Homework 3

18a. We set $x(t) = \delta(t)$ to get the impulse response of the system:

$$4h''(t) = 2\delta(t) - \delta'(t).$$

Following the algorithm on Page 166, we get that $n = 2, m = 1, m < n$. Therefore our solution looks like

$$h(t) = h_{hom}(t)u(t).$$

The homogeneous solution $h_{hom}(t)$ should satisfy $4h''_{hom}(t) = 0$.

$$h_{hom}(t) = C_1 t + C_2, h(t) = (C_1 t + C_2)u(t).$$

Now we need to determine constants of integration. In order to do that we integrate original equation once from $t = 0^-$ to $t = 0^+$.

$$4h'(0+) - 4h'(0-) = 2\int_{0-}^{0+} \delta(t)dt - \int_{0-}^{0+} \delta'(t)dt = 2,$$

$$h'(t) = C_1 u(t) + (C_1 t + C_2)\delta(t), \qquad h'(0-) = 0, h'(0+) = C_1.$$

Therefore we get $4C_1 = 2, \quad C_1 = 0.5$. Now we integrate the initial equation twice:

$$4h(0+) - 4h(0-) = 2\int_{0-}^{0+}\int_{-\infty}^{t} \delta(s)dsdt - \int_{0-}^{0+}\int_{-\infty}^{t} \delta'(s)dsdt = -1,$$

$$h(0-) = 0, h(0+) = C_2, \ 4C_2 = -1, \ C_2 = -0.25.$$

The final answer is: $h(t) = (0.5t - 0.25)u(t)$.

18b. We set $x(t) = \delta(t)$ to get the impulse response of the system:

$$h''(t) + 9h(t) = -6\delta'(t).$$

Following the algorithm on Page 166, we get that $n = 2, m = 1, m < n$. Therefore our solution looks like $h(t) = h_{hom}(t)u(t)$.

The homogeneous solution $h_{hom}(t)$ should satisfy $h''_{hom}(t) + 9h_{hom}(t) = 0$.

$$h_{hom}(t) = C_1 \cos 3t + C_2 \sin 3t\,, h(t) = (C_1 \cos 3t + C_2 \sin 3t)u(t).$$

Now we need to determine constants of integration. In order to do that we integrate original equation once from $t = 0^-$ to $t = 0^+$.

$$h'(0+) - h'(0-) + 9\int_{0-}^{0+} h(t)dt = -6\int_{0-}^{0+} \delta'(t)dt = 0,$$

$$h'(t) = (-3C_1 \sin 3t + 3C_2 \cos 3t)u(t) + (C_1 \cos 3t + C_2 \sin 3t)\delta(t),$$

$$h'(0-) = 0, h'(0+) = 3C_2, \qquad 9\int_{0-}^{0+} h(t)dt = 0,.$$

Therefore we get $3C_2 = 0$, $\quad C_2 = 0$. Now we integrate the initial equation twice:

$$h(0+) - h(0-) + 9\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt = -6\int_{0-}^{0+}\int_{-\infty}^{t} \delta'(s)dsdt = -6,$$

$$h(0-) = 0,\ \ h(0+) = C_1,\ \ 9\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt = 0,\ \ C_1 = -6.$$

The final answer is: $h(t) = -6\cos 3t\, u(t)$.

20. Assume that $f_1(x) \neq 0$ for $0 \leq x \leq 4$ and $f_2(x) \neq 0$ for $-3 \leq x \leq -1$. From definition of convolution $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(s)f_2(x-s)ds$.

$$0 \leq s \leq 4, \text{ since } f_1 \neq 0 \text{ when its argument is in this range ;}$$

$$-3 \leq x - s \leq -1, \text{ since } f_2 \neq 0 \text{ when its argument is in this range ;}$$

Combining both inequalities we get $-3 \leq x \leq 3$, which is the answer of the problem, because x is the argument of convolution.

23. Recall that $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), n$ – integer, $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(s)f_2(x-s)ds$.

Answers: A – 2,4; B – none; C – 6; D – 5; E – 1,3,8.

Let us consider the general case $A * \delta_N(t) * rect(t * M) = A * \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\tau - Nn) rect\big(M * (t-\tau)\big)d\tau = A * \sum_{n=-\infty}^{\infty} rect(M * (t - Nn))$. A — amplitude, N – period, M — time scaling.

x_1: $A = 1, N = 2, M = \frac{1}{2}$. This means that each rectangular function is non-zero over interval of length 2. Also the period of rect functions is 2. So they stand side by side of each other without breaks and do not overlap. Since amplitude is 1, the sum of such rectangular functions is just a line $y = 1$, which corresponds to graph E.

x_2: $x_2 = 4 * x_1$. So the answer is a line $y = 4$, which corresponds to graph A.

