MAE140 - Linear Circuits - Winter 16
Final, March 16, 2016

Instructions

(i) The exam is open book. You may use your class notes and textbook. You may use a hand calculator with no communication capabilities.

(ii) You have 180 minutes

(iii) Do not forget to write your **name** and **student number**

(iv) On the questions for which the answers are given, please provide detailed derivations

(v) The exam has 5 questions for a total of 50 points and 2 bonus points

Good luck!

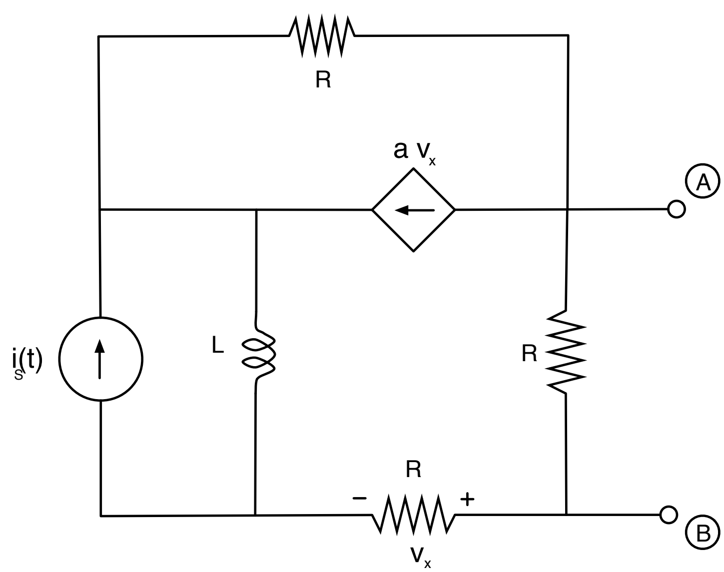


Figure 1: Circuit for Question 1.

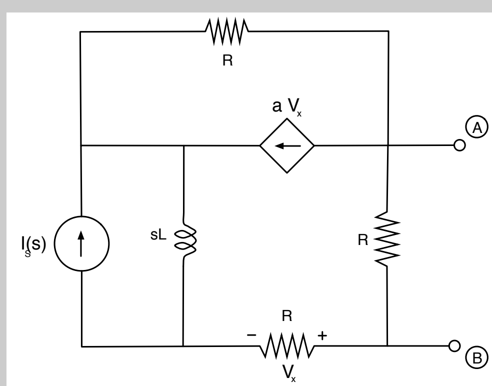
1. **Equivalent Circuits**

Part I: [2 points] Assuming zero initial conditions, transform the circuit in Figure 1 into the s-domain.

Solution:

Since all initial conditions are zero, it is easy to transform the circuit to the s-domain.

(2 points)

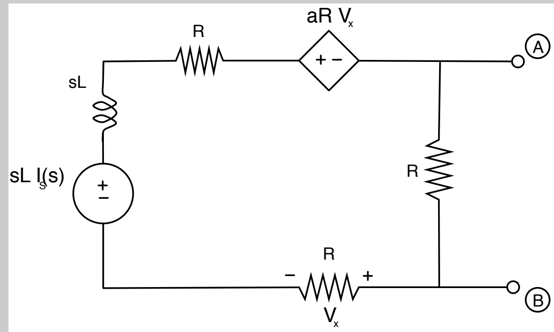


Part II: [4 points] Find the open-circuit voltage in the circuit obtained in Part I (as seen from terminals Ⓐ and Ⓑ). Be mindful of the presence of the dependent source.

Solution:

Both current sources are in parallel with impedances, so we use source transformation twice to get the circuit on the right

(1 point)



The open-circuit voltage $V_{AB}(s)$ is exactly the voltage drop seen by the vertical impedance R. By voltage division, we deduce that

$$V_{AB}(s) = \frac{R}{3R + sL}(sLI_S(s) - aRV_x(s)) \qquad \textbf{(1 point)}$$

We need to determine what the value of $V_x(s)$ is (i.e., we need to handle the dependent source). This is in fact easy to do by noting that, by voltage division,

$$V_x(s) = \frac{R}{3R + sL}(sLI_S(s) - aRV_x(s)) \qquad \textbf{(1 point)}$$

(In fact, $V_x(s) = V_{AB}(s)$). Solving for V_x, we obtain

$$V_x(s) = \frac{sRL}{3R + aR^2 + sL} I_S(s)$$

Therefore, the open-circuit voltage is

$$V_{AB}(s) = \frac{sRL}{3R + aR^2 + sL} I_S(s) \qquad \textbf{(1 point)}$$

Part III: [4 points] Find the s-domain Thévenin equivalent of the circuit obtained in Part I as seen from terminals Ⓐ and Ⓑ. Be mindful of the presence of the dependent source.

