our case $\alpha = 0$ $\beta = \beta$

$$F_1(s) = \frac{1}{s} + \frac{\frac{1}{\sqrt{2}}e^{j135}}{s+j\beta} + \frac{\frac{1}{\sqrt{2}}e^{-j135}}{s-j\beta}$$

$$f_1(t) = u(t) + 2\left|\frac{1}{\sqrt{2}}\right|e^{-\alpha t}\cos(\beta t + \angle k)\,u(t)$$

$$= u(t) + 2\left(\frac{1}{\sqrt{2}}\right)e^{-0t}\cos(\beta t + 135)\,u(t)$$

$$\boxed{f_1(t) = u(t) + \sqrt{2}\cos(-\beta t + 135)\,u(t)}$$

$f(0) = 1 + \cos(-135)\sqrt{2}$

$= 0$

$\cos\theta = \cos(-\theta)$

$$\boxed{f_1(t) = u(t) + \sqrt{2}\cos(\beta t - 135)\,u(t)}$$

b) $F_2(s) = \frac{s(s+\beta)}{s^2+\beta^2}$

Converting to proper rational function

$$s^2+\beta^2 \overline{\big) s^2+\beta s}^{\;1}$$
$$\underline{(-)s^2 (-)\beta^2}$$
$$0+\beta s-\beta^2$$

$$F_2(s) = 1 + \frac{\beta(s-\beta)}{s^2+\beta^2} = 1 + \frac{k_1}{s+j\beta} + \frac{k_1^*}{s-j\beta}$$

$$k_1 = s+j\beta\, F(s)\Big|_{s=-j\beta} = \frac{\beta(-j\beta-\beta)}{-2j\beta} = \frac{\beta^2(-j-1)}{-2j\beta}$$
$$= \frac{\beta}{2} j(-j-1)$$
$$= \frac{\beta}{2}(1-j) = \frac{\beta}{\sqrt{2}} e^{-j45}$$

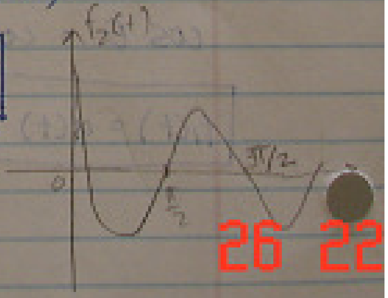
$$k_1^* = \frac{\beta}{\sqrt{2}} e^{j45}$$

$$F_2(s) = 1 + \frac{\frac{\beta}{\sqrt{2}} e^{-j45}}{s+j\beta} + \frac{\frac{\beta}{2} e^{j45}}{s-j\beta}$$

$$f_2(t) = \delta(t) + 2\left|\frac{\beta}{\sqrt{2}}\right| e^{-\alpha t} \cos(\beta t + \angle k)\, u(t)$$
$$= \delta(t) + 2\left|\frac{\beta}{\sqrt{2}}\right| e^{-0t} \cos(\beta t + 45)\, u(t)$$

$$\boxed{f_2(t) = \delta(t) + \beta\sqrt{2} \cos(-\beta t + 45)\, u(t)}$$

$$\boxed{f_2(t) = \delta(t) + \beta\sqrt{2} \cos(\beta t - 45)\, u(t)}$$



9.26) a) $F(s) = \frac{(s+40)^2}{(s+10)^2(s+100)} \Rightarrow F(s) = \frac{1}{(s+10)}\left[\frac{(s+40)^2}{(s+10)(s+100)}\right]$

inside of the bracket is not strictly proper. We need to use long division

$$F(s) = \frac{1}{s+10}\left[1 + \frac{-30(s-20)}{(s+10)(s+100)}\right]$$

$$= \frac{1}{s+10}\left[1 + \frac{10}{(s+10)} - \frac{40}{s+100}\right]$$

$$= \frac{1}{s+10} + \frac{10}{(s+10)^2} - \frac{40}{(s+10)(s+100)}$$

$$= \frac{1}{s+10} + \frac{10}{(s+10)^2} - \frac{4}{9}\frac{1}{(s+10)} + \frac{4}{9}\frac{1}{(s+100)}$$

$$= \frac{5}{9}\frac{1}{s+10} + \frac{4}{9}\frac{1}{(s+100)} + \frac{10}{(s+10)2}$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \frac{5}{9}e^{-10t}u(t) + \frac{4}{9}e^{-100t}u(t) + 10te^{-10t}u(t)$$

