MAE143 A - Signals and Systems - Winter 11
Midterm, February 2nd

Instructions

(i) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 50 minutes

(iii) Do not forget to write your name, student number, and instructor

1. **Signals**

Consider the following mathematical description of a continuous-time signal

$$x(t) = u(t-1) - (1 - e^{-(t-2)})u(t-2) - \delta(t+1).$$

Sketch the plot of the following derived signals:

(a) (2 points) $x(t)$

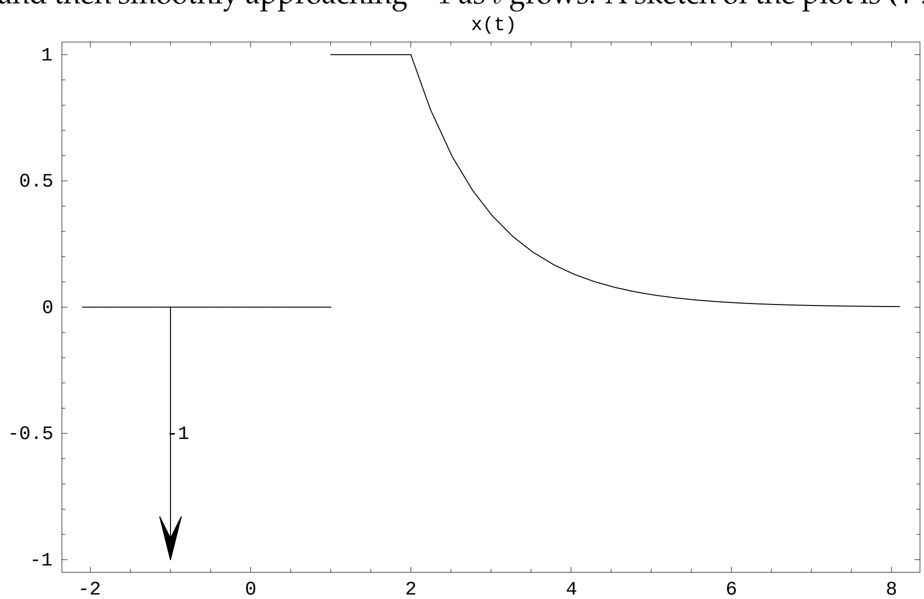
(b) (2 points) $x(2-t)$

(c) (2 points) $x(t/2)$

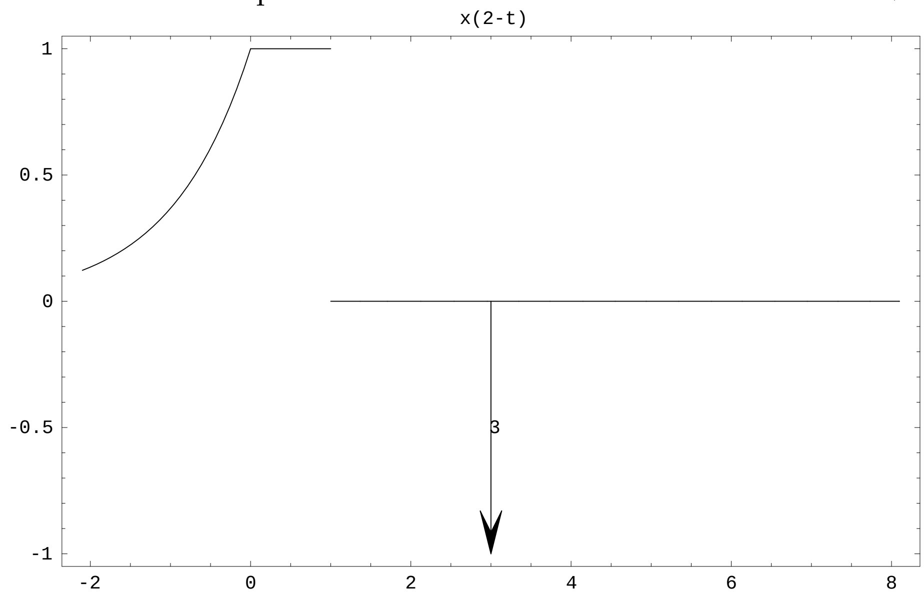
Solution:

NOTE: In this question we are not being picky about the value of $u(t)$ at $t = 0$!

