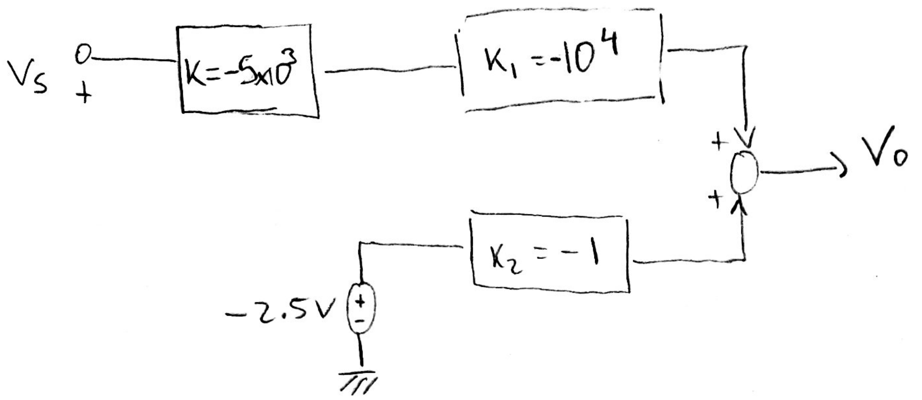
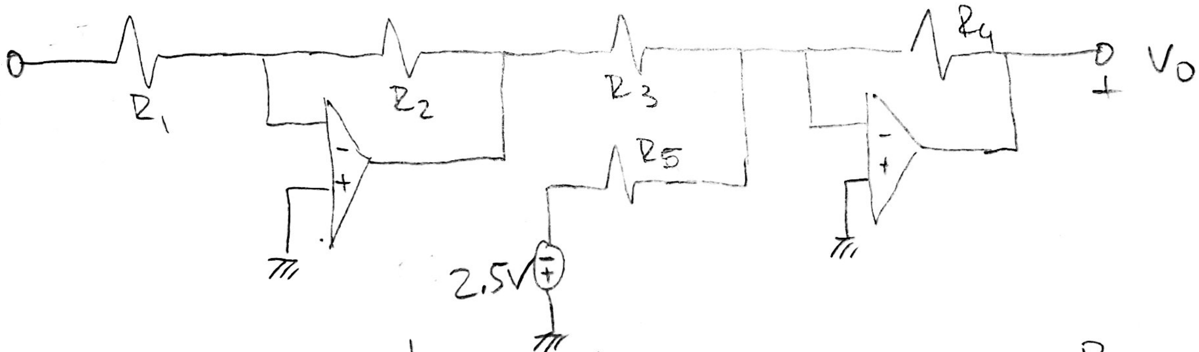
Homework 6:

4.69) Since we need a value of $5\times10^7 v_s$, we use two stages. We show the idea in the next diagram:



In the first stage, we use an inverting amplifier. For the second stage we use an inverting summer. The circuit is show next:

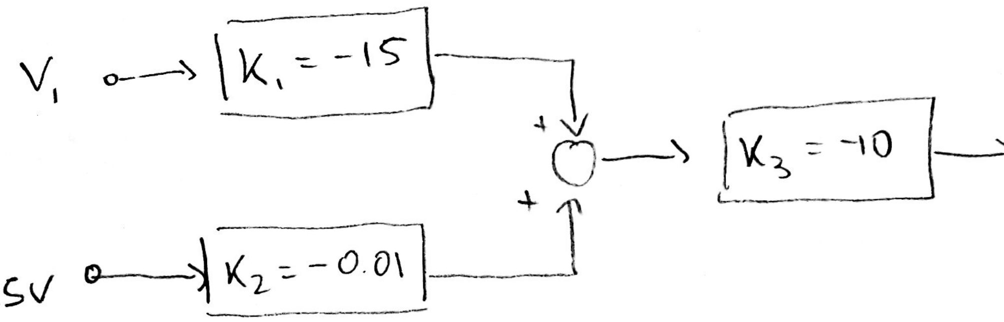


The gain for the first inverting amp. is $K=\frac{-R_2}{R_1}$. We pick $R_2 = 5\times10^6\,\Omega$ and $R_1 = 1\times10^3$, so $K=-5\times10^3$.