9.26 b) $F(s) = \dfrac{(s+10)^2}{(s+40)^2(s+100)} = \dfrac{1}{s+40}\left[\dfrac{s+10}{(s+40)(s+100)}\right]$

$$\Rightarrow F(s) = \frac{1}{(s+40)}\left[1 - \frac{120s+3900}{(s+40)(s+100)}\right]$$

$$= \frac{1}{(s+40)}\left[1 + 15\frac{1}{(s+40)} - 135\frac{1}{(s+100)}\right]$$

$$= \frac{1}{(s+40)} + 15\frac{1}{(s+40)^2} - 135\frac{1}{(s+40)(s+100)}$$

$$= \frac{1}{(s+40)} + 15\frac{1}{(s+40)^2} - \frac{9}{4}\frac{1}{s+40} + \frac{9}{4}\frac{1}{(s+100)}$$

$$= \frac{-5}{4}\frac{1}{(s+40)} + 15\frac{1}{(s+40)^2} + \frac{9}{4}\frac{1}{(s+100)}$$

$$\mathcal{L}^{-1}\{F(s)\} = -5/_4\, e^{-40t} + t15e^{-40t} + \frac{9}{4}e^{-100t}$$

4.31

Use the LT to find the $y(t)$ that satisfies the following 1st order DEs:

a) $50\frac{dy}{dt} + 250y = 0$ with $y(0-) = 10$

b) $\frac{dy}{dt} + 20y = 40u(t)$ with $y(0-) = -10$

$\Rightarrow$

a) $\mathcal{L}\left\{50\frac{dy}{dt} + 250y = 0\right\} = 50\left(sY(s) - y(0-)\right) + 250Y(s) = 0$

$$50sY(s) - 500 + 250Y(s) = 0$$

$$Y(s)(50s + 250) = 500$$

$$Y(s) = \frac{500}{50s + 250}$$

$$= \frac{10}{s+5}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s+5}\right\}$$

$$= \boxed{10e^{-5t}u(t)}$$

b) $\mathcal{L}\left\{\frac{dy}{dt} + 20y = 40u(t)\right\} = \left(sY(s) - y(0-)\right) + 20Y(s) = \frac{40}{s}$

$$sY(s) + 10 + 20Y(s) = \frac{40}{s}$$

$$(s+20)Y(s) = \frac{40}{s} - 10$$

$$Y(s) = \frac{\frac{40}{s} - 10}{s+20} = \frac{10(4-s)}{s(s+20)} = \frac{A}{s} + \frac{B}{s+20}$$

3) 9.31 b) contd.

$$A(s+20) + Bs = 10(4-s)$$

If $s=0$,

$$20A = 40$$

$$\underline{\underline{A = 2}}$$

If $s=-20$,

$$-20B = 240$$

$$\underline{\underline{B = -12}}$$

$$\therefore Y(s) = \frac{2}{s} + \frac{-12}{s+20}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2u(t) - 12e^{-20t}u(t)$$

$$= \boxed{\left[2-12e^{-20t}\right]u(t)}$$

9.46) Use the initial and final value ~~problem~~ properties to find the initial and final values of the waveforms corresponding to the transforms in p. 9-23. If either property is not applicable, explain why.

a.) $F_1(s) = \frac{(s+4)(s+8)}{s(s+2)(s+6)}$

Initial value: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$

$$= \lim_{s \to \infty} \frac{(s+4)(s+8)}{(s+2)(s+6)}$$

$$= \lim_{s \to \infty} \frac{s^2+12s+32}{s^2+8s+12}$$

$$= \lim_{s \to \infty} \frac{1 + 12/s + 32/s^2}{1 + 8/s + 12/s^2} = \underline{\underline{1}}$$

Final value: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

$$= \lim_{s \to 0} \frac{(s+4)(s+8)}{(s+2)(s+6)} = \frac{(\cancel{4}^{2})(8)}{(\cancel{2})(6)} = \frac{8}{3}$$

b.) $F_2(s) = \frac{3s^4+10s^2+4}{s(s^2+1)(s^2+4)}$

Initial value: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \frac{3s^4+10s^2+4}{(s^2+1)(s^2+4)}$

$$= \lim_{s \to \infty} \frac{3s^4+10s^2+4}{s^4+5s^2+4}$$

$$= 3 /\!/$$

Final value: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{3s^4+10s^2+4}{(s^2+1)(s^2+4)}$

Final value theorem does not apply since the poles are on the j-axis $(s = \pm j,\ s = \pm 2j)$

9.48) a) $F_1(s) = \dfrac{s(s+5)}{s^2+6s+9}$

We cannot apply initial value property because $F_1(s)$ is not strictly proper.

Final Value Property

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s F_1(s) = \lim_{s\to 0} \frac{s^2(s+5)}{s^2+6s+9} = 0$$

b) $F_2(s) = \dfrac{10(s^2+10s-20)}{s(s^2+100)}$

Final value property can not be applied because ~~[illegible]~~ the poles of the equation are on the imaginary axis $\boxed{\begin{matrix} s_1 = -10j \\ s_2 = +10j \end{matrix}}$ $s_3 = 0$

$\downarrow$

simple pole on origin is acceptable

Initial value property

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s F(s) = \lim_{s\to\infty} \frac{10(s^2+10s-20)}{s^2+100}$$

$$= \lim_{s\to\infty} \left(10 + \frac{10(10s-120)}{s^2+100} \right) = 10$$