Solution:

We need to find V_T and Z_T. It turns out that we computed the open-circuit voltage in Part II, so in fact we know that

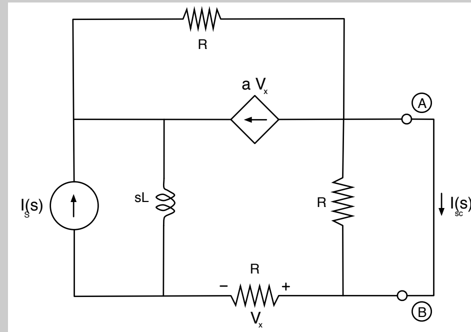
$$V_T(s) = \frac{sRL}{3R + aR^2 + sL} I_S(s) \qquad \textbf{(.5 point)}$$

To find the Thévenin impedance, we need to compute the short-circuit current (turning off sources is not an option because of the presence of the dependent source)

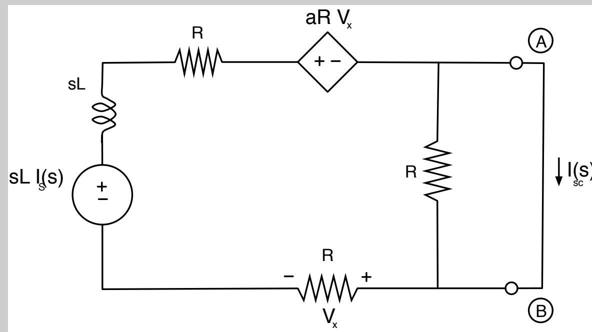
(.5 point)

The circuit then looks like the plot on the right

(.5 point)

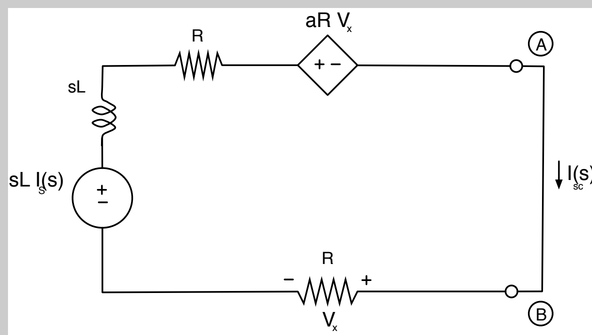


The same source transformations as we did in Part II get us the equivalent circuit on the right



Since the vertical impedance R is in parallel with a short circuit, it does not play any role (no current flows through it). Therefore, we are actually dealing with

(.5 point)



Therefore, the short-circuit current can be expressed as

$$I_{sc}(s) = \frac{sLI_S(s) - aRV_x(s)}{2R + sL} \qquad (\textbf{.5 point})$$

As before, we need to determine V_x (i.e., handle the presence of the dependent source). From the circuit, and using voltage division, we see that

$$V_x(s) = \frac{R}{2R + sL}(sLI_S(s) - aRV_x(s)) \qquad (\textbf{1 point})$$

Solving for V_x, we obtain

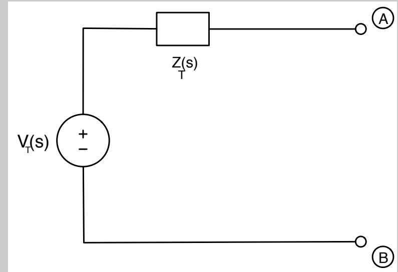
$$V_x(s) = \frac{sRL}{2R + aR^2 + sL} I_S(s)$$

Therefore, the short-circuit current is

$$I_{sc}(s) = \frac{sLI_S(s) - aRV_x(s)}{2R + sL} = \frac{sL}{2R + aR^2 + sL} I_S(s)$$

Finally, the Thévenin impedance is

$$Z_T(s) = \frac{V_T(s)}{I_{sc}(s)} = \frac{R(2R + aR^2 + sL)}{3R + aR^2 + sL}$$



(.5 point)