For the second stage, we pick $R_3 = 1\times10^3\,\Omega$, $R_4 = 10\times10^6\,\Omega$, and $R_5 = 10\times10^6\,\Omega$. We get the gains $K_1 = -\frac{R_4}{R_3} = -10.000$ and

$$K_2 = -\frac{R_4}{R_5} = -\frac{10\times10^6}{10\times10^6} = -1.$$

4.73) The output should be $V_2 = 150V_1 + 1.5\,V$. For that, we use a summer connected to an inverted amplifier, i.e.,



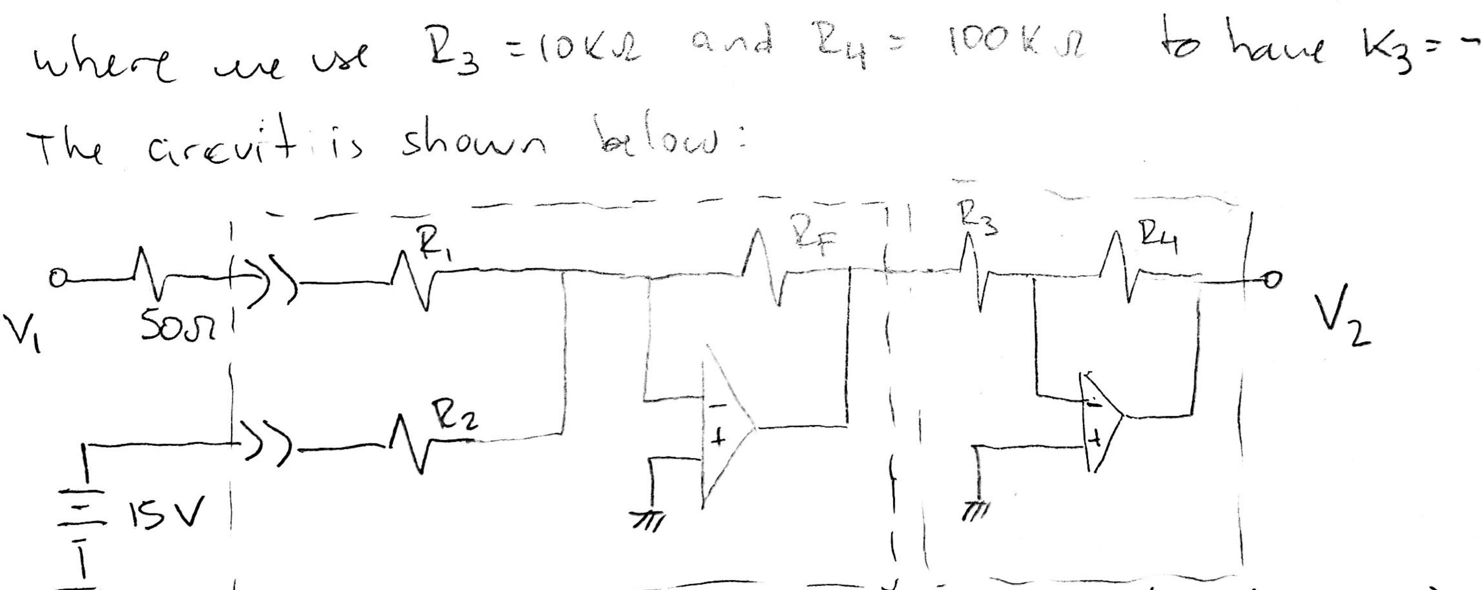
$$V_2 = (-15)(-10)V_1 + (-0.01)(10)15 = 150V_1 + 1.5V$$

We design the summer to have gains $K_1 = -15$ and $K_2 = -0.01$. For that, recall that $K_1 = -\frac{R_F}{R_1}$ and $K_2 = -\frac{R_F}{R_2}$ (see the circuit below)

We pick $R_F = 150K\Omega$, $R_1 = 10K\Omega$, and $R_2 = 15M\Omega$.

For the inverting amplifier, recall that $K_3 = -\frac{R_4}{R_3}$, where we use $R_3 = 10K\Omega$ and $R_4 = 100K\Omega$ to have $K_3 = -10$

The circuit is shown below:



Notice that the input resistance of 50Ω is insignificant compared with $R_1 = 10K\Omega$.

