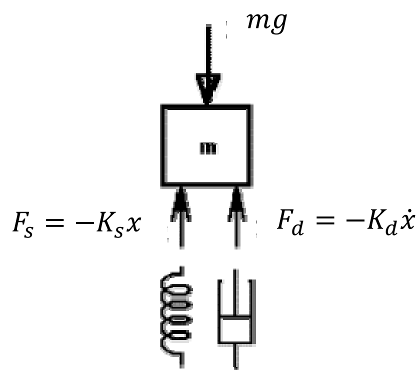
MAE 143A – Homework 4

Chapter 4



12. $\sum F = ma$

$$-mg - K_s[y(t) - x(t) - d_o] - K_d[y'(t) - x'(t)] = my''(t) \qquad (1)$$

a) When car is at rest, set all derivates in (1) equal to zero which yields

$$-mg - K_s[y(t) - x(t) - d_o] = 0$$

$$y(t) - x(t) = -\frac{mg}{K_s} + d_o$$

$$= -\frac{(1500\ kg)\left(9.81\frac{m}{s}\right)}{7500\ N\cdot\frac{s}{m}} + 0.6\ m$$

$$= 0.404\ \mathrm{m}$$

b) Rearrange (1) as

$$my''(t) + K_d[y'(t) - x'(t)] + K_s\left[y(t) - x(t) - d_o + \frac{mg}{K_s}\right] = 0 \qquad (2)$$

Define new variable as $z(t) = y(t) - x(t) - d_o + \frac{mg}{K_s}$

$$z'(t) = y'(t) - x'(t)$$

$$z''(t) = y''(t) - x''(t)$$

Substitute into (2) which yields

$$m[z''(t) + x''(t)] + K_d z'(t) + K_s z(t) = 0$$

$$mz''(t) + K_d z'(t) + K_s z(t) = -mx''(t)$$

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = -x''(t) \qquad (3)$$

Case 2: m=n, add impulse term to general solution

Solve for the homogenous equation

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = 0$$

$$\lambda^2 + \frac{K_d}{m}\lambda + \frac{K_s}{m} = 0 \quad \rightarrow \quad \lambda_{1,2} = -\frac{K_d \pm \sqrt{{K_d}^2 - 4K_s}}{2m} = -6.667 \pm 2.357i$$

$$z(t) = e^{-6.667t}[K_1 \cos(2.537t) + K_2 \sin(2.357t)]$$

$$h_z(t) = e^{-6.667t}[K_1 \cos(2.357t) + K_2 \sin(2.357t)]u(t)$$

The impulse response will take the form of

$$h(t) = e^{-6.667t}[K_1 \cos(2.357t) + K_2 \sin(2.357t)]u(t) + K_\delta \delta(t)$$

Method 1:

Solve simplified problem first

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = x(t)$$

$$z'(0^+) - z'(0^-) + \frac{K_d}{m}[z(0^+) - z(0^-)] + \frac{K_s}{m}\int_{0^-}^{0^+} z(t)dt = \int_{0^-}^{0^+} x(t)dt = 1$$

$$z'(0^+) + \frac{K_d}{m} z(0^+) = 1 \quad \rightarrow \quad -6.667K_1 + 2.357K_2 + \frac{K_d}{m}K_1 = 1$$

$$z(0^+) - z(0^-) + \frac{K_d}{m}\int_{0^-}^{0^+} z(t)dt = 0$$

$$z(0^+) = 0 \quad \rightarrow \quad K_1 = 0$$

Thus $K_1 = 0 \quad K_2 = 0.424$ which yields the homogenous solution as

$$h_z(t) = [0.424e^{-6.667t} \sin(2.357t)]u(t)$$

Then define new variable $g(t)$ where $b_0 = -1$ to solve for (3)

$$g(t) = z(t) - b_0 x(t)$$

$$g(t) = z(t) + x(t)$$

$$g'(t) = z'(t) + x'(t)$$

$$g''(t) = z''(t) + x''(t)$$

Substitute into (3) which yields

$$g''(t) - x''(t) + \frac{K_d}{m}[g'(t) - x'(t)] + \frac{K_s}{m}[g(t) - x(t)] = -x''(t)$$

$$g''(t) + \frac{K_d}{m} g'(t) + \frac{K_s}{m} g(t) = \frac{K_s}{m} x(t) + \frac{K_d}{m} x'(t)$$