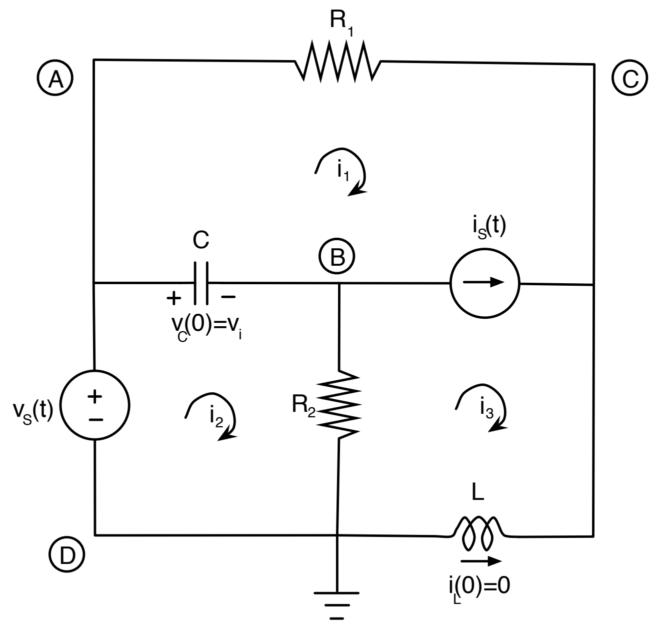
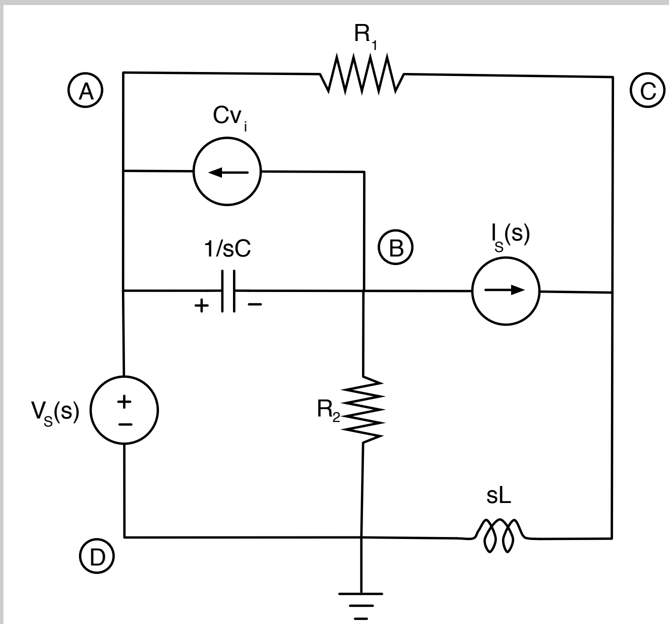


Figure 2: Nodal and Mesh Analysis Circuit for Question 2.

2. **Nodal and Mesh Analysis**

Part I: [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure and transform them into current sources. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

Solution:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity of the capacitor. Since the initial condition of the inductor is zero, no need to worry about adding a source for it.

(1 point for correct circuit; 1 point for correct initial condition)

For nodal analysis, the presence of the voltage source poses a problem. However, the choice of ground provides a solution for it. In fact, we have $V_D = 0$ and

$$V_A(s) = V_S(s) \qquad \textbf{(1 point)}$$

(this is method #2).

We next write KCL node equations for nodes B and C. For convenience, we use the shorthand notation $G_1 = 1/R_1$ and $G_2 = 1/R_2$. For node B, we have

$$sC(V_B(s) - V_A(s)) + G_2 V_B(s) = -Cv_i - I_S(s) \qquad \textbf{(1 point)}$$

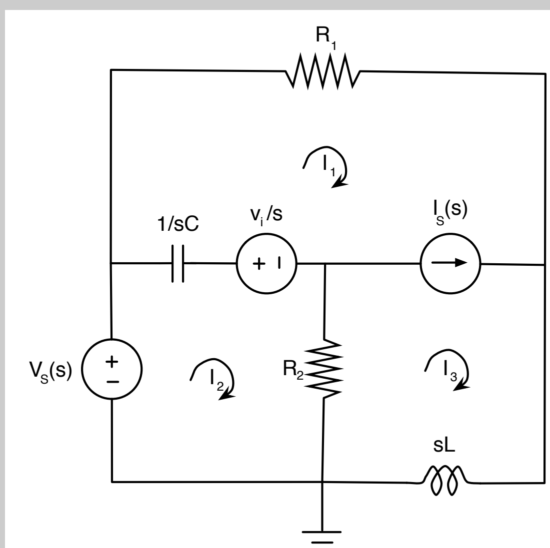
For node C, we have

$$G_1(V_C(s) - V_A(s)) + \frac{1}{sL} V_C(s) = I_S(s) \qquad \textbf{(1 point)}$$

This gives a total of 3 independent equations in 3 unknowns ($V_A(s)$, $V_B(s)$, $V_C(s)$). Alternatively, one can take the expression for $V_A(s)$ and substitute it in the other equations to arrive at 2 independent equations in 2 unknowns ($V_B(s)$, $V_C(s)$).

Part II: [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the currents shown in the figure. Use the initial conditions indicated in the figure and transform them into voltage sources. Make sure your final answer has the same number of independent equations as unknown variables. No need to solve any equations!

Solution:



In the above figure, we have transformed the circuit into the s-domain, taking good care of respecting the polarity of the capacitor. Again, no need to worry about the initial condition of the inductor because it is zero.

(1 point for correct circuit; 1 point for correct initial condition)

For mesh-current analysis, the presence of the current source is a problem that must be dealt with. In this case, we need to use a supermesh (because the current source is not in parallel with an impedance and because it belongs to more than one mesh). Therefore, we set

$$I_3(s) - I_1(s) = I_S(s) \qquad \textbf{(1 point)}$$

(this is method #3).

