MAE143 A - Signals and Systems - Winter 11
Final

Instructions

(i) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 3 hours

(iii) Do not forget to write your name, student number, and instructor

1. **Filtering and sampling.** The continuous-time signal $x(t) = 100\,\mathrm{sinc}^2(100t)$ goes through an ADC block that samples signals at a frequency $f_s = 190$ Hz. Answer the following questions

   (a) (1 point) Compute the Fourier transform of $x(t)$. What is its bandwidth?

   (b) (1 point) Would you be able to reconstruct the original signal out of the samples taken by the ADC block? Why?

   (c) (2 points) Consider a system whose impulse response is given by

   $$h(t) = 150\,\mathrm{sinc}(150t)$$

   What kind of filter is this? What is its cutoff frequency in Hz? Plot the magnitude and phase of the transfer function.

   (d) Suppose the signal $x(t)$ passes first through the system in (c) to produce the signal $y(t)$ and then goes through the ADC block to produce the samples, see Figure 1.

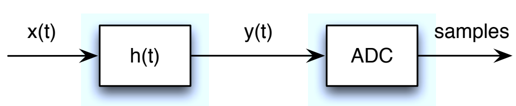


Figure 1: Block diagram for question 3, part (d).

   i. (1 point) What is the bandwidth of $y(t)$? Would you be able to reconstruct the signal $y(t)$ out of the samples taken by the ADC block? Justify your answer.

   ii. (1 point) Why would one call the system in (c) an antialiasing filter?

   (e) (1 point (bonus)) With knowledge of $y(t)$, would you be able to recover the original signal $x(t)$? Why?

Solution: (a) Using the table of basic transforms and properties (time scaling), the Fourier transform of $x(t)$ is

$$X(f) = \mathrm{tri}\left(\frac{f}{100}\right)$$ **(+ .5 point)**

The bandwidth of this signal is therefore $f_B = 100$ Hz. **(+ .5 point)**

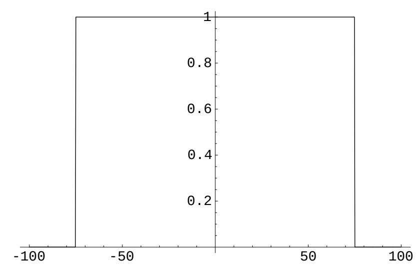
(b) No, because the sampling frequency is below the Nyquist rate, $f_s = 190 < 2f_B = 200$. **(+ 1 point)**

(c) We compute the transfer function of the system as

$$H(f) = \text{rect}\left(\frac{f}{150}\right) \qquad \textbf{(+ 1 point)}$$

Therefore, this is an ideal lowpass filter with cutoff frequency $f_c = 75$ Hz. **(+ .5 point)**

The phase plot of the filter is trivial (identically zero). The magnitude plot looks like



(+ .5 point)

(d.i) After the signal $x(t)$ goes through the ideal filter, all frequencies above the cutoff frequency (75 Hz) get cut. Therefore, the bandwidth of $y(t)$ is 75 Hz. **(+ .5 point)**

Since the sampling frequency of the ADC block is $190 > 150 = 2 * 75$, we will be able to reconstruct the signal $y(t)$ out of the samples produced by the ADC block. **(+ .5 point)**

(d.ii) The reason for the antialiasing name comes from the effect that this filter has on the signal. Since it makes sure that no frequency larger than $\frac{f_s}{2}$ will get into the ADC block, it eliminates the possibility of aliasing when performing the sampling. **(+ 1 point)**

(e) No, an ideal filter is not invertible. **(+ 1 extra point)**

2. **Continuous-time Fourier Series (CTFS) and system response.** Consider a periodic function described over one fundamental period ($T_0 = 4$) by

$$x(t) = \begin{cases} 1 - t, & t \in [0, 1], \\ 0, & t \in [-2, 0] \cup [1, 2]. \end{cases}$$

Do the following

(a) (.5 points) Plot $x(t)$ over the interval $[-6, 6]$. Could this signal correspond to the impulse response of a causal system? Why?