Set $x(t) = h_z(t)$ to solve for homogenous solution of (3)

$$h_g(t) = \left[\frac{K_s}{m} x(t) + \frac{K_d}{m} x'(t)\right] u(t) = \left[\frac{K_s}{m} h_z(t) + \frac{K_d}{m} h_z{}'(t)\right] u(t)$$

$$h_g(t) = \left\{\frac{K_s}{m}[0.424e^{-6.667t}(\sin(2.357t))] + \frac{K_d}{m}[-2.826e^{-6.667t}\sin(2.357t) + e^{-6.667t}\cos(2.357t)]\right\} u(t)$$

$$h_g(t) = \{50[0.424e^{-6.667t}(\sin(2.357t))] + 13.333[-2.826e^{-6.667t}\sin(2.357t) + e^{-6.667t}\cos(2.357t)]\}u(t)$$

$$h_g(t) = e^{-6.667t}[13.333\cos(2.357t) - 16.479\sin(2.357t)]u(t)$$

$h(t) = h_g(t) - b_0\delta(t)$

Therefore the impulse response is

$h(t) = e^{-6.667t}[13.333\cos(2.357t) - 16.479\sin(2.357t)]u(t) - \delta(t)$

Method 2:

Let $z(t) = h(t)$ and $x(t) = \delta(t)$ for (3), then

$$h"(t) + \frac{K_d}{m}h'(t) + \frac{K_s}{m}h(t) = -\delta"(t)$$

Integrate from $t = 0^+ to\ t = 0^-$:

(i) $h'(0^+) + h'(0^-) + \frac{K_d}{m}[h(0^+) + h(0^-)] + \frac{K_s}{m}\int_{0^-}^{0^+} h(t)dt = -[\delta'(0^+) - \delta'(0^-)]$

⇨ $h'(0^+) + \frac{K_d}{m}h(0^+) + \frac{K_s}{m}\int_{0^-}^{0^+} h(t)dt = 0$

⇨ $(-6.667K_1 + 2.357K_2) + \frac{K_d}{m}K_1 + \frac{K_s}{m}K_\delta = 0$ (a)

(ii) $h(0^+) + h(0^-) + \frac{K_d}{m}\int_{0^-}^{0^+} h(t)dt + \frac{K_s}{m}\int_{0^-}^{0^+}\int_{-\infty}^{t} h(t)dt = -[\delta(0^+) - \delta(0^-)]$

⇨ $K_1 + \frac{K_d}{m}K_\delta = 0$ (b)

(iii) $\int_{0^-}^{0^+} h(t)dt + \frac{K_d}{m}\int_{0^-}^{0^+}\int_{-\infty}^{t} h(t)dt + \frac{K_s}{m}\int_{0^-}^{0^+}\int_{-\infty}^{t}\int_{-\infty}^{s} h(s)dsdt = -\int_{0^-}^{0^+}\delta(t)dt$

⇨ $K_\delta = -1$ (c)

Solving equations (a) (b) (c) we get $K_1 = 13.333, K_2 = 16.497, K_\delta = -1$

Therefore the impulse response is

$h(t) = e^{-6.667t}[13.333\cos(2.357t) - 16.479\sin(2.357t)]u(t) - \delta(t)$

c) Apply convolution integral $h_c = 0.15\,m$

$$z(t) = 0.15h(t) * u(t) = 0.15\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

$$z(t) = 0.15\int_{-\infty}^{\infty}\{e^{-6.67\tau}[13.33\cos(2.36\tau) - 16.5\sin(2.36\tau)]u(\tau) - \delta(\tau)\}u(t-\tau)d\tau$$

Use following properties to solve for z(t)

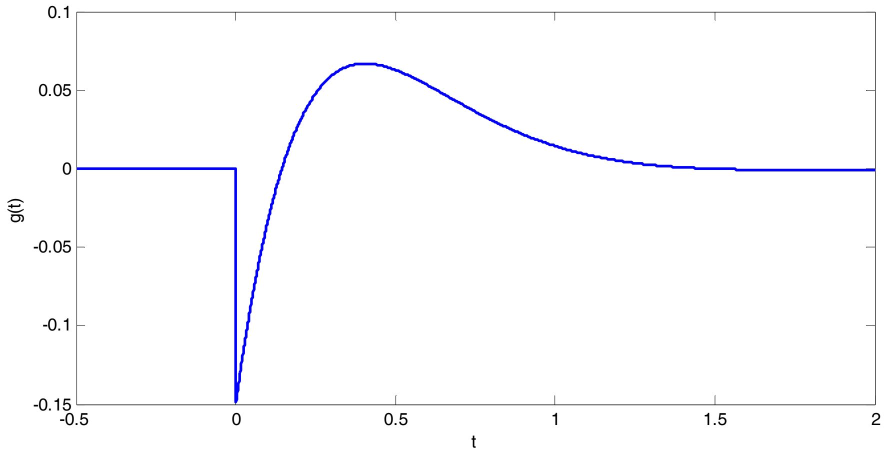
$$\int_0^t e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$$