KVL for the supermesh looks like

$$R_1 I_1(s) + sLI_3(s) + R_2(I_3(s) - I_2(s)) + \frac{1}{sC}(I_1(s) - I_2(s)) - \frac{v_i}{s} = 0 \qquad \textbf{(1 point)}$$

For mesh 2, KVL takes the form

$$R_2(I_2(s) - I_3(s)) - V_S(s) + \frac{1}{sC}(I_2(s) - I_1(s)) + \frac{v_i}{s} = 0 \qquad \textbf{(1 point)}$$

This gives a total of 3 independent equations in 3 unknowns ($I_1(s)$, $I_2(s)$, $I_3(s)$).

Part III: [1 bonus point] Express the transform of the capacitor voltage in terms of your unknown variables of Part I and also in terms of your unknown variables of Part II.

Solution:

We just need to be careful to not lose track of the transform of the capacitor voltage. In the case of Part I, because we use a current source to account for the initial condition, we have

$$V_{capacitor}(s) = V_A(s) - V_B(s) \qquad \textbf{(.5 bonus point)}$$

In the case of Part II, because we use a voltage source to account for the initial condition, we actually have

$$V_{capacitor}(s) = \frac{1}{sC}(I_2(s) - I_1(s)) + \frac{v_i}{s} \qquad \textbf{(.5 bonus point)}$$

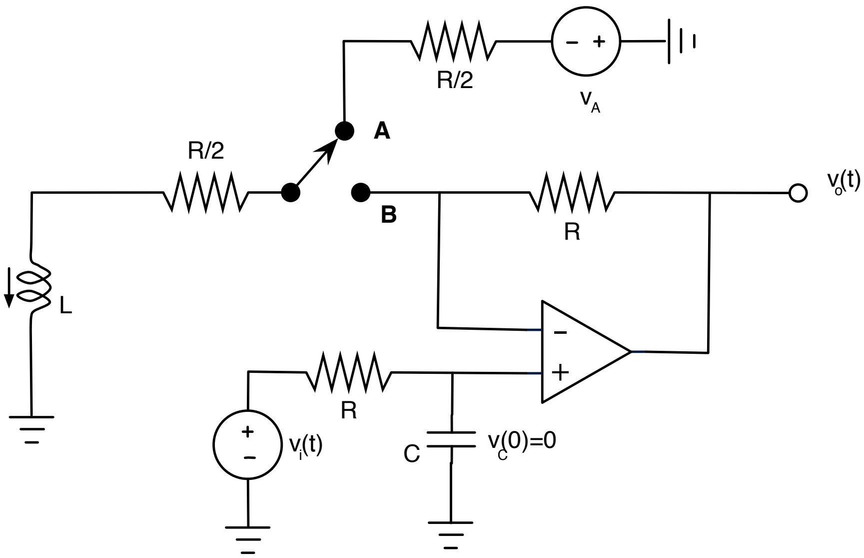


Figure 3: RCL circuit for Laplace Analysis for Question 3.

3. **Laplace Domain Circuit Analysis**

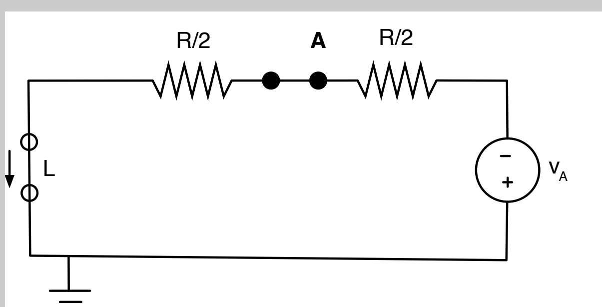
Part I: [2 points] Consider the circuit depicted in Figure 3. The value v_A of the voltage source at the top is constant. The switch is kept in position **A** for a very long time. At $t = 0$ it is moved to position **B**. Show that the initial condition for the inductor is given by

$$i_L(0^-) = -\frac{v_A}{R}.$$

[Show your work]

Solution: To find the initial condition, we substitute the inductor by a short circuit.

[1 point for correct circuit; 1 point for substituting inductor by short circuit]



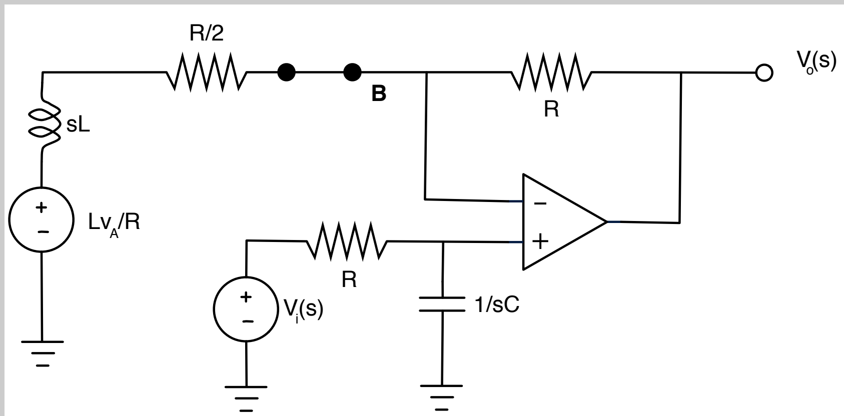
Therefore, we deduce that

$$i_L(0^-) = -\frac{v_A}{R}$$

Part II: [4 points] Use this initial condition to transform the circuit into the s-domain for $t \geq 0$. Use an equivalent model for the inductor in which the initial condition appears as a voltage source.