9.6) $f(t) = A\left((B + \alpha t)e^{-\alpha t}\right)u(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{ABe^{-\alpha t}u(t) + \alpha Ate^{-\alpha t}u(t)\}$$

$$= \mathcal{L}\{ABe^{-\alpha t}u(t)\} + \mathcal{L}\{\alpha Ate^{-\alpha t}u(t)\} \quad \text{(Linearity)}$$

$$= AB\,\mathcal{L}\{e^{-\alpha t}u(t)\} + \alpha A\,\mathcal{L}\{te^{-\alpha t}u(t)\} \quad \text{(Linearity)}$$

$$= \frac{AB}{s+\alpha} + \frac{\alpha A}{(s+\alpha)^2} \quad \text{(table)}$$

$$= \frac{AB(s+\alpha) + \alpha A}{(s+\alpha)^2}$$

$$= \frac{A\left(B(s+\alpha) + \alpha A\right)}{(s+\alpha)^2}$$

Poles: two poles located at $s = -\alpha$

Zeros: $Bs + \alpha(B+1) = 0$

$$s = -\frac{\alpha}{B}(B+1)$$

Therefore, it has one zero located at $s = \frac{-\alpha}{B}(B+1)$

and other at $+\infty$

9.7) $f(t) = (5 - 5\cos(500t))\,u(t)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{5u(t)\} - \mathcal{L}\{5\cos(500t)\,u(t)\}$ (linearity)

$= \dfrac{5}{s} - \dfrac{5s}{s^2+500^2}$ (table)

$= \dfrac{5(s^2+500^2) - 5s^2}{s(s^2+500^2)} = \dfrac{500^2}{s(s^2+500^2)}$

Poles: $s = 0$
$s = \pm 500j$

Zeros: Three zeros at $+\infty$

9.9) $f(t) = \delta'(t) + \delta(t) - e^{-t}u(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\delta'(t)\} + \mathcal{L}\{\delta(t)\} - \mathcal{L}\{e^{-t}u(t)\} \quad \text{(linearity)}$$

$$= \mathcal{L}\left\{\frac{d}{dt}\delta(t)\right\} + 1 - \frac{1}{s+1}$$

(table)

$$= s + 1 - \frac{1}{s+1}$$

$$= \frac{s(s+1) + (s+1) - 1}{s+1}$$

$$= \frac{s^2 + s + s + \cancel{1} - \cancel{1}}{s+1}$$

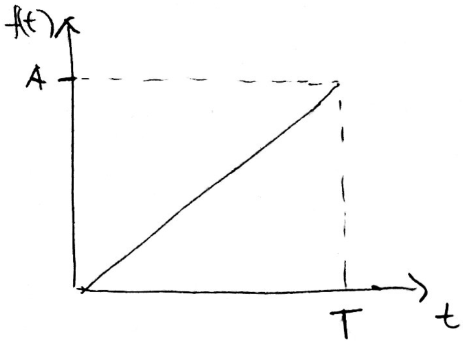
$$= \frac{s^2 + 2s}{s+1}$$

$$= \frac{s(s+2)}{s+1}$$

Zeros: $s = 0$
$s = -2$

Poles: $s = -1$
$s = +\infty$

9.17)



The function is a ramp with slope $\frac{A}{T}$.

Recall that $u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$ and $u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t \geq T \end{cases}$

Then, we can describe the function as:

a) $f(t) = \frac{A}{T} t\, u(t) - \frac{A}{T} t\, u(t-T)$

b) We use t-domain translation to find the Laplace transform, but first we put the function in a more managable way:

$$f(t) = \frac{A}{T} t\, u(t) - \frac{A}{T}(t+T-T)\, u(t-T)$$

$$= \frac{A}{T} t\, u(t) - \frac{A}{T}(t-T)\, u(t-T) - \frac{A\cancel{T}}{\cancel{T}} u(t-T)$$

$$\mathcal{L}\{f(t)\} = \frac{A}{Ts^2} - \frac{A}{Ts^2} e^{-Ts} - \frac{A}{s} e^{-Ts}$$

$$= \frac{A - Ae^{-Ts} - TAse^{-Ts}}{Ts^2}$$

9.17) part (c):

$$F(s) = \int_{0^-}^{+\infty} f(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{T} \frac{A}{T}\, t\, u(t)\, e^{-st}\, dt$$

Next, we use integration by parts, i.e., given two function w and v, we have

$$\int w(x)\, v'(x)\, dx = w(x)\, v(x) - \int v(x)\, w'(x)\, dx.$$

In our case we use $w(t) = \frac{A\,t}{T}$ and $v(t) = -\frac{1}{s} e^{-st}$.