x_3: $A = \frac{1}{4}, N = \frac{1}{2}, M = 1/2$. This means that each rectangular function is non-zero over interval of length 2. The period of the functions is ½ . These functions overlap and exactly 4 rectangular functions cover each point, their value add up to the line $y = 4$. After scaling the result with factor of $A = \frac{1}{4}$ we obtain the final result graph E.

x_4: $x_4 = 4 * x_3$. So the answer is a line $y = 4$, which corresponds to graph A.

x_5: $A = 1, N = 2, M = 2$. Each rectangular function is non-zero over interval of length ½ . Also the period of rectangular functions is 2. This means that there is a space between each 2 functions which is equal to $2 - \frac{1}{2} = \frac{3}{2}$. Since amplitude is 1, the sum of such rectangular functions corresponds to graph D.

x_6: $x_6 = 4 * x_5$. So the answer is graph C.

x_7: $A = \frac{1}{4}, N = \frac{1}{2}, M = 2$. Each rectangular function is non-zero over interval of length ½ . The period of rectangular functions is ½ , they do not overlap and span the whole domain without breaks. The amplitude is ¼ , the final graph is supposed to be a line $y = \frac{1}{4}$ that does not correspond to any given graph.

x_8: $x_8 = 4 * x_7$. So the answer is a line $y = 1$ or graph E.

27a. The system equation is: $y''(t) + ay'(t) + by(t) = x(t)$.

To find the impulse response $h(t)$ we solve $h''(t) + ah'(t) + bh(t) = \delta(t)$. We start with computing solution of the corresponding homogeneous equation

$$h''_{hom}(t) + ah'_{hom}(t) + bh_{hom}(t) = 0.$$

$$\lambda^2 + a\lambda + b = 0, \ \ \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}, \quad h_{hom} = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

For this equation $h(t) = h_{hom}(t)u(t) = \left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\right)u(t)$. The stability of the system will be guaranteed if the impulse response is absolutely integrable. This requires real parts of λ to be negative. It will prevent response from growing.

Let us proceed with solving for unknown constants C_1, C_2. For this purpose we integrate the original equation from $t = 0^-$ to $t = 0^+$ once and twice:

$$\begin{cases} h'(0^+) - h'(0^-) + ah(0^+) - ah(0^-) + b\displaystyle\int_{0-}^{0+} h(t)dt = \int_{0-}^{0+} \delta(t)dt = 1, \\ h(0^+) - h(0^-) + a\displaystyle\int_{0-}^{0+} h(t)dt + b\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt = \int_{0-}^{0+}\int_{-\infty}^{t} \delta(s)dsdt = 0. \end{cases}$$

We notice that all integrals in the left hand sides are equal to zero because the impulse response $h(t)$ does not have singularities. Also, $h(0^-) = h'(0^-) = 0$ because the only excitation of the system occurs at time $t = 0$. Therefore one gets

$$\begin{cases} h'(0^+) + ah(0^+) = 1, \\ h(0^+) = 0. \end{cases} \Rightarrow \begin{cases} h'(0^+) = 1, \\ h(0^+) = 0. \end{cases}$$

One needs to be careful evaluating $h(0^+),\ h'(0^+)$ considering complex or real $\lambda_{1,2}$. Values of parameters are set to be $a = 0.5, b = -0.1, \lambda_1 = 0.1531, \lambda_2 = -0.6531$. Because of the λ_1 we already expect system to be unstable. Substituting those values we get

$h(0^+) = C_1 + C_2 = 0,\ h'(0^+) = 0.1531C_1 - 0.6531C_2 = 1,\ \Rightarrow\ C_1 = 1.2404, C_2 = -1.2404.$
$h(t) = 1.2404(e^{0.1531t} - e^{-0.6531t})u(t)$.

$$\int_{-\infty}^{\infty} |h(t)|dt = 1.2404\int_0^{\infty}(e^{0.1531t} - e^{-0.6531t})dt = \infty.$$ Not BIBO stable.

If we redefine the response such that real parts of λ are all negative, system will be stable. For example, pair $a = 0.5, b = 1$ would determine a stable system.

27b. The system equation is: $y''(t) + ay'(t) + by(t) = x''(t)$.