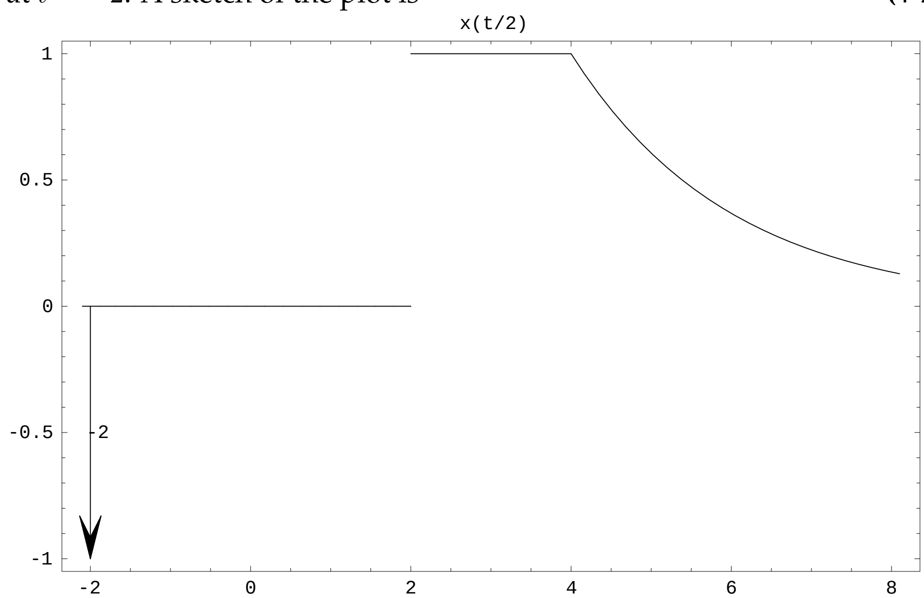
(a) Start with the third summand, which is a negative impulse at time $t = -1$. This is the only nonzero value taken by the function for $t < 1$. At $t = 1$, the first summand, a unit step, which start adding 1 to the value of the function. Finally, the second summand is zero for $t < 2$, so $x(t) \equiv 1$ for $1 < t < 2$. For $t \geq 2$, this summand is negative, with value 0 at $t = 2$, and then smoothly approaching -1 as t grows. A sketch of the plot is **(+ 2 points)**



(b) The function $x(-t)$ has the same plot as the one above except that it is mirrored with respect to the vertical axis at 0. For example, $x(-t)$ at $t = 1$ is the impulse. The final function $x(2-t)$ is then shifted in time by two seconds. For example, $x(2-t)$ at $t = 3$ is the impulse. A sketch of the plot is **(+ 2 points)**



(c) The function $x(t/2)$ has its time streched by a factor of 2. For example, the impulse happens at $t = -2$. A sketch of the plot is **(+ 2 points)**



2. **System Properties**

   A system takes as input the signal $x(t)$ and produces as output $y(t)$. Provide a detailed answer to the following question regarding properties of the systems:

   (a) (2 points) If $y(t) = \tan^{-1}(x(t))$, is the system linear? Is it invertible?

(b) (2 points) If $y(t) = \frac{1}{t}\int_0^t x(\tau)\,d\tau$ is the system linear? Is it time-invariant?

(c) (2 points) If $y'(t) + y(t) = x(t)$ is the system linear? Is it BIBO stable?

Solution: **(a)** The system is not linear because the function $\tan^{-1}$ is not, i.e.,

$$\tan^{-1}(x_1(t) + x_2(t)) \neq \tan^{-1}(x_1(t)) + \tan^{-1}(x_2(t)). \qquad \textbf{(+ 1 point)}$$

The system is invertible: given the response $y(t) = \tan^{-1}(x(t)) \in (-\pi, \pi]$, we can determine the excitation by taking the $\tan$, i.e.,

$$\tan(y(t)) = \tan(\tan^{-1}(x(t))) = x(t). \qquad \textbf{(+ 1 point)}$$

Note that the system $y(t) = \tan(x(t))$ is not invertible!

(b) The system is linear, i.e., homogeneous and additive. Additivity follows from

$$\frac{1}{t}\int_0^t (x_1(\tau) + x_2(\tau))\,d\tau = \frac{1}{t}\int_0^t x_1(\tau)\,d\tau + \frac{1}{t}\int_0^t x_2(\tau)\,d\tau = y_1(t) + y_2(t). \qquad \textbf{(+ .5 point)}$$

Homogeneity follows from

$$\frac{1}{t}\int_0^t Kx(\tau)\,d\tau = K\frac{1}{t}\int_0^t x(\tau)\,d\tau = Ky(t). \qquad \textbf{(+ .5 point)}$$

The system is not time-invariant, because

$$\frac{1}{t}\int_0^t x(\tau - t_0)\,d\tau = \frac{1}{t}\int_{-t_0}^{t-t_0} x(\nu)\,d\nu \neq \frac{1}{t-t_0}\int_0^{t-t_0} x(\tau)\,d\tau = y(t-t_0). \qquad \textbf{(+ 1 point)}$$

(c) The system is linear since the ODE describing it is linear of the form $y' = Ay + Bu$, with $A = -1$ and $B = 1$. **(+ 1 point)**

The unique eigenvalue of the homogeneous equation is -1, and therefore, the system is BIBO stable. **(+ 1 point)**

3. **Impulse Response**

An LTI system is described by the ODE

$$y''(t) + y'(t) = x(t)$$

(a) (3 points) Compute the impulse response $h(t)$. Use your answer to determine if the system is BIBO stable.

(b) (3 points) Use the impulse response $h(t)$ and the convolution formula to compute $y(t)$ when $x(t) = e^{-2t}u(t)$.