(b) (2 points) Show that the harmonic numbers of $x(t)$ with representation time $T_F = T_0$ are

$$X[0] = \frac{1}{8},$$
$$X[k] = \frac{1}{\pi^2 k^2}\left(1 - j\frac{\pi}{2}k - e^{-j\frac{\pi}{2}k}\right), \qquad k \in \mathbb{Z} \setminus \{0\}.$$

(c) (1.5 points) Compute the transfer function of the LTI system

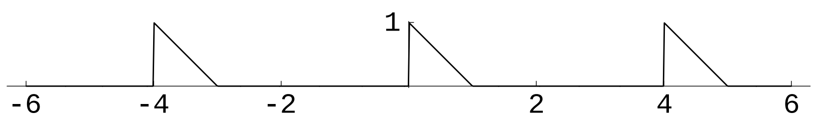
$$\dot{y}(t) + y(t) = x(t)$$

What is the impulse response? Is the system BIBO stable?

(d) (2 points) Show, using the CTFS from (b) and the transfer function from (c), that the response $y(t)$ of the LTI system to the input signal $x(t)$ is

$$y(t) = X[0] + \sum_{k=1}^{\infty} \frac{4}{\sqrt{4+\pi^2 k^2}} |X[k]| \cos\left(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}) + \angle X[k]\right)$$

Solution: (a) Over $[-2, 2]$, the function $x(t)$ is only nonvanishing on $[0, 1]$, where it looks like a unit triangle. It has fundamental period $T_0 = 4$. Therefore, the plot of $x(t)$ over the interval $[-6, 6]$ is



(+ .25 point)

The function cannot be the impulse response of a causal system because there are values of time before $t = 0$ for which the function is nonzero. Since the impulse is exerted at $t = 0$, a causal system cannot anticipate the future to make this happen. **(+ .25 point)**

(b) To compute the Fourier series of $x(t)$, we need to compute the harmonic numbers

$$X[k] = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-j2\pi k f_0 t} dt \qquad \textbf{(+ .5 point)}$$

We select $t_0 = -T_0/2 = -2$ (any choice would give the same answer since the function is periodic). For $k = 0$, we get

$$X[0] = \frac{1}{4} \int_0^1 (1-t) dt = \frac{1}{8}. \qquad \textbf{(+ .5 point)}$$

Next, we reason for the case when $k \neq 0$. Over the interval $[-2, 2]$, the function is only non-vanishing in $[0, 1]$. Then,

$$X[k] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi k f_0 t} dt = \frac{1}{4} \int_0^1 (1-t) e^{-j\frac{\pi}{2}kt} dt$$

Note that, using integration by parts,

$$\int_a^b t e^{-ct} dt = -\frac{1}{c} t e^{-ct}\Big|_a^b + \frac{1}{c} \int_a^b e^{-ct} dt = -\frac{1}{c} t e^{-ct}\Big|_a^b - \frac{1}{c^2} e^{-ct}\Big|_a^b \qquad \textbf{(+ .5 point)}$$

Substituting above, we get (for $c = j\frac{\pi}{2}k$)

$$\begin{aligned}
X[k] &= \frac{1}{4}\left(-\frac{1}{c}e^{-ct} + \frac{1}{c}te^{-ct} + \frac{1}{c^2}e^{-ct}\right)\Big|_0^1 \\
&= \frac{1}{4}\left(-\frac{1}{c}e^{-c} + \frac{1}{c}e^{-c} + \frac{1}{c^2}e^{-c}\right) - \frac{1}{4}\left(-\frac{1}{c} + 0 + \frac{1}{c^2}\right) \\
&= \frac{1}{4c^2}\left(e^{-c} + c - 1\right) = \frac{1}{\pi^2 k^2}\left(1 - j\frac{\pi}{2}k - e^{-j\frac{\pi}{2}k}\right) && \textbf{(+ .5 point)}
\end{aligned}$$

(c) We use the Laplace transform to get

$$sY(s) + Y(s) = X(s) \qquad \textbf{(+ .25 point)}$$

Therefore, the transfer function is

$$H(x) = \frac{Y(s)}{X(s)} = \frac{1}{s+1} \qquad \textbf{(+ .25 point)}$$

The impulse response of the system can be computed by calculating the inverse Laplace transform of the transfer function. Hence,

$$h(t) = \mathcal{L}^{-1}(H(s)) = e^{-t}u(t) \qquad \textbf{(+ .5 point)}$$

The system is BIBO stable because $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ **(+ .5 point)**

(d) Since we have the CTFS of $x(t)$ and the transfer function of the system, we know that the response is

$$y(t) = \sum_{k=-\infty}^{\infty} X[k] H\left(j\frac{\pi k}{2}\right) e^{j\frac{\pi}{2}kt} \qquad \textbf{(+ .5 point)}$$