Do you recognize the resulting circuit as one of the basic op-amp building blocks? Express the output response transform $V_o(s)$ as a function of $V_i(s)$ and v_A.

Solution: We add one voltage source in series for the inductor to take care of its initial condition, paying special attention to the direction of the current. No need to worry about the initial condition of the capacitor because it is zero.



(1 point for correct circuit; 1 point for correct polarity)

We recognize this circuit as one of the basic op-amp building blocks. In fact, this is a differential amplifier.

(1 point)

Given the above, the output response transform is

$$\begin{aligned} V_o(s) &= \frac{1/sC}{1/sC + R}\frac{3R/2 + sL}{R/2 + sL}V_i(s) - \frac{R}{R/2 + sL}\frac{L}{R}v_A \\ &= \frac{3R + 2Ls}{(1 + RCs)(R + 2Ls)}V_i(s) - \frac{2L}{R + 2Ls}v_A \end{aligned} \quad \textbf{(1 point)}$$

Part III: [4 points] Use partial fractions and inverse Laplace transforms to show that the output voltage $v_o(t)$ when $v_A = 1\,V$, $v_i(t) = e^{-1500t}u(t)\,V$, $C = 1$ mF, $L = 1$ mH, and $R = 1\,\Omega$ is

$$v_o(t) = (e^{-500t} - 2e^{-1000t})u(t).$$

Solution: From our answer to Part II, and substituting the values for the impedances and the sources, the Laplace transform of the output voltage is

$$\begin{aligned} V_o(s) &= \frac{3 + 2\cdot 10^{-3}s}{(1 + 10^{-3}s)(1 + 2\cdot 10^{-3}s)}\frac{1}{s + 1500} - \frac{1}{500 + s} \\ &= \frac{1000}{(1000 + s)(500 + s)} - \frac{1}{500 + s} \end{aligned} \quad \textbf{(1 point)}$$

Using partial fractions, we get

$$V_o(s) = \frac{A}{1000 + s} + \frac{B}{500 + s} - \frac{1}{500 + s} \quad \textbf{(1 point)}$$

Using the residue method to compute A and B, we obtain

$$V_o(s) = -\frac{2}{1000+s} + \frac{2}{500+s} - \frac{1}{500+s} = -\frac{2}{1000+s} + \frac{1}{500+s} \qquad \textbf{(1 point)}$$

The output voltage is then

$$v_o(t) = (e^{-500t} - 2e^{-1000t})u(t) \qquad \textbf{(1 point)}$$

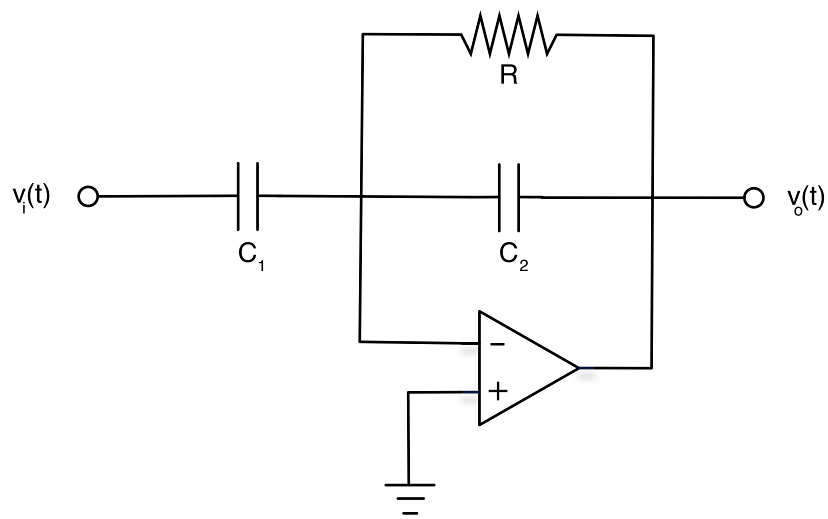
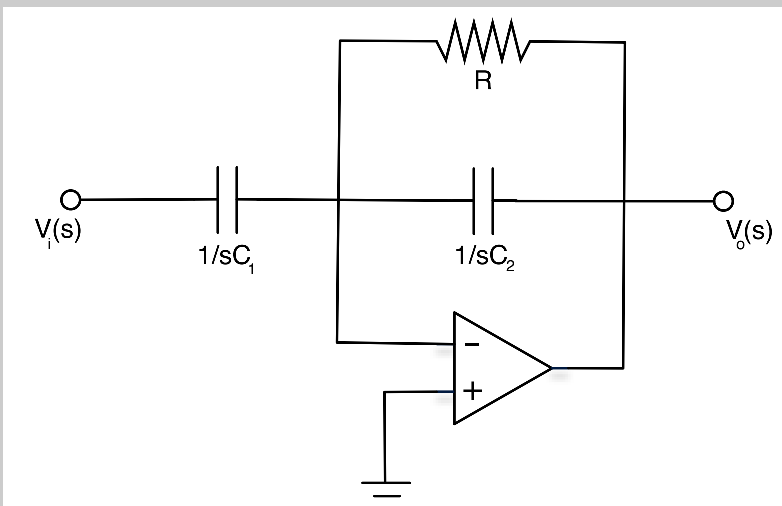


Figure 4: Frequency Response Analysis for Question 4.