To find the impulse response $h(t)$ we solve $h''(t) + ah'(t) + bh(t) = \delta''(t)$. The homogeneous solution is the same as in Part a. However the impulse response will contain additional term

$$h(t) = h_{hom}(t)u(t) + C\delta(t) = \left(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\right)u(t) + C\delta(t)$$

since in this problem the orders of highest derivatives of left and right hand sides of the equation coincide. In order to determine three constants C, C_1, C_2 the original equation is integrated three times from $t = 0^-$ to $t = 0^+$:

$$\begin{cases} h'(0^+) - h'(0^-) + ah(0^+) - ah(0^-) + b\displaystyle\int_{0-}^{0+} h(t)dt = \delta'(0^+) - \delta'(0^-) = 0, \\ h(0^+) - h(0^-) + a\displaystyle\int_{0-}^{0+} h(t)dt + b\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt = \delta(0^+) - \delta(0^-) = 0, \\ \displaystyle\int_{0-}^{0+} h(t)dt + a\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt + b\int_{0-}^{0+}\int_{-\infty}^{t}\int_{-\infty}^{s} h(w)dwdsdt = \int_{0-}^{0+} \delta(t)dt = 1. \end{cases}$$

We notice that all double and triple integrals in the left hand sides are equal to zero because the impulse response $h(t)$ has $\delta(t)$ singularity and it disappears after first integration. Also, $h(0^-) = h'(0^-) = 0$ because the only excitation of the system occurs at time $t = 0$. Therefore one gets

$$\begin{cases} h'(0^+) + ah(0^+) + b\displaystyle\int_{0-}^{0+} h(t)dt = 0, \\ h(0^+) + a\displaystyle\int_{0-}^{0+} h(t)dt = 0, \\ \displaystyle\int_{0-}^{0+} h(t)dt = 1. \end{cases} \Rightarrow \begin{cases} h'(0^+) = b - a^2, \\ h(0^+) = -a, \\ \displaystyle\int_{0-}^{0+} h(t)dt = 1. \end{cases}$$

The bottom equation in the system defines $C = 1$. Again, values of parameters are set to be $a = 0.5, b = -0.1, \lambda_1 = 0.1531, \lambda_2 = -0.6531$.

$$\begin{cases} h'(0^+) = -0.35, \\ h(0^+) = -0.5. \end{cases}$$

$h(0^+) = C_1 + C_2 = -0.5$, $h'(0^+) = 0.1531C_1 - 0.6531C_2 = -0.35$, $\Rightarrow$ $C_1 = -0.8392, C_2 = 0.3392$.
$h(t) = (-0.8392e^{0.1531t} + 0.3392e^{-0.6531t})u(t) + \delta(t)$. Again, system is BIBO unstable.

28. The system equation is: $y''(t) + \frac{2}{3}y'(t) + \frac{1}{8}y(t) = x(t)$. We let $x(t) = \delta(t)$ to find the impulse response $h(t)$.

The homogeneous solution of this equation is $h_{hom}(t) = e^{-\frac{t}{3}}(C_1 \sin\frac{t}{6\sqrt{2}} + C_2 \cos\frac{t}{6\sqrt{2}})$.

The impulse response accordingly has the form $h(t) = e^{-\frac{t}{3}}(C_1 \sin\frac{t}{6\sqrt{2}} + C_2 \cos\frac{t}{6\sqrt{2}})u(t)$.

Integrating equations once and twice from $t = 0^-$ to $t = 0^+$ one gets equations

$$\begin{cases} h'(0^+) - h'(0^-) + \frac{2}{3}[h(0^+) - h(0^-)] + \frac{1}{8}\int_{0-}^{0+} h(t)dt = \int_{0-}^{0+} \delta(t)dt = 1, \\ h(0^+) - h(0^-) + \frac{2}{3}\int_{0-}^{0+} h(t)dt + \frac{1}{8}\int_{0-}^{0+}\int_{-\infty}^{t} h(s)dsdt = \int_{0-}^{0+}\int_{-\infty}^{t} \delta(s)dsdt = 0. \end{cases}$$

Note that all integrals in the left hand sides are equal to zero because solution $y(t)$ does not have impulses in it. Terms computed at $t = 0^-$ are equal to zero as well, since solution is equal to zero for all negative t's.

$$\begin{cases} h'(0^+) + \frac{2}{3}[h(0^+)] = 1, \\ h(0^+) = 0. \end{cases} \Rightarrow \begin{cases} h'(0^+) = 1, \\ h(0^+) = 0. \end{cases}$$

We evaluate $h(t = 0^+) = C_2 = 0$, then $h'(t = 0^+) = C_1 * \frac{1}{6\sqrt{2}} = 1, C_1 = 6\sqrt{2}$. The impulse response of the system is equal to $h(t) = 6\sqrt{2}\, e^{-\frac{t}{3}} \sin\frac{t}{6\sqrt{2}} u(t)$.

Stability check: System is BIBO stable if its impulse response is absolutely integrable. Let us check this condition

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} 6\sqrt{2}\, e^{-\frac{t}{3}} | \sin\frac{t}{6\sqrt{2}} | dt \le 6\sqrt{2}\int_{0}^{\infty} e^{-\frac{t}{3}}dt = 18\sqrt{2} < \infty$$

Conclusion: system is BIBO stable. Note that the same result one can get investigating the eigenvalues of the original equation. One can calculate them $\lambda_{1,2} = -0.3333 \pm 0.1179i$ and notice that their real parts are negative which means that system is BIBO stable.

32a.