Let us compute the magnitude and the phase of $H\left(j\frac{\pi k}{2}\right)$. Note that

$$H\left(j\frac{\pi k}{2}\right) = \frac{1}{1 + j\frac{\pi k}{2}}$$

Therefore,

$$\left|H\left(j\frac{\pi k}{2}\right)\right| = \frac{2}{\sqrt{4 + \pi^2 k^2}}, \qquad \angle H\left(j\frac{\pi k}{2}\right) = 0 - \arctan\left(\frac{\pi k}{2}\right) \qquad \textbf{(+ .5 point)}$$

Then, we can write the response $y(t)$ as

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} \frac{2}{\sqrt{4+\pi^2 k^2}} X[k] e^{j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}))} \\
&= X[0] + \sum_{k=-\infty}^{-1} \frac{2}{\sqrt{4+\pi^2 k^2}} X[k] e^{j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}))} + \sum_{k=1}^{\infty} \frac{2}{\sqrt{4+\pi^2 k^2}} X[k] e^{j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}))} \\
&= X[0] + \sum_{k=1}^{\infty} \frac{2}{\sqrt{4+\pi^2 k^2}} X[-k] e^{-j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}))} + \sum_{k=1}^{\infty} \frac{2}{\sqrt{4+\pi^2 k^2}} X[k] e^{j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}))}
\end{aligned}$$

(+ .5 point)

Using the facts that $|X[k]| = |X[-k]|$ and $\angle X[-k] = -\angle X[k]$, we write

$$
\begin{aligned}
y(t) &= X[0] + \sum_{k=1}^{\infty} \frac{2}{\sqrt{4+\pi^2k^2}} |X[k]| \left(e^{-j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}) + \angle X[k])} + e^{j(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}) + \angle X[k])} \right) \\
&= X[0] + \sum_{k=1}^{\infty} \frac{4}{\sqrt{4+\pi^2k^2}} |X[k]| \cos\left(\frac{\pi}{2}kt - \arctan(\frac{\pi k}{2}) + \angle X[k] \right)
\end{aligned}
$$

(+ .5 point)

3. **Continuous-time Fourier Transform (CTFT).** Consider the periodic signal

$$z(t) = \text{tri}(t-1) * \delta_2(t)$$

Do the following

(a) (1 point) Plot $z(t)$ over the interval $[-6, 6]$. What is the fundamental period of $z(t)$?

(b) (2 points) Show, using only the tables of basic CTFT transforms and basic CTFT properties, that the Fourier transform of $z(t)$ is

$$Z(f) = \frac{1}{2}\delta(f) - \sum_{k=-\infty}^{\infty} \frac{2}{\pi^2(2k-1)^2} \delta\left(f - \frac{2k-1}{2}\right)$$

and sketch a plot of its magnitude $|Z(f)|$.

(c) (3 points) Consider now the signal

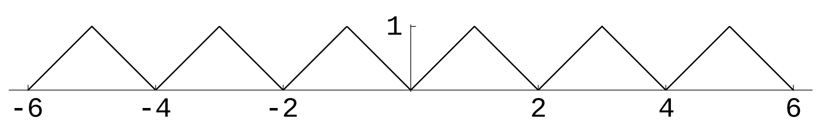
$$u(t) = z(t)\,\text{rect}\left(\frac{t-1}{4}\right)$$

Plot the signal $u(t)$ on the interval $[-6, 6]$. Compute the Fourier transform of $u(t)$ using the tables of basic CTFT transforms and basic CTFT properties together with the solution to (b).

(d) (1 point (bonus)) The effect in the time domain that the multiplication by the rectangle function has on $z(t)$ is called 'windowing.' Can you explain what is the effect of windowing in the spectrum of $z(t)$?

Solution: (a) The signal $z(t)$ is a time-shifted unit triangle function with fundamental period $T_0 = 2$. **(+ .5 point)**

Its plot looks like



(+ .5 point)