It follows:

$$F(s) = -\frac{A}{Ts}\, e^{-st} \Big|_{t=0}^{t=T} - \int_0^T -\frac{A}{Ts}\, e^{-st}\, dt$$

$$= \frac{-A}{s}\, e^{-sT} + \left(-\frac{A}{Ts^2}\, e^{-st} \Big|_0^T \right)$$

$$= \frac{-A}{s}\, e^{-sT} - \frac{A}{Ts^2}\, e^{-sT} + \frac{A}{Ts^2}$$

which gives the same result as in part (b)

9.22)

a) $F_1(s) = \dfrac{s}{(s+10)(s+40)}$

We decompose $F(s)$ into a partial fraction expansion:

$$\frac{s}{(s+10)(s+40)} = \frac{K_1}{s+10} + \frac{K_2}{s+40}$$

$$K_1 = (s+10)\,F(s)\Big|_{s=-10} = \frac{(s+10)\,s}{(s+10)(s+40)} = \frac{-10}{-10+40} = \frac{-10}{30} = -\frac{1}{3}$$

$$K_2 = (s+40)\,F(s)\Big|_{s=-40} = \frac{s}{s+10}\Big|_{s=-40} = \frac{-40}{-40+10} = \frac{-40}{-30} = +\frac{4}{3}$$

$$F_1(s) = \left(-\frac{1}{3}\right)\left(\frac{1}{s+10}\right) + \frac{3}{4}\left(\frac{1}{s+40}\right)$$

$$f_1(t) = \left(-\frac{1}{3}e^{-10t} + \frac{4}{3}e^{-40t}\right)u(t)$$

b) $F_2(s) = \dfrac{(s+1)(s+10)}{s(s+100)(s+1000)}$

We use partial fraction: $F_2(s) = \dfrac{K_1}{s} + \dfrac{K_2}{s+100} + \dfrac{K_3}{s+1000}$

$$K_1 = s\,F_2(s)\Big|_{s=0} = \frac{(s+1)(s+10)}{(s+100)(s+1000)}\Big|_{s=0} = \frac{10}{100000} = \frac{1}{10000}$$

$$K_2 = (s+100)\,F_2(s)\Big|_{s=-100} = \frac{(s+1)(s+10)}{s(s+1000)}\Big|_{s=-100} = \frac{(-99)(-90)}{-100(900)} = -\frac{99}{1000}$$

$$K_3 = (s+1000)\,F_2(s)\Big|_{s=-1000} = \frac{(s+1)(s+10)}{s(s+100)}\Big|_{s=-1000} = \frac{(-999)(-990)}{-1000(-900)} = \frac{98901}{90000}$$

$$F_2(s) = \frac{1/10000}{s} - \frac{99/1000}{s+100} + \frac{98901/90000}{s+1000}$$

$$f_2(t) = \frac{1}{10000}u(t) - \frac{99}{1000}e^{-100t}u(t) - \frac{98901}{90000}e^{-1000t}u(t)$$

9.31) a) $F_1(s) = \dfrac{(s+10^6)(s+10^7)}{s(s+10^5)(s+10^8)}$

We use partial fraction:

$$F_1(s) = \frac{K_1}{s} + \frac{K_2}{(s+10^5)} + \frac{K_3}{s+10^8}$$

$$K_1 = \frac{(s+10^6)(s+10^7)}{(s+10^5)(s+10^8)}\Big|_{s=0} = \frac{(10^6)(10^7)}{(10^5)(10^8)} = 1$$

$$K_2 = \frac{(s+10^6)(s+10^7)}{s(s+10^8)}\Big|_{s=-10^5} = \frac{(-10^5+10^6)(-10^5+10^7)}{-10^5(-10^5+10^8)} = -\frac{33}{37}$$

$$K_3 = \frac{(s+10^6)(s+10^7)}{s(s+10^5)}\Big|_{s=-10^8} = \frac{(-10^8+10^6)(-10^8+10^7)}{-10^8(-10^8+10^5)} = \frac{33}{37}$$

$$F_1(s) = \frac{1}{s} - \frac{33/37}{s+10^5} + \frac{33/37}{s+10^8}$$

$$f_1(t) = u(t) - \frac{33}{37}e^{-10^5 t}u(t) + \frac{33}{37}e^{-10^8 t}u(t)$$

b) $F_2(s) = \dfrac{5(s^4+10s^2+4)}{s(s^2+1)(s^2+4)}$

We use partial fraction:

$$F_2(s) = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{s^2+4}$$

$$5(s^4+10s^2+4) = A(s^2+1)(s^2+4) + (Bs+C)s(s^2+4) + (Ds+E)s(s^2+1)$$
$$= A(s^4+5s^2+4) + Bs^4+4Bs^2+Cs^3+4Cs + (Ds^4+Ds^2+Es^3+Es)$$