$$\int_0^t e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)]$$

$$z(t) = 0.15\left\{13.33\frac{e^{-6.67t}}{50}[-6.67\cos(2.36\tau) + 2.36\sin(2.36\tau)] - 16.5\frac{e^{-6.67t}}{50}[-6.67\sin(2.36\tau) - 2.36\cos(\tau)]\right\} - 0.15u(t)$$

$$z(t) = 0.15\{e^{-3.33t}[2.81\sin(2.36t) - \cos(2.36t)] + 1\}u(t) - 0.15u(t)$$

$$z(t) = 0.15e^{-3.33t}[2.81\sin(2.36t) - \cos(2.36t)]u(t)$$



20. $y(t) = \int_{-\infty}^{t/3} x(\lambda)\, d\lambda$

Time-Invariance:

Let $x_1(t) = g(t)$. Then $y_1(t) = \int_{-\infty}^{\frac{t}{3}} g(\lambda) d\lambda$.

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \int_{-\infty}^{\frac{t}{3}} g(\lambda - t_0) d\lambda = \int_{-\infty}^{\frac{t}{3} - t_o} g(u) du$

Substitute t with $t - t_o$ into $y_1(t)$ yields $y_1(t - t_0) = \int_{-\infty}^{\frac{t}{3}} g(\lambda) d\lambda$

$y_2(t) \neq y_1(t - t_0)$, thus the system is time-variant.

Stability:

If $x(t)$ is bounded, then $y(t) = \int_{-\infty}^{\frac{t}{3}} x(\lambda)\, d\lambda = x(\lambda) \int_{-\infty}^{t/3} d\lambda$ increases without bound therefore the system is unstable.

Invertibility:

$y'(t) = x\left(\frac{t}{3}\right)$ → $x(t) = y'(3t)$

Thus the system is invertible.

Chapter 6

18c. $-y"(t) + 3y'(t) = 3x(t) + 5x"(t)$

Rewrite as $y"(t) - 3y'(t) = -3x(t) - 5x"(t)$ (1)

Case 2: m=n, add impulse term to general solution

Solve for the homogeneous solution

$y"(t) - 3y'(t) = 0$

$\lambda^2 - 3\lambda = 0 \quad \rightarrow \quad \lambda_{1,2} = 0,3$

$y(t) = K_1 + K_2 e^{3t}$

$h_z(t) = (K_1 + K_2 e^{3t})u(t)$

Method 1:

Solve simplified problem first

$y"(t) - 3y'(t) = x(t)$

$$y'(0^+) - y'(0^-) - 3[y(0^+) - y(0^-)] = \int_{0^-}^{0^+} x(t)dt = 1$$

$$y'(0^+) - 3y(0^+) = 1 \quad \rightarrow \quad 3K_2 - 3(K_1 + K_2) = 1$$

$$y(0^+) - y(0^-) - 3\int_{0^-}^{0^+} y(t)dt = \int_{0^-}^{0^+} \int_{-\infty}^{t} x(s)dsdt = 0$$

$$y(0^+) = 0 \quad \rightarrow \quad K_1 + K_2 = 0$$

Thus, $K_1 = -\frac{1}{3} \quad K_2 = \frac{1}{3}$

$$h_z(t) = \left[-\frac{1}{3}(1 - e^{3t})\right]u(t)$$

Define new variable $g(t)$ where $b_0 = -5$ to solve for (1)

$g(t) = y(t) - b_0 x(t)$

$g(t) = y(t) + 5x(t)$

$g'(t) = y'(t) + 5x'(t)$

$g"(t) = y"(t) + 5x"(t)$

Substitute into (1) which yields

$g"(t) - 5x"(t) - 3[g'(t) - 5x'(t)] = -3x(t) - 5x"(t)$

$g"(t) - 3g'(t) = -3x(t) - 15x'(t)$

$h_g(t) = [-3h_z - 15h_z'(t)]u(t)$

$$h_g(t) = \left[-3\frac{1}{3}(1 - e^{3t}) - 15e^{3t}\right]u(t) = (1 - 16e^{3t})u(t)$$