4. **Frequency Response Analysis**

Part I: [1 point] Assuming zero initial conditions, transform the circuit in Figure 4 into the s-domain.

Solution: Since all initial conditions are zero, there is no need to add an independent source for the capacitors. Therefore, the circuit in the s-domain looks like



(1 point)

Part II: [2 points] Show that the transfer function from $V_i(s)$ to $V_o(s)$ is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{RC_1 s}{1 + RC_2 s}.$$

[Show your work]

Solution: This is clearly an inverting op-amp. **(1 point)**
Therefore the output response transform can be expressed as

$$V_o(s) = -\frac{R||1/sC_2}{1/sC_1} V_i(s) = -\frac{RC_1 s}{1 + RC_2 s} V_i(s). \qquad \textbf{(1 point)}$$

from which the answer follows.

Part III [5.5 points] Let $R = 1\ \text{k}\Omega$, $C_1 = 20\ \mu\text{F}$ and $C_2 = 1\ \mu\text{F}$. Compute the gain and phase functions of $T(s)$. What are the DC gain and the ∞-freq gain? What are the corresponding values of the phase function? What is the cut-off frequency ω_c and its phase? Sketch plots for the gain and phase functions. What type of filter is this one?
[Explain your answer]

Solution: For the given values of R, C_1 and C_2, the transfer function takes the form

$$T(s) = -\frac{20 \cdot 10^{-3} s}{1 + 10^{-3} s} = -\frac{20s}{1000 + s}$$

The frequency response is then the complex function

$$T(j\omega) = \frac{-20j\omega}{1000 + j\omega}, \quad \omega \geq 0$$

Its magnitude is the gain function,

$$|T(j\omega)| = \frac{|-20j\omega|}{|1000 + j\omega|} = \frac{20\omega}{\sqrt{10^6 + \omega^2}} \qquad \textbf{(.5 point)}$$

And its phase is

$$\angle T(j\omega) = \angle(-20j\omega) - \angle(1000 + j\omega) = \frac{3\pi}{2} - \arctan\left(\frac{\omega}{1000}\right) \qquad \textbf{(.5 point)}$$

At $\omega = 0$, we obtain

$$|T(j0)| = 0, \quad \angle T(j0) = \frac{3\pi}{2} \text{ rad}$$

(**correct DC-gain gets .5 point, correct phase gets .5 point**)

At $\omega = \infty$, we obtain

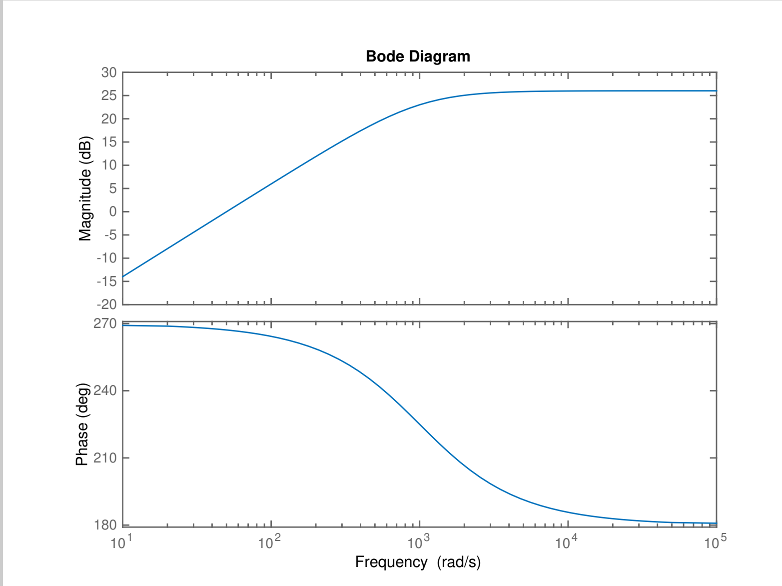
$$|T(j\infty)| = 20, \quad \angle T(j\infty) = \pi \text{ rad}$$

(**correct ∞-freq gain gets .5 point, correct phase gets .5 point**)

The cut-off frequency is defined by

$$|T(j\omega_c)| = \frac{T_{\max}}{\sqrt{2}} = \frac{20}{\sqrt{2}}.$$

Solving for it, we find $\omega_c = 1000$ rad/s. **(.5 point)**
The phase at ω_c is $\angle T(j1000) = \frac{5\pi}{4}$ rad. **(.5 point)**
With the values obtained above, you can sketch the magnitude of the frequency response as



(.5 point for gain plot, .5 point for phase plot)

This circuit is a high-pass filter. **(.5 point)**

Part IV [1.5 points] Using what you know about frequency response, compute the steady state response $v_o^{SS}(t)$ of this circuit when $v_i(t) = \frac{1}{4}\cos(500t + \frac{\pi}{4})$ using the same values of R, C_1, and C_2 as in Part III.