Equalizing terms:

$5 = A + B + D$

$0 = C + E$

$50 = 5A + 4B + D$

$0 = 4C + E$

$20 = 4A$

We have 5 equations and 5 unknowns. By solving we get:
$A = 5$, $B = \frac{25}{3}$, $C = 0$, $D = -\frac{25}{3}$, $E = 0$

9.31) Continuation.

$$F_2(s) = \frac{5}{s} + \frac{\frac{25}{3}s}{s^2+1} + \frac{-\frac{25}{3}s}{s^2+4}$$

$$f_2(t) = 5u(t) + \left(\frac{25}{3}\cos(t) - \frac{25}{3}\cos(2t)\right)u(t)$$

9.36) a) $$F_1(s) = \frac{s(s+10)(s+100)}{(s+1)(s+1000)(s+10000)} = \frac{s^3+110s^2+1000s}{s^3+11001s^2+10011000s+10^7}$$

Notice that $F_1(s)$ is not strictly proper rational function since the order of the numerator and denominator are equal.

We begin by performing a long division in order to change the function into a quotient and a remainder

$$\begin{array}{r} 1 \\ s^3+11001s^2+10011000s+10^7 \overline{\big)\, s^3+110s^2+1000s} \\ -s^3-11001s^2-10011000s-10^7 \\ \hline -10891s^2-10010000s-10^7 \end{array}$$

We have,

$$F(s) = 1 - \frac{10891s^2+10010000s+10^7}{(s+1)(s+1000)(s+10000)} = 1 + \frac{K_1}{s+1} + \frac{K_2}{s+1000} + \frac{K_3}{s+10000}$$

we use partial fraction

$$K_1 = \left.\frac{-(10891s^2+10010000s+10^7)}{(s+1000)(s+10000)}\right|_{s=-1} = \frac{-(+10891-10010000+10^7)}{(999)(9999)} = -8.91\cdot10^{-5}$$

$$K_2 = \left.\frac{-(10891s^2+10010000s+10^7)}{(s+1)(s+10000)}\right|_{s=-1000} = 99.099$$

$$K_3 = \left.\frac{-(10891s^2+10010000s+10^7)}{(s+1)(s+1000)}\right|_{s=-10000} = -1\times10^4$$

$$f(t) = \delta(t) + \left(-8.91\cdot10^{-5}e^{-t} + 99.099e^{-1000t} - 1\times10^4 e^{-10000t}\right)u(t)$$

b) $F_2(s) = \frac{(s+1000)(s+100000)}{(s+10\,000)} = \frac{s^2 + 101000s + 10^8}{s + 10\,000}$

The function is improper, so we proceed as follows:

$$\begin{array}{r|l} & s + 91000 \\ \hline s+10000 & s^2 + 101000s + 10^8 \\ & -s^2 - 10\,000s \\ \hline & +91000s + 10^8 \\ & -91000s - 910000000 \\ \hline & -81\times10^7 \end{array}$$

$$F_2(s) = s + 91000 - \frac{81\times10^7}{s+10000}$$

Using the Laplace transform found in Problem 9.9, we have

$$F_2(t) = \frac{d\,\delta(t)}{dt} + 91000\,\delta(t) - 81\times10^7 e^{-10000t} u(t)$$

9.28) a) $F_1(s) = \dfrac{600}{(s+10)(s+20)(s+30)} = \dfrac{3}{s+10} + \dfrac{K}{s+20} + \dfrac{3}{s+30}$

For graders:
For this problem, we are not required to use sum of residues.

Here we have that $F_1(s)$ is proper rational function with $m = 0$ and $n = 3$. It follows that $n > m+1$ and the sum of residues is 0.

To find K, we have $0 = 3 + K + 3 \Rightarrow K = -6$

b) $F_2(s) = \dfrac{2(s+10)}{(s+15)(s+20)} = \dfrac{K}{s+15} + \dfrac{4}{s+20}$

We have that $F_2(s)$ is proper rational function with $m = 1$ and $n = 2$. Since $n = m+1$, it follows that $2 = K + 4 \Rightarrow K = -2$