$h(t) = h_g(t) - b_0\delta(t)$

Therefore the impulse response is

$h(t) = (1 - 16e^{3t})u(t) - 5\delta(t)$

Method 2:

Let $y(t) = h(t)$ and $x(t) = \delta(t)$ for (1), then

$h''(t) - 3h'(t) = -3\delta(t) - 5\delta''(t)$

Integrate from $t = 0^+ to\ t = 0^-$:

(i) $-[h'(0^+) - h'(0^-)] + 3[h(0^+) - h(0^-)] = 3\int_{0^-}^{0^+} \delta(t)dt + 5[\delta'(0^+) - \delta'(0^-)]$

⇨ $-h'(0^+) + 3h(0^+) = 3\int_{0^-}^{0^+} \delta(t)dt$ → $-3K_2 + 3(K_1 + K_2) = 3$ (a)

(ii) $-[h(0^+) - h(0^-)] + 3\int_{0^-}^{0^+} h(t)dt = 3\int_{0^-}^{0^+}\int_{-\infty}^{t} \delta(s)dsdt + 5[\delta(0^+) - \delta(0^-)]$

⇨ $-h(0^+) + 3\int_{0^-}^{0^+} h(t)dt = 0$ → $-(K_1 + K_2) + 3K_\delta = 0$ (b)

(iii) $-\int_{0^-}^{0^+} h(t)dt + 3\int_{0^-}^{0^+}\int_{-\infty}^{t} h(s)dsdt = 3\int_{0^-}^{0^+}\int_{-\infty}^{t}\int_{-\infty}^{s} \delta(w)dwdsdt + 5\int_{0^-}^{0^+} \delta(t)dt$

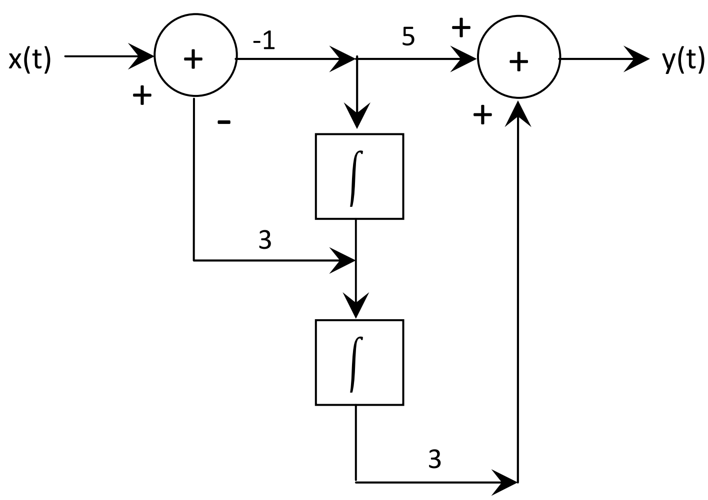
⇨ $-\int_{0^-}^{0^+} h(t)dt = 5\int_{0^-}^{0^+} \delta(t)dt$ → $-K_\delta = 5$ (c)

Solving equations (a) (b) (c) we get $K_1 = 1, K_2 = -16, K_\delta = -5$

Therefore the impulse response is

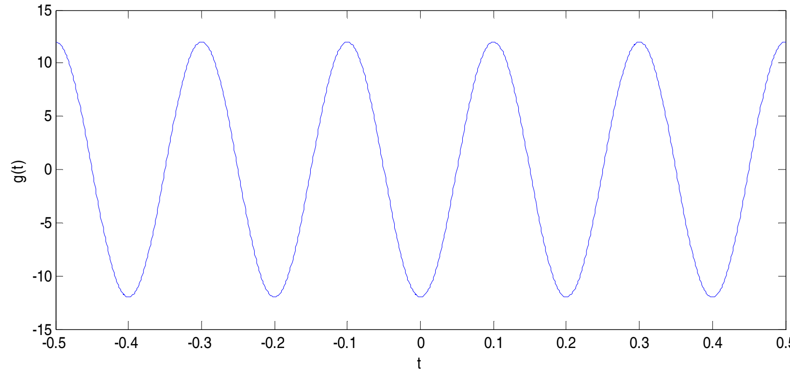
$h(t) = (1 - 16e^{3t})u(t) - 5\delta(t)$

Plot II-type block diagram with integrators:



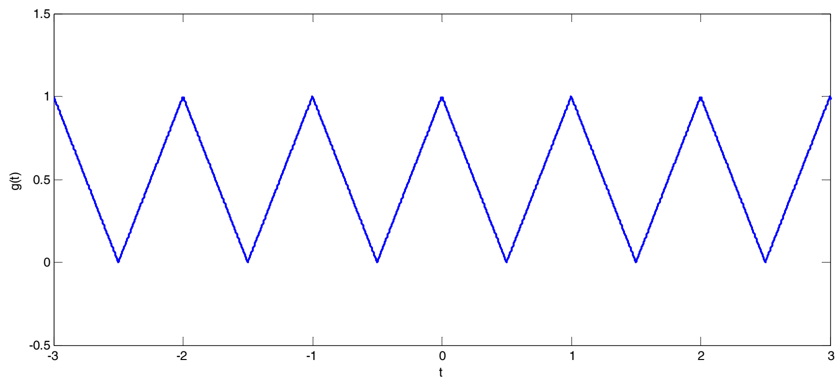
22a. $g(t) = 3\cos(10\pi t) * 4\delta(t + {}^{1}/_{10})$

$= 12cos[10\pi(t + {}^{1}/_{10})] = 12\cos(12\pi t + \pi) = -12\cos(10\pi t)$



22d. $g(t) = tri(2t) * \delta_1(t)$

$$= \sum_{n=-\infty}^{\infty} tri(2(t-n))$$



Chapter 15

33b. $G(s) = \frac{4}{(s+3)(s+8)}$

Express in partial fraction expansion.

$$G(s) = \frac{A}{s+3} + \frac{B}{s+8} = \frac{4}{(s+3)(s+8)}$$

$$A(s+8) + B(s+3) = 4$$

$$s(A+B) + 8A + 3B = 4$$

$$s(A+B) = 0,\ 8A + 3B = 4 \quad \rightarrow \quad A = \frac{4}{5}, B = -\frac{4}{5}$$

$$G(s) = \frac{\frac{4}{5}}{s+3} - \frac{\frac{4}{5}}{s+8}$$

From the Laplace transform table on pg. 591, we get

$$g(t) = \left(\tfrac{4}{5}e^{-3t} - \tfrac{4}{5}e^{-8t}\right)u(t)$$

Initial-value theorem

$$g(0^+) = \lim_{s\to\infty} sG(s) = \lim_{s\to\infty} \frac{4s}{(s+3)(s+8)} = 0$$

Final-value theorem

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{4s}{(s+3)(s+8)} = 0$$

33b. $G(s) = \frac{s}{s^2+2s+2}$

$$G(s) = \frac{s}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

From the Laplace transform table on pg. 591, we get

$$g(t) = e^{-t}[\cos(t) - \sin(t)]u(t)$$

Initial-value theorem

$$g(0^+) = \lim_{s\to\infty} sG(s) = \lim_{s\to\infty} \frac{s^2}{s^2+2s+2} = 1$$

Final-value theorem

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s^2}{s^2+2s+2} = 0$$

39a) $y''(t) + 2y'(t) + 10y(t) = x(t)$

First Laplace transform both sides of the equation.

$$s^2Y(s) - sy(0^-) - \frac{d}{dt}(y(t))_{t=0^-} + 2[sY(s) - y(0^-)] + 10Y(s) = \frac{1}{s}$$

Then solve for Y(s).

$$Y(s) = \frac{\frac{1}{s} + sy(0^-) + 2y(0^-) + \frac{d}{dt}(y(t))_{t=0^-}}{s^2+2s+10}$$

$$Y(s) = \frac{\frac{1}{s} - 5s}{s^2+2s+10} = \frac{1}{s}\frac{1-5s^2}{(s+1)^2+9}$$

Express in partial fraction expansion.

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 9}$$

$$A(s+1)^2 + 9A + s(Bs + C) = 1 - 5s^2$$

$$As^2 + 2As + A + 9A + Bs^2 + Cs = 1 - 5s^2$$

$$(A + B)s^2 + (2A + C)s + 10A = 1 - 5s^2$$

$$A + B = -5,\ 2A + C = 0,\ 10A = 1 \quad \rightarrow \quad A = \frac{1}{10}, B = -\frac{51}{10}, C = -\frac{1}{5}$$

$$Y(s) = \frac{\frac{1}{10}}{s} + \frac{-\frac{51}{10}s - \frac{1}{5}}{(s+1)^2 + 9} = \frac{1}{10}\left(\frac{1}{s} - \frac{51s + 2}{(s+1)^2 + 9}\right)$$

$$Y(s) = \frac{1}{10}\left[\frac{1}{s} - 51\frac{s+1}{(s+1)^2 + 9} + \frac{49}{3}\frac{3}{(s+1)^2 + 9}\right]$$