(c) (4 points (bonus)) Use Laplace transforms to compute the answer to the above items (a) and (b).

Solution: **(a)** Since the order of the system is $n = 2$ and the input does not appear with any derivative, $m = 0$, we are in the easiest of cases. We begin by substituting $x(t)$ by the impulse function,

$$y''(t) + y'(t) = \delta(t)$$

Now integrating both sides from $-\infty$ to $t \geq 0$ under zero initial conditions, we get

$$y'(t) + y(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

Integrating once more from $-\infty$ to $t \geq 0$ and using the zero initial conditions and the fact that $y(t)$ is impulse free, we get

$$y(t) = \int_{-\infty}^{t} u(\tau) d\tau.$$

Therefore, at $t = 0^+$, we have $y(0^+) = 0$ and $y'(0^+) = 1$.

Alternatively, you can cite the notes on computing the impulse response for justifying your choice of initial conditions. **(+ 1 point)**

Now, we need to solve the homogeneous equation

$$y''(t) + y'(t) = 0$$

with initial conditions $y(0^+) = 0$ and $y'(0^+) = 1$. Use your preferred method to solve this equation (e.g., characteristic equation or simply using the change of variables $z = y'$) to obtain the general solution

$$y(t) = k_1 - k_2 e^{-t}.$$

Fitting the initial conditions, we get

$$y(0^+) = k_1 - k_2 = 0, \qquad y'(0^+) = k_2 = 1$$

from where $k_1 = k_2 = 1$. **(+ 1 point)**

Therefore, the impulse response is

$$h(t) = (1 - e^{-t})u(t). \qquad \textbf{(+ .5 point)}$$

The system is not BIBO stable because

$$h(t) \geq (1 - e^{-1})u(t) \geq 0.6\, u(t) \quad \text{for all } t > 1$$

Therefore

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{1} |h(\tau)| d\tau + \int_{1}^{\infty} |h(\tau)| d\tau \geq \int_{1}^{\infty} |h(\tau)| d\tau \geq 0.6 \int_{1}^{\infty} u(\tau) d\tau = +\infty$$

hence the system is not BIBO stable. Here you can use any $t > 0$ to prove a lower bound on $|h(t)|$ as we did above or simply state that $\lim_{t\to\infty} |h(t)| > 0$ therefore the integral will diverge. **(+ .5 point)**

(b) We use the convolution formula to compute

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} (1-e^{-\tau})u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau \qquad \textbf{(+ 1 point)}$$

This integral is zero if $\tau \leq 0$ and $t - \tau \leq 0$. For $t \geq 0$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} (1-e^{-\tau})u(\tau)e^{-2t+2\tau}u(t-\tau)d\tau &= e^{-2t}\int_0^t (1-e^{-\tau})e^{2\tau}d\tau \\ &= e^{-2t}(\frac{1}{2}e^{2\tau} - e^{\tau})\Big|_0^t = e^{-2t}(\frac{1}{2}e^{2t} - e^t - \frac{1}{2} + 1) \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \qquad \textbf{(+ 1.5 point)}\end{aligned}$$

Therefore, $y(t) = (\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t})u(t)$. **(+ .5 point)**

(c) We use the Laplace transform to find the transfer function of the system,

$$(s^2 + s)Y(s) = X(s)$$

Therefore,

$$H(s) = \frac{1}{s + s^2} \qquad \textbf{(+ .5 point)}$$

Now compute the partial fraction expansion

$$H(s) = \frac{1}{s+s^2} = \frac{k_1}{s} + \frac{k_2}{s+1} \qquad \textbf{(+ .5 point)}$$

where the constants k_1 and k_2 are computed using the residues

$$k_1 = \lim_{s\to 0} sH(s) = \lim_{s\to 0}\frac{1}{1+s} = 1, \quad k_2 = \lim_{s\to -1}(s+1)H(s) = \lim_{s\to -1}\frac{1}{s} = -1, \quad \textbf{(+ .5 point)}$$

The impulse response is the Laplace inverse of the transfer function, hence

$$h(t) = \mathcal{L}^{-1}(H(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) = (1 - e^{-t})u(t) \qquad \textbf{(+ .5 point)}$$

Next, we find the response $y(t)$. In the frequency domain,

$$X(s) = \mathcal{L}(e^{-2t}u(t)) = \frac{1}{s+2} \qquad \textbf{(+ .5 point)}$$

and

$$Y(s) = H(s)X(s) = \frac{1}{s+s^2}\frac{1}{s+2}. \qquad \textbf{(+ .5 point)}$$

As before we produce the partial fraction expansion

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

where the constants are computed from the residues

$$k_1 = \lim_{s\to 0} sY(s) = \frac{1}{2}, \quad k_2 = \lim_{s\to -1}(s+1)Y(s) = -1, \quad k_3 = \lim_{s\to -2}(s+2)Y(s) = 1/2.$$

(+ .5 point)

Finally,

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right) = \left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right)u(t).$$

(+ .5 point)