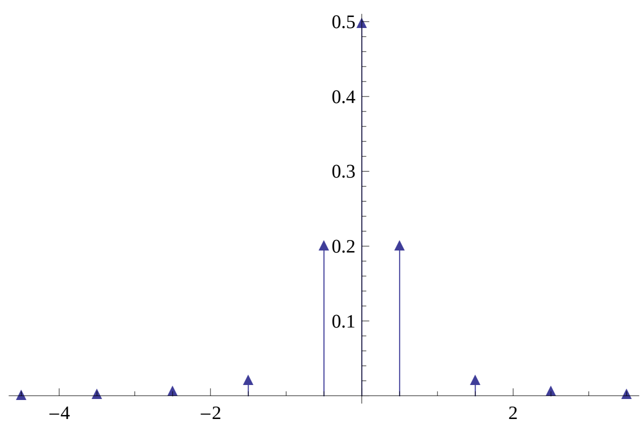
(b) Since $z(t)$ is periodic, its Fourier transform corresponds to an impulse train whose coefficients are the harmonic numbers of its Fourier series. Using the multiplication-convolution property, the Fourier transform of $z(t)$ is

$$Z(f) = \mathcal{F}(\mathrm{tri}(t-1))\mathcal{F}(\delta_2(t)) \qquad \textbf{(+ .5 point)}$$

From the table of transforms in the book, we know $\mathcal{F}(\mathrm{tri}(t)) = \mathrm{sinc}^2(f)$ and $\mathcal{F}(\delta_2(t)) = \frac{1}{2}\delta_{1/2}(f)$. Using now the time shifting property of the Fourier transform, we get

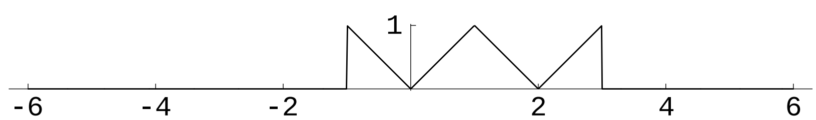
$$\begin{aligned} Z(f) &= \mathrm{sinc}^2(f) e^{-j2\pi f 1} \frac{1}{2}\delta_{1/2}(f) = \frac{1}{2} e^{-j2\pi f}\,\mathrm{sinc}^2(f)\delta_{1/2}(f) && \textbf{(+ .5 point)} \\ &= \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{-j\pi k}\,\mathrm{sinc}^2\left(\frac{k}{2}\right)\delta\left(f - \frac{k}{2}\right) \\ &= \frac{1}{2}\delta(f) + \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{-j\pi(2k-1)}\,\mathrm{sinc}^2\left(\frac{2k-1}{2}\right)\delta\left(f - \frac{2k-1}{2}\right) \\ &= \frac{1}{2}\delta(f) - \sum_{k=-\infty}^{\infty} \frac{2}{\pi^2(2k-1)^2}\delta\left(f - \frac{2k-1}{2}\right) && \textbf{(+ .5 point)} \end{aligned}$$

The plot of $|Z(f)|$ looks like



(+ .5 point)

(c) The rectangle function $\mathrm{rect}\left(\frac{t-1}{4}\right)$ is only nonzero over the interval $[-1, 3]$. The plot $u(t)$ over the interval $[-6, 6]$ is then



(+ .5 point)

The signal $u(t)$ is not periodic anymore. Using the multiplication-convolution property, the Fourier Transform of $u(t)$ is

$$U(f) = Z(f) * \mathcal{F}\left(\text{rect}\left(\frac{t-1}{4}\right)\right) \qquad \textbf{(+ .5 point)}$$

From the table of transforms in the book, we know $\mathcal{F}(\text{rect}(t)) = \text{sinc}(f)$. Using the time shifting and time scaling properties of the Fourier transform, we get

$$\begin{aligned} U(f) &= Z(f) * \left(4\,\text{sinc}(4f)e^{-j2\pi f}\right) & \textbf{(+ 1 point)} \\ &= \left(\frac{1}{2}\delta(f) - \sum_{k=-\infty}^{\infty} \frac{2}{\pi^2(2k-1)^2}\delta\left(f - \frac{2k-1}{2}\right)\right) * \left(4\,\text{sinc}(4f)e^{-j2\pi f}\right) \\ &= 2\,\text{sinc}(4f)e^{-j2\pi f} + \sum_{k=-\infty}^{\infty} \frac{8}{\pi^2(2k-1)^2} e^{-j2\pi f}\,\text{sinc}\left(4f - 2(2k-1)\right) & \textbf{(+ 1 point)} \end{aligned}$$

where we have used the fact that $e^{j\pi(2k-1)} = -1$.

(d) The effect that windowing has in the frequency domain is to spread the spectrum of the signal $z(t)$. This signal is periodic, and hence its energy is concentrated at frequencies $f = (2k-1)/2$ (deltas at these frequencies in the expression for the Fourier transform). As can be seen in our answer in (c), the Fourier transform of the signal $u(t)$ substitutes the deltas by sinc functions that spread out the energy of the original signal throughout the whole spectrum. **(+ 1 extra point)**