From the Laplace transform table on pg. 591, we get

$$y(t) = \frac{1}{10}\left[1 - e^{-t}\left(51\cos(3t) - \frac{49}{3}\sin(3t)\right)\right]u(t)$$

Additional Item:

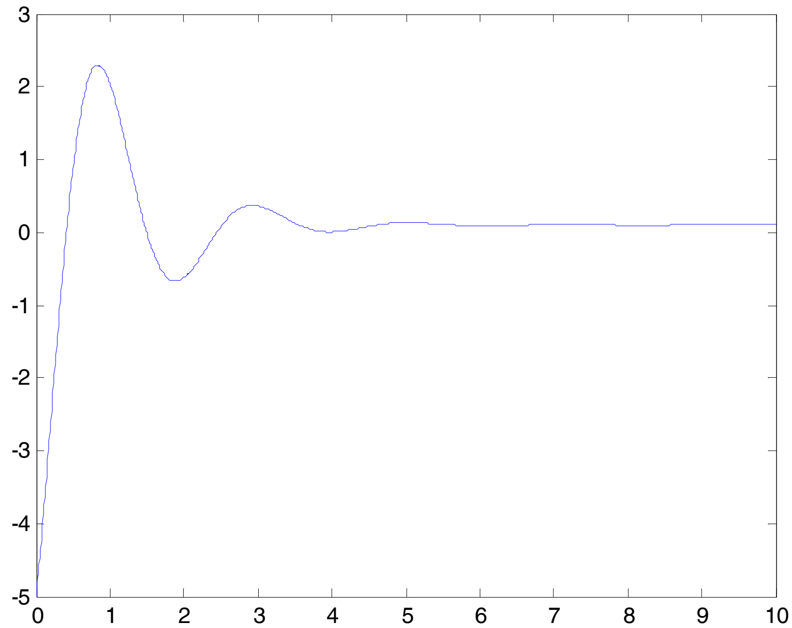
```
% Problem 39 (a): Use matlab to compute and plot the response of the
% system. You should use commands like 'tf', and 'step'.

% Method 1: use 'tf' and 'step' command - simulate response at initial
% conditions and add to step response

Y = tf([1],[1 2 10]);         % construct transfer function
a = [0 1; -10 -2]; b = [0; 1]; c = [1 0]; d = [0];
Y1 = ss(a,b,c,d);             % construct state space
y0 = [-5 10];                 % initial conditions
t = 0:0.01:10;
y1 = step(Y,t);               % step response at zero state
y2 = initial(Y1,y0,t);        % initial condition response
y = y1+y2;                    % time response with I.C.s
plot(t,y);

% Method 2: use 'lsim' command - compute time response to zero inputs

Y = tf([1],[1 2 10]);         % construct transfer function
a = [0 1; -10 -2]; b = [0; 1]; c = [1 0]; d = [0];
Y1 = ss(a,b,c,d);             % construct state space
x = sign(t);                  % x(t) = u(t)
y0 = [-5 10];                 % initial conditions
t = 0:0.01:10;
y = lsim(Y1,x,t,y0);          % time response with I.C.s
plot(t,y);
```



43. $H1 = \frac{K}{(s+1)(s+2)}$ $\quad\quad H_2 = 1$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1}{1 + H_1 H_2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{K}{(s+1)(s+2)}}{1 + \frac{K}{(s+1)(s+2)}}$$

$$H(s) = \frac{K}{s^2 + 3s + 2 + K}$$

Find the roots of the denominator

$$s = \frac{-3 \pm \sqrt{9 - 4(2+K)}}{2} = \frac{-3 \pm \sqrt{1 - 4K}}{2}$$

The system is stable for all positive real values of K.

Additional Item:

```
% Problem 43: For a fixed value of K that makes the system
% stable, use the matlab commands 'tf' and 'feedback' to compute
% the transfer function from x to y. Use 'impulse' to plot the
% impulse response of the resulting system.

% Solution: set K to any real positive value, i.e. K = 1
```

```
num = [1];
den = [1 3 2];
H = tf(num,den)
Hf = feedback(H,1)
impulse(Hf)

>> H = tf(num,den)

 Transfer function:

      1

-------------

s^2 + 3 s + 2

>> Hf = feedback(H,1)

 Transfer function:

      1

-------------

s^2 + 3 s + 3
```