Solution: To compute the steady-state response to the input $v_i(t) = \frac{1}{4}\cos(500t + \frac{\pi}{4})$, we use the frequency response values for $\omega = 500$,

$$|T(j500)| = 4\sqrt{5}$$
$$\angle T(j500) = \frac{3\pi}{2} - \arctan\left(\frac{1}{2}\right) \simeq 4.24874 \text{ rad.}$$

Therefore,

$$v_o^{SS}(t) = \frac{1}{4}|T(j500)|\cos\left(500t + \frac{\pi}{4} + \angle T(j500)\right) \qquad \textbf{(1 point for correct expression)}$$
$$= \sqrt{5}\cos(500t + 5.0314) \qquad \textbf{(.5 point for correct values)}$$

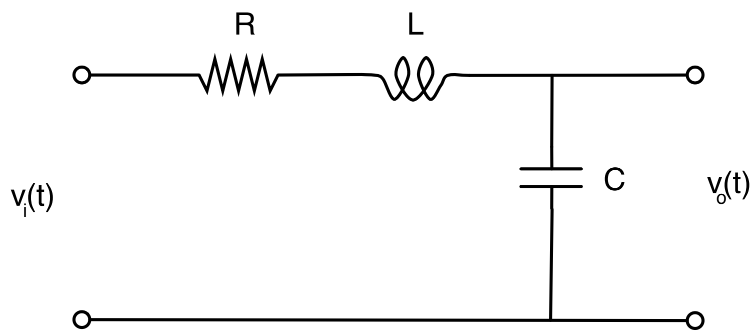


Figure 5: Circuit for Question 5.

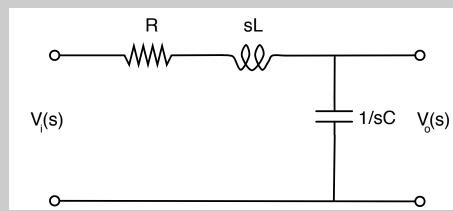
5. **Loading and the Chain Rule**

Consider the circuit in Figure 5.

Part I: [2 points] Assuming zero initial conditions, plot the circuit in the s-domain. Find the transfer function $T(s)$ and determine its poles and zeros assuming $R > 2\sqrt{L/C}$.

Solution:

Since there are no initial conditions to take care of, the circuit in the s-domain looks like



(.5 point)

We can easily determine the transfer function by using voltage division.

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R + sL} = \frac{1}{1 + RCs + LCs^2} \qquad \textbf{(.5 point)}$$

There are no finite zeros. The transfer function has two zeros at $s = \infty$. **(.5 point)**

The poles are of T are

$$p_1 = \frac{-RC + \sqrt{(RC)^2 - 4LC}}{2LC}, \quad p_2 = \frac{-RC - \sqrt{(RC)^2 - 4LC}}{2LC}. \qquad \textbf{(.5 point)}$$

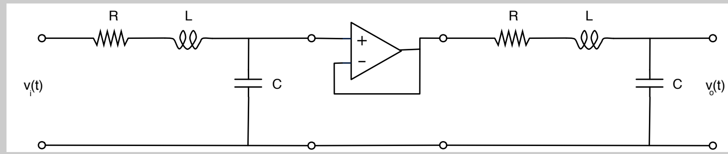
Since $R > 2\sqrt{L/C}$, both poles are real and negative.

Part II: [3 points] A student with a rusty recollection of MAE140 connected two identical copies of this circuit in series and was surprised to observe that the transfer function of the resulting circuit is not $T(s) \cdot T(s) = T(s)^2$. Could you explain to him why this is so and suggest an easy way to solve the problem so that he obtains $T(s)^2$ as the transfer function of the resulting circuit? Properly justify your answer.

Solution: The problem is loading. If we connect two exact copies of the circuit in Figure 5 in series, stage 2 will draw current from stage 1, and load it, invalidating the chain rule.

(1 point)

An easy way to solve the problem is to add a voltage follower in between the two copies of the circuit in Figure 5, as depicted on the right. **(1 point)**



The infinite input impedance of the op-amp avoids loading stage 1. The zero output impedance of the op-amp avoids stage 3 loading stage 2. Therefore, the chain rule applies. Since the transfer function of the voltage follower is 1, the overall transfer function is $T(s) \cdot 1 \cdot T(s) = T(s)^2$.

(1 point)

Part III: [2 points] We could help the student of Part II even further by designing a different circuit with the same transfer function that does not have the same problem. To make the computations concrete, let $R = 100\ \Omega$, $C = 10\ \mu F$ and $L = 5$ mH, and compute the corresponding numerical values for the poles of the transfer function $T(s)$. Next, write $T(s)$ as a product of the form

$$T(s) = \frac{-A}{s+\alpha} \cdot \frac{-B}{s+\beta}$$

and determine appropriate values for $A > 0$, $B > 0$, $\alpha > 0$, and $\beta > 0$.

Solution: With the values of R, C, and L, the transfer function can be written as

$$T(s) = \frac{1}{1 + 10^{-3}s + 5 \cdot 10^{-8}s^2} = \frac{2 \cdot 10^7}{2 \cdot 10^7 + 2 \cdot 10^4 s + s^2}$$

The poles are then

$$p_1 = 10^4(-1 + \sqrt{.8}) \simeq -1055.73, \quad p_2 = 10^4(-1 - \sqrt{.8}) \simeq -18944.3 \qquad \textbf{(1 point)}$$

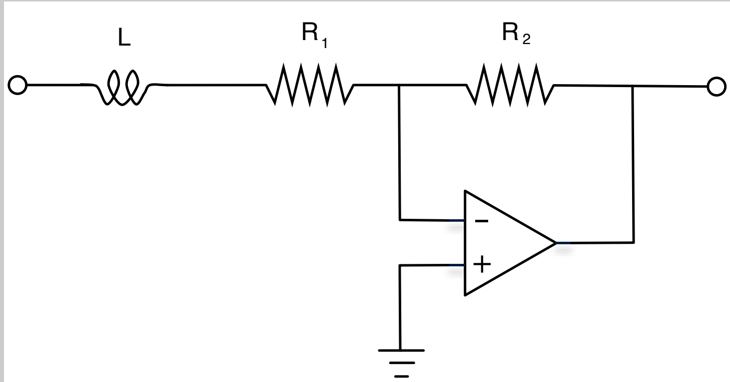
Therefore, $2 \cdot 10^7 + 2 \cdot 10^4 s + s^2 = (s - p_1)(s - p_2)$. We use this information to factorize the denominator of the transfer function as

$$T(s) = \frac{-A}{s - p_1} \cdot \frac{-B}{s - p_2}$$

So we choose $\alpha = -p_1$ and $\beta = -p_2$. Any A and B whose product is $2 \cdot 10^7$ will do, for instance, $A = 20000$ and $B = 1000$. **(1 point)**

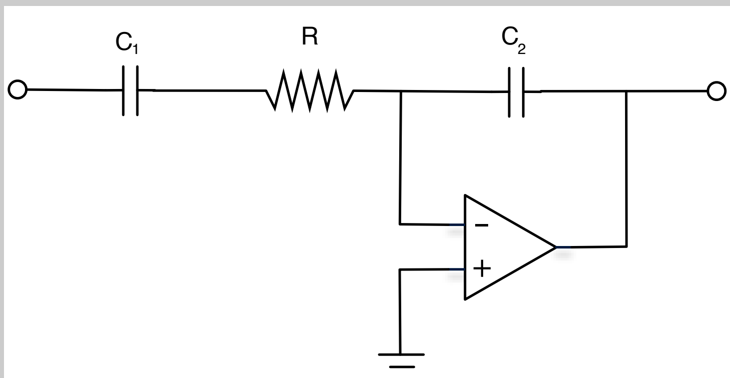
Part IV: [3 points] Design an inverting op-amp circuit whose transfer function is $\frac{-A}{s+\alpha}$ and another inverting op-amp circuit whose transfer function is $\frac{-B}{s+\beta}$, with the numerical values for A, B, α, β you chose in Part III. What is the transfer function of the series connection of these two inverting op-amps?

Solution: There are several ways to do this. For instance, to get $\frac{-A}{s+\alpha}$, we use an inductor and two resistors together with the op-amp, like this



with $R_1 = \alpha$ Ohms, $R_2 = A$ Ohms, and $L = 1$ H. **(1 point)**

To get $\frac{-B}{s+\beta} = \frac{-B/s}{1+\beta/s}$, we change our design and use two capacitors and one resistor together with the op-amp, like this



with $R = 1$ Ohms, $C_1 = 1/\beta$ F, and $C_2 = 1/B$ F. **(1 point)**

The transfer function of the series connection of the two inverting op-amps is the product of both (there is no loading since stage 2 is connected to the output of an op-amp in stage 1), which is precisely $T(s)$! **(1 point)**

Part V: [1 bonus point] Let us call circuit "B" the series connection of the two inverting op-amps that you obtained in Part IV. Would the student with the rusty recollection of MAE140 have the same problem he had in Part II if he were to connect two identical copies of circuit "B" in series? Why?

Solution: No, he would not, because the zero output impedance of the op-amp avoids loading. So if the student connected two identical copies of circuit "B" in series, he would obtain a circuit whose transfer function is in fact $T(s) \cdot T(s) = T(s)^2$, as he originally intended. **(1 point)**